Chapter 0
From the Ground Up

0.1 Basic Problems

0.1 Consider the following problems about trigonometric and polar forms.

(a) Let \( z = 6 e^{j\pi/4} \) find  
   i. \( \Re e(z) \),  
   ii. \( \Im m(z) \)

(b) If \( z = 8 + j3 \) and \( v = 9 - j2 \), is it true that
   i. \( \Re e(z) = 0 \).  
   ii. \( \Im m(v) = -0.5j(v - v^*) \)?  
   iii. \( \Re e(z + v^*) = \Re e(z + v) \)?  
   iv. \( \Im m(z + v^*) = \Im m(z - v) \)?

Answers: (a) \( \Re e(z) = 3\sqrt{2} \); \( \Im m(z) = 3\sqrt{2} \); (b) Yes to all.

Solution

(a) \( z = 6e^{j\pi/4} = 6 \cos(\pi/4) + j6 \sin(\pi/4) \)
   i. \( \Re e(z) = 6 \cos(\pi/4) = 3\sqrt{2} \)
   ii. \( \Im m(z) = 6 \sin(\pi/4) = 3\sqrt{2} \)

(b)  
   i. Yes, \( \Re e(z) = 0.5(z + z^*) = 0.5(2\Re e(z)) = \Re e(z) = 8 \)
   ii. Yes, \( \Im m(v) = -0.5j(v - v^*) = -0.5j(2\Im m(v)) = \Im m(v) = -2 \)
   iii. Yes, \( \Re e(z + v^*) = \Re e(\Re e(z) + \Re e(v^*) + \Im m(z) - \Im m(v)) = \Re e(z + v) = 17 \)
   iv. Yes, \( \Im m(z + v^*) = \Im m(17 + j5) = \Im m(z - v) = \Im m(-9 + j5) = 5 \)
0.2 Using the vectorial representation of complex numbers it is possible to get some interesting inequalities.

(a) Is it true that for a complex number $z = x + jy$ we have that $|x| \leq |z|$? Show it geometrically by representing $z$ as a vector.

(b) The so called triangle inequality says that for any complex (or real) numbers $z$ and $v$ we have that $|z + v| \leq |z| + |v|$. Show this geometrically.

(c) If $z = 1 + j$ and $v = 2 + j$ is it true that

(i) $|z + v| \leq |z| + |v|$? (ii) $|z - v| \leq |z| + |v|$?

Answer: (a) $|x| = |z||\cos(\theta)|$ and since $|\cos(\theta)| \leq 1$ then $|x| \leq |z|$; (c) yes to both.

Solution

(a) Representing the complex number $z = x + jy = |z|e^{j\theta}$ then $|x| = |z||\cos(\theta)|$ and since $|\cos(\theta)| \leq 1$ then $|x| \leq |z|$, the equality holds when $\theta = 0$ or when $z = x$, i.e., it is real.

(b) Adding two complex numbers is equivalent to adding two vectors to create a triangle with two sides the two vectors being added and the other side the vector resulting from the addition. Unless the two vector being added have the same angle, in which case $|z| + |v| = |z + v|$, it holds that $|z| + |v| > |z + v|$.

(c) The answer to both is yes. Indeed,

(a) $|z + v| = \sqrt{13} \leq |z| + |v| = \sqrt{2} + \sqrt{5}$

(b) $|z - v| = |1 - 1| = 1 \leq |z| + |v| = \sqrt{2} + \sqrt{5}$
0.3 Use Euler’s identity in the following problems.

(a) Find trigonometric identities in terms of \( \sin(\alpha), \sin(\beta), \cos(\alpha), \cos(\beta) \) for

(i) \( \cos(\alpha + \beta) \)  
(ii) \( \sin(\alpha + \beta) \)

(b) Is it true that

\[
\int_0^1 e^{j2\pi t} dt = 0 ?
\]

(c) Is it true that

i. \((-1)^n = \cos(\pi n)\) for any integer \(n\)?
ii. \(e^{j0} + e^{j\pi/2} + e^{j\pi} + e^{j3\pi/2} = 0\)? (sketch a figure)

**Answer:** (a) \( \cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta) \); (c) Yes to both.

**Solution**

(a) i. \( 2 \cos(\alpha + \beta) = e^{j(\alpha+\beta)} + e^{-j(\alpha+\beta)} = (e^{j\alpha} e^{j\beta}) + (e^{j\alpha} e^{j\beta})^* = 2 \Re(e^{j\alpha} e^{j\beta}) \) and

\[
\Re[e^{jn} e^{j\beta}] = \Re[(\cos(\alpha) + j \sin(\alpha))(\cos(\beta) + j \sin(\beta))] = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)
\]

so that

\[
\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)
\]

ii. \( 2j \sin(\alpha + \beta) = e^{j\alpha} e^{j\beta} - (e^{j\alpha} e^{j\beta})^* = 2j \Im[e^{j\alpha} e^{j\beta}], \) and the imaginary is

\[
\sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta) = \sin(\alpha + \beta)
\]

(b) \[
\int_0^1 e^{j2\pi t} dt = \frac{e^{j2\pi} - 1}{j2\pi} = 0
\]

also \[
\int_0^1 e^{j2\pi t} dt = \int_0^1 \cos(2\pi t) dt + j \int_0^1 \sin(2\pi t) dt = 0 + j0
\]

since the integrals of the sinusoids are over a period.

(c) i. Yes, \((-1)^n = (e^{j\pi})^n = e^{jn\pi} = \cos(n\pi) + j \sin(n\pi) = \cos(n\pi) \) since \(\sin(n\pi) = 0\) for any integer \(n\).

ii. Yes, \(e^{j0} = -e^{j\pi} \) and \(e^{j\pi/2} = -e^{j3\pi/2} \) so they add to zero.
0.4 Consider the following problems related to computation with complex numbers.

(a) Find and plot all roots of

\[
\begin{align*}
(i) & \quad z^3 = -1, \\
(ii) & \quad z^2 = 1 \\
(iii) & \quad z^2 + 3z + 1 = 0
\end{align*}
\]

(b) Suppose you want to find the natural log of a complex number \( z = |z|e^{i\theta} \), which is

\[
\log(z) = \log(|z|e^{i\theta}) = \log(|z|) + \log(e^{i\theta}) = \log(|z|) + j\theta
\]

If \( z \) is negative it can be written as \( z = |z|e^{j\pi} \) and we can find \( \log(z) \) by using the above derivation. The natural log of any complex number can be obtained this way also. Justify each one of the steps in the above equation and find

\[
(i) \log(-2), \quad (ii) \log(1 + j1), \quad (iii) \log(2e^{j\pi/4})
\]

(c) Let \( z = -1 \) find

\[
(i) \log(z), \quad (ii) e^{\log(z)}.
\]

(d) Let \( z = 2e^{j\pi/4} = 2(\cos(\pi/4) + j\sin(\pi/4)) \)

\[
(i) \text{ find } z^2/4, \quad (ii) \text{ is it true that } (\cos(\pi/4) + j\sin(\pi/4))^2 = \cos(\pi/2) + j\sin(\pi/2) = j^2?
\]

Answers: (a) (i) \( z_k = e^{j\pi(2k+1)/3} \); (b) \( \log(2e^{j\pi/4}) = \log(2) + j\pi/4 \); (c) \( e^{\log(z)} = -1 \); (d) (i) \( z^2/4 = j \).

Solution

(a) We have

i. \( z^3 = -1 = e^{j\pi(2k+1)} \) for \( k = 0, 1, 2 \), so roots are \( z_k = e^{j\pi(2k+1)/3}, k = 0, 1, 2 \), i.e., on a circle of unit radius and separated \( \pi/3 \) with one at \(-1\).

ii. \( z^2 = 1 = e^{j2\pi k} \) for \( k = 0, \pm 1, \pm 2, \cdots \), so roots are \( z_k = e^{j\pi k} = 1 \), i.e., on a circle of unit radius and separated \( \pi \), one at zero and the other at \(-1\).

iii. \( z^2 + 3z + 1 = (z + 1.5)^2 + (1 - 1.5^2) = 0 \) so the roots are \( z_{1,2} = \pm \sqrt{(1.5^2 - 1)} - 1.5 \)

(b) The log (using the Naperian base) of a product is the sum of the logs of the terms in the product, and the log and the exponential are the opposite of each other so the last term in the equation. Using the given expression for the log of a complex number (log of real numbers is a special case):

\[
\begin{align*}
\log(-2) &= \log(2e^{j\pi}) = \log(2) + j\pi \\
\log(1 + j1) &= \log(\sqrt{2}e^{j\pi/4}) = \log(\sqrt{2}) + j\pi/4 = 0.5\log(2) + j\pi/4 \\
\log(2e^{j\pi/4}) &= \log(2) + j\pi/4
\end{align*}
\]

(c) \( z = -1 = e^{j\pi} \) thus

i. \( \log(z) = \log(1e^{j\pi}) = \log(1) + j\pi = j\pi \)

ii. From above result, \( e^{\log(z)} = e^{j\pi} = -1 \)

(d) i. \( z^2/4 = (2e^{j\pi/4})^2/4 = 4e^{j\pi/2}/4 = j \)

ii. Yes, \( (\cos(\pi/4) + j\sin(\pi/4))^2 = (j/2)^2 \), and \( (z/2)^2 = (\cos(\pi/4) + j\sin(\pi/4))^2 = \cos(\pi/2) + j\sin(\pi/2) = j \)
0.2 Problems using MATLAB

0.5 Sampling — Consider a signal \( x(t) = 4\cos(2\pi t) \) defined for \(-\infty < t < \infty\). For the following values of the sampling period \( T_s \), generate a discrete-time signal \( x(n) = x(nT_s) = x(t)|_{t=nT_s} \).

(i) \( T_s = 0.1 \), (ii) \( T_s = 0.5 \), (iii) \( T_s = 1 \)

Determine for which of the given values of \( T_s \) the discrete-time signal has lost the information in the continuous-time signal. Use MATLAB to plot \( x(t) \) (use \( T_s = 10^{-4} \) and the \textit{plot} function) and the resulting discrete-time signals (use the \textit{stem} function). Superimpose the analog and the discrete-time signals for \( 0 \leq t \leq 3 \); use \textit{subplot} to plot the four figures as one figure. You also need to figure out how to label the different axes and how to have the same scales and units.

Answers: For \( T_s = 1 \) the discrete signal has lost information.

Solution

As we will see later, the sampling period of \( x(t) \) with a frequency of \( \Omega_{\text{max}} = 2\pi f_{\text{max}} = 2\pi \) should satisfy the Nyquist sampling condition

\[
\frac{1}{T_s} \geq 2f_{\text{max}} = 2 \text{ samples/sec}
\]

so \( T_s \leq 1/2 \) (sec/sample). Thus when \( T_s = 0.1 \) the continuous-time and the discrete-time signals look very much like each other, indicating the signals have the same information — such a statement will be justified in the chapter on sampling where we will show that the continuous-time signal can be recovered from the sampled signal. It is clear that when \( T_s = 1 \) the information is lost. Although it is not clear from the figure that when we let \( T_s = 0.5 \) the discrete-time signal keeps the information, this sampling period satisfies the Nyquist sampling condition and as such the original signal can be recovered from the sampled signal. The following MATLAB script is used.

```matlab
% Pr. 0.5
clear all; clf
T=3; Tss= 0.0001; t=[0:Tss:T];
xa=4*cos(2*pi*t); % continuous-time signal
xamin=min(xa);xamax=max(xa);
figure(1)
subplot(221)
plot(t,xa); grid
title('Continuous-time Signal'); ylabel('x(t)'); xlabel('t sec')
axis([0 T 1.5 *xamin 1.5*xamax])
N=length(t);
for k=1:3,
 if k==1,Ts= 0.1; subplot(222)
t1=[0:Ts:T]; n=1:Tsl/Tss; xd=zeros(1,N); xd(n)=4*cos(2*pi*t1);
plot(t,xa); hold on; stem(t,xd);grid;hold off
axis([0 T 1.5*xamin 1.5*xamax]); ylabel('x(0.1 n)'); xlabel('t')
 elseif k==2, Ts=0.5; subplot(223)
t2=[0:Ts:T]; n=1:Tsl/Tss; xd=zeros(1,N); xd(n)=4*cos(2*pi*t2);
plot(t,xa); hold on; stem(t,xd); grid; hold off
axis([0 T 1.5*xamin 1.5*xamax]); ylabel('x(0.5 n)'); xlabel('t')
 else,Ts=1; subplot(224)
t3=[0:Ts:T]; n=1:Tsl/Tss; xd=zeros(1,N); xd(n)=4*cos(2*pi*t3);
plot(t,xa); hold on; stem(t,xd); grid; hold off
axis([0 T 1.5*xamin 1.5*xamax]); ylabel('x(n)'); xlabel('t')
end
```

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Figure 2: Problem 5: Analog continuous-time signal (top left); continuous-time and discrete-time signals superposed for $T_s = 0.1$ sec (top right) and $T_s = 0.5$ sec and $T_s = 1$ sec (bottom left to right).
0.6 Differential and difference equations — Find the ordinary differential equation relating a current source $i_s(t) = \cos(\Omega_0 t)$ with the current $i_L(t)$ in an inductor, with inductance $L = 1$ Henry, connected in parallel with a resistor of $R = 1$ Ω (see Fig. 3). Assume a zero initial current in the inductor.

![RL circuit diagram](image)

Figure 3: Problem 6: RL circuit: input $i_s(t)$, output $i_L(t)$.

(a) Obtain a discrete equation from the ordinary differential equation using the trapezoidal approximation of an integral.

(b) Create a MATLAB script to solve the difference equation for $T_s = 0.01$ and for three frequencies for $i_s(t)$, $\Omega_0 = 0.005\pi$, $0.05\pi$ and $0.5\pi$. Plot the input current source $i_s(t)$ and the approximate solution $i_L(nT_s)$ in the same figure. Use the MATLAB function `plot`. Use the MATLAB function `filter` to solve the difference equation (use `help` to learn about `filter`).

(c) Solve the ordinary differential equation using symbolic MATLAB when the input frequency is $\Omega_0 = 0.5\pi$. Compare $i_s(t)$ and $i_L(t)$.

Answer: $\frac{di_L(t)}{dt} + i_L(t) = i_s(t)$

Solution

(a) According to Kirchoff’s current law

$$i_s(t) = i_R(t) + i_L(t) = \frac{v_L(t)}{R} + i_L(t)$$

but $v_L(t) = L \frac{di_L(t)}{dt}$ so that the ordinary differential equation relating the input $i_s(t)$ to the output current in the inductor $i_L(t)$ is

$$\frac{di_L(t)}{dt} + i_L(t) = i_s(t)$$

after replacing $L = 1$ and $R = 1$. Notice that this d.e. is the dual of the one given in the Chapter, so that the difference equation is

$$i_L(nT_s) = \frac{T_s}{2 + T_s} [i_s(nT_s) + i_s((n-1)T_s)] + \frac{2 - T_s}{2 + T_s} i_L((n-1)T_s) \quad n \geq 1$$

$$i_L(0) = 0$$

(b)(c) The scripts to solve the difference and ordinary differential equations are the following.

```matlab
% Pr. 0.6
clear all
```
Figure 4: Problem 6: Left (top to bottom): solution of difference equation for $\Omega_0 = 0.005$, 0.05, 0.5 (rad/sec). Right: input (top), solution of ordinary differential equation (bottom).

```matlab
% solution of difference equation
Ts=0.01;
t=[0:Ts:100];
figure(4)
for k=0:2;
    if k==0, subplot(311)
    elseif k==1, subplot(312)
    else, subplot(313)
    end
W0= 0.005*10^k*pi; % frequency of source
is=cos(W0*t); % source
a=[1 (-2+Ts)/(2+Ts)]; % coefficients of i_L(n), i_L(n-1)
b=[Ts/(2+Ts) Ts/(2+Ts)]; % coefficients of i_s(n), i_s(n-1)
il=filter(b,a,is); % current in inductor computed by
                  % MATLAB function ‘filter’
H=1/sqrt(1+W0^2)*ones(1,length(t)); % filter gain at W0
plot(t,is,t,il,'r',t,H,'g'); xlabel('t'); ylabel('i_s(t),i_L(t)')
axis([0 100 1.1*min(is) 1.1*max(is)])
legend('input current','output current','filter gain'); grid
pause(0.1)
end
%%%%
% solution of ordinary differential equation for cosine input of frequency 0.5pi
clear all
syms t x y
x=cos(0.5*pi*t);
y=dsolve('Dy+y=cos(0.5*pi*t)','y(0)=0','t')
figure(5)
subplot(211)
ezplo(x,[0 100]);grid
subplot(212)
ezplo(y,[0 100]);grid
axis([0 100 -1 1])
```
0.7 More integrals and sums — Although sums behave like integrals, because of the discrete nature of sums one needs to be careful with the upper and lower limits more than in the integral case. To illustrate this consider the separation of an integral into two integrals and compare it with the separation of a sum into two sums. For the integral we have that

\[ \int_{0}^{1} t \, dt = \int_{0}^{0.5} t \, dt + \int_{0.5}^{1} t \, dt \]

Show that this is true by computing the three integrals. Then consider the sum

\[ S = \sum_{n=0}^{100} n \]

find this sum and determine which of the following is equal to this sum

(i) \( S_1 = \sum_{n=0}^{50} n + \sum_{n=50}^{100} n \)

(ii) \( S_2 = \sum_{n=0}^{50} n + \sum_{n=51}^{100} n \)

Use symbolic MATLAB function \texttt{symsum} to verify your results. 

\textbf{Answers:} \( S = 5050 \), \( S_1 = 5100 \), \( S_2 = 5050 \).

\textbf{Solution}

The indefinite integral equals \( 0.5t^2 \). Computing it in \([0, 1]\) gives the same value as the sum of the integrals computed between \([0, 0.5]\) and \([0.5, 1]\).

As seen before, the sum

\[ S = \sum_{n=0}^{100} n = \frac{100(101)}{2} = 5050 \]

while

\[ S_1 = S + 50 = 5100 \]

\[ S_2 = S \]

the first sum has an extra term when \( n = 50 \) while the other does not. To verify this use the following script:

```matlab
% Pr. 0.7
clear all
N=100;
syms n,N
S=symsum(n,0,N)
S1=symsum(n,0,N/2)+symsum(n,N/2,N)
S2=symsum(n,0,N/2)+symsum(n,N/2+1,N)
giving
S = 5050
S1 = 5100
S2 = 5050
```

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0.8 Complex functions of time — Consider the complex function $x(t) = (1 + jt)^2$ for $-\infty < t < \infty$.

(a) Find the real and the imaginary parts of $x(t)$ and carefully plot them with MATLAB. Try to make MATLAB plot $x(t)$ directly, what do you get? Does MATLAB warn you? Does it make sense?

(b) Compute the derivative $y(t) = dx(t)/dt$ and plot its real and imaginary parts, how do these relate to the real and the imaginary parts of $x(t)$?

(c) Compute the integral

$$\int_0^1 x(t)dt$$

(d) Would the following statement be true? (remember * indicates complex conjugate)

$$\left(\int_0^1 x(t)dt\right)^* = \int_0^1 x^*(t)dt$$

Answers: $\int_0^1 x(t)dt = 2/3 + j1$; statement is true.

Solution

(a)(b) Since $x(t) = (1 + jt)^2 = 1 + j2t + j^2t^2 = 1 - t^2 + j2t$,

its derivative with respect to $t$ is

$$y(t) = \frac{dx(t)}{dt} = -2t + 2j = \frac{d\text{Re}[x(t)]}{dt} + j\frac{d\text{Im}[x(t)]}{dt}$$

% Pr. 0.8
clear all
t=[-5: 0.001:5];
x=(1+j*t).^2;
xr=real(x);
xi=imag(x);
figure(7)
subplot(211)
plot(t,xr); title('Real part of x(t)'); grid
subplot(212)
plot(t,xi); title('Imaginary part of x(t)'); xlabel('Time'); grid
% Warning when plotting complex signals
figure(8)
disp('Read warning. MATLAB is being nice with you, this time!')
plot(t,x); title('COMPLEX Signal x(t)'); xlabel('Time')

When plotting the complex function $x(t)$ as function of $t$, MATLAB ignores the imaginary part. One should not plot complex functions as functions of time as the results are not clear when using MATLAB. See Fig. 5 for plots.

(c) Using the rectangular expression of $x(t)$ we have

$$\int_0^1 x(t)dt = \int_0^1 (1 - t^2 + 2jt)dt = \int_0^1 (1 - t^2)dt + 2j\int_0^1 t dt = t - t^3/3 + j(2t^2/2)|_0^1 = 2/3 + j1$$

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(d) The integral

\[ \int_0^1 x^*(t) dt = \int_0^1 (1 - t^2 - 2jt) dt = \int_0^1 (1 - t^2) dt - 2j \int_0^1 t dt = t - \frac{t^3}{3} - j \frac{2t^2}{2} \bigg|_0^1 = \frac{2}{3} - j1 \]

which is the complex conjugate of the integral calculated in (c). So yes, the expression is true.

Figure 5: Problem 8: Real and imaginary parts of \( x(t) \) (left); complex signal \( x(t) \) ignoring imaginary part.
0.9 Hyperbolic sinusoids — In filter design you will be asked to use hyperbolic functions. In this problem we relate these functions to sinusoids and obtain a definition of these functions so that we can actually plot them.

(a) Consider computing the cosine of an imaginary number, i.e., use
\[ \cos(x) = \frac{e^{jx} + e^{-jx}}{2} \]
let \( x = j\theta \) and find \( \cos(x) \). The resulting function is called the hyperbolic cosine or
\[ \cos(j\theta) = \cosh(\theta) \]

(b) Consider then the computation of the hyperbolic sine \( \sinh(\theta) \), how would you do it? Carefully plot it as a function of \( \theta \).

(c) Show that the hyperbolic cosine is always positive and bigger than 1 for all values of \( \theta \)

(d) Show that \( \sinh(\theta) = -\sinh(-\theta) \)

(e) Write a MATLAB script to compute and plot these functions between \(-10\) and 10.

Answers: \( \sinh(\theta) = -j \sin(j\theta) \) and odd.

Solution
(a) If \( x = j\theta \)
\[ \cos(j\theta) = \frac{1}{2}(e^{-\theta} + e^{\theta}) = \cosh(\theta) \]

(b) The hyperbolic sine is defined as
\[ \sinh(\theta) = \frac{1}{2}(e^\theta - e^{-\theta}) \]
which is connected with the circular sine as follows
\[ \sin(j\theta) = \frac{1}{2j}(e^{-\theta} - e^{\theta}) = j \sinh(\theta) \Rightarrow \sinh(\theta) = -j \sin(j\theta) \]

(c) Since \( e^{\pm\theta} > 0 \) then \( \cosh(\theta) = \cosh(-\theta) > 0 \), the smallest value is for \( \theta = 0 \) which gives \( \cosh(0) = 1 \)

(d) Indeed,
\[ \sinh(-\theta) = \frac{1}{2}(e^{-\theta} - e^{\theta}) = -\sinh(\theta) \]

% Pr. 0.9
clear all
theta=sym('theta');
x= 0.5*(exp(-theta)+exp(theta));
y= 0.5*(exp(theta)-exp(-theta));
figure(9)
subplot(211)
ezplot(x,[-10,10])
grid
subplot(212)
ezplot(y,[-10,10])
grid
Figure 6: Problem 9: $\cosh(\theta)$ (top) and $\sinh(\theta)$ (bottom).
1.1 Consider a finite support signal $x(t) = t$, $0 \leq t \leq 1$, and zero elsewhere.

(a) Plot $x(t + 1)$ and $x(-t + 1)$. Add these signals to get a new signal $y(t)$. Do it graphically and verify your results analytically.

(b) How does $y(t)$ compare to the signal $\Lambda(t) = (1 - |t|)$ for $-1 \leq t \leq 1$ and zero otherwise? Plot them. Compute the integrals of $y(t)$ and $\Lambda(t)$ for all values of $t$ and compare them.

Answers: $x(-t + 1)$ is $x(-t)$ delayed by 1; $y(t)$ is even, $y(0) = 2$.

Solution

(a) If $x(t) = t$ for $0 \leq t \leq 1$, then $x(t + 1)$ is $x(t)$ advanced by 1, i.e., shifted to the left by 1 so that $x(0) = 0$ occurs at $t = -1$ and $x(1) = 1$ occurs at $t = 0$. 

Figure 1.1: Problem 1: Original signal $x(t)$, shifted versions $x(t + 1)$, $x(-t)$ and $x(-t + 1)$. 

1
The signal \( x(-t) \) is the reversal of \( x(t) \) and \( x(-t + 1) \) would be \( x(-t) \) advanced to the right by 1. Indeed,

\[
\begin{align*}
1 & \quad x(0) \\
0 & \quad x(1) \\
-1 & \quad x(2)
\end{align*}
\]

The sum \( y(t) = x(t + 1) + x(-t + 1) \) is such that at \( t = 0 \) it is \( y(0) = 2 \); \( y(t) = x(t + 1) \) for \( t < 0 \); and \( y(t) = x(-t + 1) \) for \( t > 0 \). Thus,

\[
\begin{align*}
y(t) &= x(t + 1) = t + 1 & 0 \leq t + 1 < 1 \quad \text{or} \quad -1 \leq t < 0 \\
y(0) &= 2 \\
y(t) &= x(-t + 1) = -t + 1 & 0 \leq -t + 1 < 1 \quad \text{or} \quad 0 < t \leq 1
\end{align*}
\]

or

\[
y(t) = \begin{cases} 
  t + 1 & -1 \leq t < 0 \\
  2 & t = 0 \\
  -t + 1 & 0 < t \leq 1
\end{cases}
\]

(b) Except for the discontinuity at \( t = 0 \), \( y(t) \) looks like the even triangle signal \( \Lambda(t) \), their integrals are identical as the discontinuity of \( y(t) \) does not add any area.

Figure 1.2: Problem 1: Triangular signal \( y(t) \) with discontinuity at the origin.
1.2 Given the causal full-wave rectified signal \( x(t) = |\sin(2\pi t)|u(t) \)

(a) Find the even component of \( x(t) \), call it \( x_e(t) \) and plot it. Is \( x_e(t) \) periodic? If so, what is its fundamental fundamental period \( T_e \)? Would the odd component of \( x(t) \) or \( x_o(t) \) be periodic too?

(b) Are \( x_e(t) \) and \( x_o(t) \) causal signals? Explain.

**Answers:** \( x_e(t) \) is periodic, \( x_o(t) \) is not, both are non-causal.

**Solution**

(a) \( x(t) \) is called causal because it is zero for \( t < 0 \), it repeats every 0.5 sec. for \( t \geq 0 \).

\[
\begin{align*}
&\text{Figure 1.3: Problem 2.} \\
\end{align*}
\]

(b) Even component \( x_e(t) = 0.5[x(t) + x(-t)] \) is periodic of fundamental period \( T_e = 0.5 \). The odd component is \( x_o(t) = 0.5[x(t) - x(-t)] \) is not periodic.

(c) \( x_e(t) \) and \( x_o(t) \) are non-causal signals as they are different from zero for negative times.
1.3 Is it true that (if not true, give correct answer)

(a) for any positive integer $k$
\[
\int_0^k e^{j2\pi t} dt = 0?
\]

(b) for a periodic signal $x(t)$ of fundamental period $T_0$
\[
\int_0^{T_0} x(t) dt = \int_{t_0}^{t_0+T_0} x(t) dt
\]
for any value of $t_0$? Consider, for instance, $x(t) = \cos(2\pi t)$.

(c) $\cos(2\pi t)\delta(t - 1) = 1$?

(d) If $x(t) = \cos(t)u(t)$ then $dx(t)/dt = -\sin(t)u(t) + \delta(t)$?

(e) the integral
\[
\int_{-\infty}^{\infty} [e^{-t}u(t)] \delta(t-2) d\tau = e^{-2}
\]

(f) If $x(t) = \cosh(t)u(t) = 0.5(e^t + e^{-t})u(t)$ is $dx(t)/dt = \sinh(t)u(t) + \delta(t)$?

(g) that the power of the causal full-rectified signal $x(t) = |\sin(t)|u(t)$ is twice the power of its even component $x_e(t) = 0.5[x(t) + x(-t)]$?

**Answers:** (a) and (b) are true; $\cos(2\pi t)\delta(t-1) = \delta(t-1)$; (g) yes, $P_{x_e} = 0.5P_x$.

**Solution**

(a) Yes, expressing $e^{j2\pi t} = \cos(2\pi t) + j\sin(2\pi t)$, periodic of fundamental period $T_0 = 1$, then the integral is the area under the cosine and sine in one or more periods (which is zero) when $k \neq 0$ and integer. If $k = 0$, the integral is also zero.

(b) Yes, whether $t_0 = 0$ (first equation) or a value different from zero, the two integrals are equal as the area under a period is the same. In the case $x(t) = \cos(2\pi t)$, both integrals are zero.

(c) It is not true, $\cos(2\pi t)\delta(t - 1) = \cos(2\pi)\delta(t - 1) = \delta(t - 1)$.

(d) It is true, considering $x(t)$ the product of $\cos(t)$ and $u(t)$ its derivative is
\[
\frac{dx(t)}{dt} = \frac{d\cos(t)}{dt}u(t) + \cos(t)\frac{du(t)}{dt} = -\sin(t)u(t) + \cos(0)\delta(t)
\]

(e) Yes,
\[
\int_{-\infty}^{\infty} [e^{-t}u(t)] \delta(t-2) d\tau = \int_{0}^{\infty} [e^{-2}] \delta(t-2) d\tau = e^{-2}
\]

(f) Yes,
\[
\frac{dx(t)}{dt} = 0.5[e^t u(t) + e^{-t} \delta(t)] + 0.5[-e^{-t} u(t) + e^{-t} \delta(t)]
\]
\[
= 0.5[e^t - e^{-1}]u(t) + \delta(t) = \sinh(t)u(t) + \delta(t)
\]
(g) The even component $x_e(t)$ is a periodic full-wave rectified signal of amplitude $1/2$ and fundamental period $T_1 = \pi$.

Power of $x(t)$

$$P_x = 0.5 \left[ \frac{1}{\pi} \int_0^{\pi} x^2(t)dt \right]$$

Power of $x_e(t)$

$$P_{xe} = \frac{1}{\pi} \int_0^{\pi} (0.5x(t))^2dt = 0.5P_x$$
The signal
\[ x(t) = \begin{cases} 
-t & -2 \leq t \leq 0 \\
 t & 0 \leq t \leq 2
\end{cases} \]
can be written as \( x(t) = |t|p(t) \).

(a) Carefully plot \( x(t) \) and define \( p(t) \), then find \( y(t) = dx(t)/dt \) and carefully plot it.

(b) Calculate
\[ \int_{-\infty}^{t} y(\tau)d\tau \]
and comment on how your result relates to \( x(t) \).

(c) Is it true that
\[ \int_{-\infty}^{\infty} x(\tau)d\tau = 2 \int_{0}^{2} x(\tau)d\tau \]?

**Answers:** \( y(t) = 2\delta(t + 2) - u(t + 2) + 2u(t) - u(t - 2) - 2\delta(t - 2) \); yes, it is true.

**Solution**

(a) See Fig. 4a
\[ x(t) = |t| \left[ u(t + 2) - u(t - 2) \right] \] Derivative \( p(t) \)

![Figure 1.4: Problem 4](image)

\[ y(t) = \frac{dx(t)}{dt} = 2\delta(t + 2) - u(t + 2) + 2u(t) - u(t - 2) - 2\delta(t - 2) \]

(b) Integral
\[ \int_{-\infty}^{t} y(\tau)d\tau' = \begin{cases} 
0 & t < -2 \\
-t & -2 \leq t < 0 \\
 t & 0 \leq t < 2 \\
0 & t \geq 2
\end{cases} \]

which equals \( x(t) \).

(c) Yes, because \( x(t) \) is an even function of \( t \).
1.2 Problems using MATLAB

1.5 Sampling signal and impulse signal — Consider the sampling signal

\[ \delta_T(t) = \sum_{k=0}^{\infty} \delta(t - kT) \]

which we will use in the sampling of analog signals later on.

(a) Plot \( \delta_T(t) \). Find

\[ ss_T(t) = \int_{-\infty}^{t} \delta_T(\tau)d\tau \]

and carefully plot it for all \( t \). What does the resulting signal \( ss(t) \) look like? This signal has been called the “stairway to the stars.” Explain.

(b) Use MATLAB function \( \text{stairs} \) to plot \( ss_T(t) \) for \( T = 0.1 \). Determine what signal would be the limit as \( T \to 0 \).

(c) A sampled signal is

\[ x_s(t) = x(t)\delta_T(t) = \sum_{k=0}^{\infty} x(kT_s)\delta(t - kT_s) \]

Let \( x(t) = \cos(2\pi t)u(t) \) and \( T_s = 0.1 \). Find the integral

\[ \int_{-\infty}^{t} x_s(t)dt \]

and use MATLAB to plot it for \( 0 \leq t \leq 10 \). In a simple way this problem illustrates the operation of a discrete to analog converter which converts a discrete-time into a continuous-time signal (its cousin is the digital to analog converter or DAC).

Answers: (a) \( ss(t) = \sum_{k=0}^{\infty} u(t - kT) \); (c) \( \int_{-\infty}^{t} x_s(t)dt = \sum_{k=0}^{\infty} x(kT_s)u(t - kT_s) \).

Solution

(a) The sampling signal is a sequence of unit impulses at uniform times \( kT \), for \( t > 0 \). The integral

\[ ss_T(t) = \int_{-\infty}^{t} \sum_{k=0}^{\infty} \delta(t - kT)dt = \sum_{k=0}^{\infty} \int_{-\infty}^{t} \delta(t - kT)dt = \sum_{k=0}^{\infty} u(t - kT) \]

This signal is called the “stairway to the stars” \( ss_T(t) \) for obvious reasons.

(b)(c) The following script will display the signal \( ss(t) \) and the conversion to analog (just uncomment the desired signal and comment the not desired signal).

% Pr. 1_5 parts (b) and (c)
clear all; clf
T=0.1; t=0:T:10;
%f=t; % part (b)
f=cos(2*pi*t); % part (c)
figure(6)
stairs(t,f);grid
%axis([0 10 0 10]) % part (b)
axis([0 10 -1.1 1.1]) % part (c)
hold on
plot(t,f,'r')
hold off

When \( f(t) = t + 1 \) the output looks like a digitized line with unit slope and cut at 1 (see figure below), similarly when \( f(t) = \cos(2\pi t) \) the output looks like a digitized sinusoid.
Figure 1.5: Problem 5: ‘Digitized’ ramp and cosine signals using ‘stairway to stars’
Chapter 2

Continuous–time Systems

2.1 Basic Problems

The input-output relationship of a system is

\[ y(t) = \text{sign}(x(t)) = \begin{cases} 
1 & \text{if } x(t) \geq 0 \\
-1 & \text{if } x(t) < 0 
\end{cases} \]

where \( x(t) \) is the input and \( y(t) \) the output.

(a) Let the input be \( x(t) = \sin(2\pi t)u(t) \), plot the corresponding output \( y(t) \). What is the output of the system if we double the input? That is, what is the output \( y_1(t) \) if the input \( x_1(t) = 2x(t) = 2\sin(2\pi t)u(t) \)? Is the system linear or not?

(b) Is the given system time-invariant?

Answers: output \( y(t) = \sum_{k=0}^{\infty} p(t-k) \) with \( p(t) = u(t) - 2u(t-0.5) + u(t-1) \).

Solution

(a) \( x(t) = \sin(2\pi t)u(t) \) is zero for \( t < 0 \) and repeats periodically every \( T_0 = 1 \). Thus,

\[ y(t) = \sum_{k=0}^{\infty} p(t-k), \quad p(t) = u(t) - 2u(t-0.5) + u(t-1) \]

If the input is \( 2x(t) \) the output is the same as before, so the system is non-linear.

(b) The system is time-invariant: if input \( x(t - \tau) \) the output is \( y(t - \tau) \) for any value of \( \tau \).
2.2 Consider the following problems about the properties of systems.

(a) A system is represented by the ordinary differential equation \( dz(t)/dt = w(t) - w(t - 1) \) where \( w(t) \) is the input and \( z(t) \) the output.

i. How is this system related to an averager having an input/output equation

\[
z(t) = \int_{t-1}^{t} w(\tau)d\tau + 2
\]

ii. Is the system represented by the given ordinary differential equation LTI?

(b) If we consider the current \( i(t) \) the input of a capacitor, with a zero initial voltage across it, the voltage across the capacitor—or the output—is given by

\[
v_C(t) = \int_{0}^{t} i(\tau)d\tau
\]

i. Is this capacitor time-invariant? If not, what conditions would you impose to make it time-invariant?

ii. If we let \( i(t) = u(t) \) what is the corresponding voltage \( v_C(t) \)? If we delay the current source so that the input is \( i(t - 1) \), find the corresponding voltage across the capacitor and indicate if this result shows the capacitor is time invariant.

(c) An amplitude modulation system has an input/output equation

\[
y(t) = x(t) \sin(2\pi t) \quad -\infty < t < \infty
\]

i. Let \( x(t) = u(t) \), plot the corresponding output \( y(t) = \sin(2\pi t)u(t) \)

ii. If the input is delayed, i.e., the input is \( x_1(t) = x(t - 0.5) = u(t - 0.5) \), plot the corresponding output \( y_1(t) = x_1(t) \sin(2\pi t) \), and use this result to determine whether the given AM system is time-invariant.

**Answer:**

(a) \( z(t) \) solution of differential equation with initial condition of 2; (b) If \( i(t) = 0 \) for \( t < 0 \), system is TI; (c) If \( x(t - 0.5) = u(t - 0.5) \) the output is \( y_1(t) = \sin(2\pi t)u(t - 0.5) \).

**Solution**

(a) Derivative

\[
\frac{dz(t)}{dt} = w(t) - w(t - 1)
\]

which excludes the initial condition of 2. System is LTI if initial condition is zero.

(b) i. If input is \( i(t - \mu) \) then the output is letting \( \eta = \tau - \mu \)

\[
\int_{0}^{t} i(\tau - \mu)d\tau = \int_{-\mu}^{0} i(\eta)d\eta + \int_{0}^{t-\mu} i(\eta)d\eta = v_C(t - \mu)
\]

that is, provided that \( i(t) = 0 \) for \( t < 0 \), the system is time-invariant.
ii. If \( i(t) = u(t) \) then \( v_c(t) = \int_0^t u(\tau)d\tau = r(t) \). If we shift the inputs \( i_1(t) = i(t - 1) = u(t - 1) \) the previous output is shifted, so system is time-invariant.

(c) If \( x(t) = u(t) \) then \( y(t) = \sin(2\pi t)u(t) \) while corresponding to \( x(t - 0.5) = u(t - 0.5) \) is \( y_1(t) = \sin(2\pi t)u(t-0.5) \) indicating the system is not time-invariant as \( y_1(t) \) is not \( y(t-0.5) \).

---

Figure 2.1: Problem 2(c)
2.3 The following problems relate to linearity, time–invariance and causality of systems.

(a) A system is represented by the equation \( z(t) = v(t)f(t) + B \) where \( v(t) \) is the input, \( z(t) \) the output, \( f(t) \) a function and \( B \) a constant.

i. Let \( f(t) = A \), a constant. Is the system linear if \( B \neq 0 \)? linear if \( B = 0 \)? Explain.

ii. Let \( f(t) = \cos(\Omega_0 t) \) and \( B = 0 \) is this system linear? time-invariant?

iii. Let \( f(t) = u(t) - u(t-1) \), the input \( v(t) = u(t) - u(t-1), B = 0 \), find the corresponding output \( z(t) \). Let then the input be delayed by 2, i.e., the input is \( u(t-2) - u(t-3) \) and \( f(t) \) and \( B \) be the same, determine the corresponding output. Using these results, is the system time invariant?

(b) An averager is defined as
\[
y(t) = \frac{1}{T} \int_{t-T}^{t} x(\tau)d\tau
\]
where \( T > 0 \) is the averaging interval, \( x(t) \) and \( y(t) \) are the system input and output.

i. Determine if the averager is a linear system.

ii. Let \( T = 1, x(t) = u(t) \), calculate and plot the corresponding output, delay then the input to get \( x(t-2) = u(t-2) \), and calculate and plot the corresponding output.

From this example, does the system seem time invariant? Explain. Can you show it in general?

iii. Is this system causal? Give an example to verify your assertion.

**Answer:** (a) If \( f(t) = v(t) = u(t) - u(t-1) \) then \( z(t) = u(t) - u(t-1) \), and if input is \( v_1(t) = u(t-2) - u(t-3) \) output is \( z_1(t) = v_1(t)f(t) = 0 \); (b) If \( x(t) = u(t) \) then \( y(t) = r(t) - r(t-1) \) and if \( x_1(t) = u(t-2) \) the corresponding output is \( y_1(t) = y(t-2) \).

**Solution**

(a)  
\[ z(t) = Av(t) + B, \text{ system is linear if } B = 0, \text{ non-linear otherwise.} \]

ii. \( z(t) = v(t) \cos(\Omega_0 t) \) is linear but time-varying.

iii. If \( f(t) = v(t) = u(t) - u(t-1), B = 0 \) then \( z(t) = u(t) - u(t-1) \), and if we shift \( v(t) \) so the input is \( v_1(t) = u(t-2) - u(t-3) \) the output is \( z_1(t) = v_1(t)f(t) = 0 \) which is different from \( z(t-2) \), so the system is time-varying.

(b)  
\[ \frac{1}{T} \int_{t-T}^{t} [A x_1(\tau) + B x_2(\tau)]d\tau = \frac{A}{T} \int_{t-T}^{t} x_1(\tau)d\tau + \frac{B}{T} \int_{t-T}^{t} x_2(\tau)d\tau = A y_1(t) + B y_2(t) \]

where \( y_i(t), i = 1, 2, \) are the corresponding outputs to \( x_i(t), i = 1, 2. \)
ii. If \( x(t) = u(t) \) then \( y(t) = r(t) - r(t - 1) \) and if \( x_1(t) = u(t - 2) \) the corresponding output is \( y_1(t) = y(t - 2) \). System is time-invariant.

In general, if \( x_1(t) = x(t - \lambda) \) the output is

\[
\frac{1}{T} \int_{t-T}^{t} x_1(\tau) \, d\tau = \frac{1}{T} \int_{t-T-\lambda}^{t-\lambda} x(\nu) \, d\nu
\]

(using \( \nu = \tau - \lambda \)) which is the same as \( y(t - \lambda) \), so time-invariant.

iii. \( y(t) \) depends on present and past inputs, and zero if input is zero, so the system is causal.
2.4 Consider the analog averager,

\[ y(t) = \frac{1}{T} \int_{t-T/2}^{t+T/2} x(\tau) d\tau \]

where \( x(t) \) is the input and \( y(t) \) the output.

(a) Find the impulse response \( h(t) \) of the averager. Is this system causal?

(b) Let \( x(t) = u(t) \), find the output of the averager.

**Answer:** \( h(t) = (1/T)[u(t + T/2) - u(t - T/2)] \); non-causal system.

**Solution**

(a) Letting \( x(t) = \delta(t) \) the impulse response is

\[
\begin{align*}
    h(t) &= \frac{1}{T} \int_{t-T/2}^{t+T/2} \delta(\tau) d\tau \\
    &= \frac{1}{T} \int_{t-T/2}^{t} \delta(\tau) d\tau + \frac{1}{T} \int_{t}^{t+T/2} \delta(\tau) d\tau 
\end{align*}
\]

If \( t > 0 \), and \( t - T/2 < 0 \) the first integral includes 0, while the second does not. Thus

\[
h(t) = \frac{1}{T} \int_{t}^{t+T/2} \delta(\tau) d\tau = \frac{1}{T} \quad t > 0 \text{ and } t - T/2 < 0, \text{ or } 0 < t < T/2
\]

Likewise when \( t < 0 \) then \( t - T/2 < -T/2 \) and \( t + T/2 < T/2 \) the reverse of the previous case happens and so

\[
h(t) = 0 + \frac{1}{T} \int_{t}^{t+T/2} \delta(\tau) d\tau = \frac{1}{T} \quad t < 0 \text{ and } t + T/2 > 0, \text{ or } -T/2 < t < 0
\]

so that

\[
h(t) = \frac{1}{T} [u(t + T/2) - u(t - T/2)]
\]

indicating that the system is non-causal as \( h(t) \neq 0 \) for \( t < 0 \).

(b) If \( x(t) = u(t) \) then the output of the averager is

\[
y(t) = \frac{1}{T} \int_{t-T/2}^{t+T/2} u(\tau) d\tau
\]

If \( t + T/2 < 0 \) then \( y(t) = 0 \) since the argument of the unit step signal is negative. If \( t + T/2 \geq 0 \) and \( t - T/2 < 0 \) then

\[
y(t) = \int_{0}^{t+T/2} u(\tau) d\tau = \frac{1}{T} (t + T/2)
\]

and finally when \( t - T/2 \geq 0 \)

\[
y(t) = \frac{1}{T} \int_{t-T/2}^{t+T/2} u(\tau) d\tau = 1
\]
The unit-step response of the noncausal averager is

\[ y(t) = \begin{cases} 
0 & t < -T/2 \\
\frac{1}{T}(t + T/2) & -T/2 \leq t < T/2 \\
1 & t \geq T/2 
\end{cases} \]
2.5 The input-output equation characterizing a system of input $x(t)$ and output $y(t)$ is

$$y(t) = e^{-2t}y(0) + 2\int_0^t e^{-2(t-\tau)}x(\tau)d\tau \quad t \geq 0$$

and zero otherwise.

(a) Find the ordinary differential equation that also characterizes this system.

(b) Suppose $x(t) = u(t)$ and any value of $y(0)$, we wish to determine the steady-state response of the system. Is the value of $y(0)$ of any significance, i.e., do we get the same steady-state response if $y(0) = 0$ or $y(0) = 1$? Explain.

(c) Compute the steady-state response when $y(0) = 0$, and $x(t) = u(t)$. To do so first find the impulse response of the system, $h(t)$ using the given input–output equation, and then find $y(t)$ by computing the convolution graphically to determine the steady-state response.

(d) Suppose the input is zero, is the system depending on the initial condition BIBO stable?

Find the zero-input response $y(t)$ when $y(0) = 1$. Is it bounded?

**Answers:** (a) $dy(t)/dt + 2y(t) = 2x(t)$; (d) Yes.

**Solution**

(a) To find the differential equation let $y(0) = 0$, so that

$$y(t) = 2e^{-2t}\int_0^t e^{2\tau}x(\tau)d\tau$$

and its derivative is

$$\frac{dy(t)}{dt} = -4e^{-2t}\int_0^t e^{2\tau}x(\tau)d\tau + 2e^{-2t}e^{2t}x(t) = -2y(t) + 2x(t)$$

giving the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = 2x(t)$$

(b) Finding the integral after replacing $x(\tau) = u(\tau)$ we have that

$$y(t) = y(0)e^{-2t} + 2e^{-2t}\left.\frac{e^{2\tau}}{2}\right|_0^t = y(0)e^{-t} + (1 - e^{-2t})$$

so that in the steady state, i.e., when $t \to \infty$, the output is 1, independent of the value of the initial condition.

(c) From the given input-output equation, letting $x(t) = \delta(t)$ and $y(0) = 0$ the output is the impulse response of the system

$$h(t) = 2\int_0^t e^{-2(t-\tau)}\delta(\tau)d\tau = 2e^{-2t}\int_0^t \delta(\tau)d\tau = 2e^{-2t}u(t)$$
Computing the convolution integral as

\[ y(t) = \int_{-\infty}^{\infty} x(t - \tau)h(\tau)d\tau \]

it can be obtained graphically by reflecting the input \( x(t) = u(t) \) and shifting it linearly from \(-\infty\) to \(\infty\) to get

- for \( t < 0 \) then \( y(t) = 0 \)
- for \( t \geq 0 \)

\[ y(t) = \int_{0}^{t} 2e^{-2\tau}d\tau = 1 - e^{-t} \]

and so in the steady state the output goes to 1.

(d) The zero-input response is

\[ y_{zi}(t) = y(0)e^{-2t}u(t) \]

and it is bounded for any finite value of the initial condition \( y(0) \), in particular for \( y(0) = 1 \), therefore the system that depends on the initial condition is BIBO.
2.6 The input $x(t)$ and the corresponding output $y(t)$ of a linear time-invariant (LTI) system are

$$ x(t) = u(t) - u(t-1) \quad \rightarrow \quad y(t) = r(t) - 2r(t-1) + r(t-2) $$

Determine the outputs $y_i(t)$, $i = 1, 2, 3$ corresponding to the following inputs

(i) $x_1(t) = u(t) - u(t-1) - u(t-2) + u(t-3)$,  
(ii) $x_2(t) = u(t+1) - 2u(t) + u(t-1)$,  
(iii) $x_3(t) = \delta(t) - \delta(t-1)$

Plot the inputs $x_i(t)$ and the corresponding output $y_i(t)$.

**Answers:** $y_1(t) = y(t) - y(t-2)$; $y_3(t) = dy(t)/dt = u(t) - 2u(t-1) + u(t-2)$.

**Solution**

(a) $x_1(t) = x(t) - x(t-2)$ so $y_1(t) = y(t) - y(t-2)$, two triangular pulses, the second multiplied by $-1$.

(b) $x_2(t) = x(t+1) - x(t)$ then $y_2(t) = y(t+1) - y(t)$ (they overlap between 0 and 1).

(c) $x_3(t) = \delta(t) - \delta(t-1)$ so $y_3(t) = dy(t)/dt = u(t) - 2u(t-1) + u(t-2)$. Considering that the output of $x(t)$ is $y(t)$, i.e., $y(t) = S[x(t)]$, and that the integrator and the differentiator are LTI systems Fig. 2.3 shows how to visualize the result in this problem by considering that you can change the order of the cascading of LTI systems.
2.7 Consider a system represented by a first-order ordinary differential equation:
\[
\frac{dy(t)}{dt} + ay(t) = x(t), \quad \text{with initial condition } y(0).
\]
(a) Show first that for a function \( f(t) \)
\[
\frac{d}{dt} \int_0^t f(\tau) d\tau = f(t)
\]
using the definition of the derivative.
(b) Apply the above result to show that the solution of the first-order ordinary differential equation with initial condition \( y(0) \) given above is
\[
y(t) = y(0)e^{-at} + \int_0^t e^{-a(t-\tau)} x(\tau) d\tau \quad t \geq 0
\]
(c) Another way to show the above expression is the solution of the ordinary differential equation is to multiply both sides of the ordinary differential equation by \( e^{at} \)
i. show that the left term is \( d(e^{at}y(t))/dt \),
ii. integrate the two terms to obtain the solution.

Solution

(a) By the definition of the derivative
\[
\frac{d}{dt} \left[ \int_0^t f(\tau) d\tau \right] = \lim_{h \to 0} \frac{1}{h} \left[ \int_0^{t+h} f(\tau) d\tau - \int_0^t f(\tau) d\tau \right]
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left[ \int_t^{t+h} f(\tau) d\tau \right]
\]
\[
= \lim_{h \to 0} \int_0^1 f(\tau)h = f(t)
\]
(b) \( y(t) \) satisfies the ordinary differential equation, indeed
\[
\frac{dy(t)}{dt} = -ay(0)e^{-at} + \frac{d}{dt} \left[ e^{-at} \int_0^t e^{a\tau} x(\tau) d\tau \right]
\]
\[
= -a[y(0)e^{-at} + e^{-at} \int_0^t e^{a\tau} x(\tau) d\tau] + x(t) = -ay(t) + x(t)
\]
(c) Multiplying the two terms of the ordinary differential equation by \( e^{at} \) we get
\[
\frac{e^{at}dy(t)/dt + ae^{at}y(t)}{d(e^{at}y(t))/dt} = e^{at} x(t)
\]
\[
e^{at} y(t) = \int_0^t e^{a\tau} x(\tau) d\tau + y(0)
\]
solving for \( y(t) \) we obtain the solution.
2.8 The bounded-input bounded-output stability assumes that the input is always bounded, limited in amplitude. If that is not the case, even a stable system would provide an unbounded output. Consider the analog averager, with an input-output relationship

\[ y(t) = \frac{1}{T} \int_{t-T}^{t} x(\tau) d\tau \]

(a) Suppose that the input to the averager is a bounded signal \( x(t) \), i.e., there is a finite value \( M \) such that \( |x(t)| < M \). Find the value for the bound of the output \( y(t) \) and determine whether the averager is BIBO stable or not.

(b) Let the input to the averager be \( x(t) = tu(t) \), i.e., a ramp signal, compute the output \( y(t) \) and determine if it is bounded or not. If \( y(t) \) is not bounded, does that mean that the averager is an unstable system? Explain.

Answers: \( h(t) = (1/T)[u(t) - u(t-T)] \); if input is ramp signal the output is not bounded.

Solution

(a) We can either find the impulse response or show that the output is bounded for any input signal that is bounded. To find the impulse response change the variable to \( \sigma = t - \tau \) which gives

\[ y(t) = \frac{1}{T} \int_{0}^{t} x(t-\sigma) d\sigma = \int_{0}^{t} \frac{1}{T} x(t-\sigma) d\sigma \]

which is a convolution integral with \( h(t) = (1/T)[u(t) - u(t-T)] \). This impulse response is absolutely integrable as

\[ \int_{0}^{\infty} |h(t)| dt = \int_{0}^{T} \frac{1}{T} dt = 1 \]

Likewise, if we assume that \( x(t) \) is bounded, i.e., there is a value \( M < \infty \) such that \( |x(t)| < M \), then

\[ |y(t)| \leq \frac{1}{T} \int_{t-T}^{t} |x(\tau)| d\tau \leq \frac{M}{T} \int_{t-T}^{t} d\tau = M < \infty \]

so system is BIBO stable.

(b) The ramp is not bounded, there is no \( M \) such that \( |r(t)| < M < \infty \). So when we compute the output of the stable system to the ramp we do not expect it to be bounded either. Indeed

\[ y(t) = \frac{1}{T} \int_{t-T}^{t} \tau d\tau = \frac{t^2 - (t - T)^2}{2T} = t - \frac{T}{2} \]

increases as \( t \) increases, so it is unbound.
2.2 Problems using MATLAB

2.9 An ideal low-pass filter — The impulse response of an ideal low-pass filter is \( h(t) = \frac{\sin(t)}{t} \).

(a) Given that the impulse response is the response of the system to an input \( x(t) = \delta(t) \) with zero initial conditions, can an ideal low-pass filter be used for real-time processing? Explain.

(b) Is the ideal low-pass filtering bounded-input bounded-output stable? Use MATLAB to check if the impulse response satisfies the condition for BIBO stability.

Answers: Ideal low-pass filter is non-causal.

Solution

(a) The system with the sinc impulse response is non-causal because \( h(t) \neq 0 \) for \( t < 0 \). As such an ideal low-pass filter cannot be used for real-time processing as it would require future inputs, not available in real-time. This can be seen by looking at the convolution with a causal signal \( x(t) \):

\[
y(t) = \int_{0}^{\infty} x(\tau)h(t - \tau)d\tau
\]

\[
= \int_{0}^{t} x(\tau)h(t - \tau)d\tau + \int_{t}^{\infty} x(\tau)h(t - \tau)d\tau
\]

where the upper limit of the top integral is due to \( h(\tau) \) having an infinite support, and the lower limit because \( x(t) \) is causal. The second bottom integral shows that the output depends on future values of the input.

Figure 2.4: Problem 9: absolute value of sinc function and its integral values for \( t = n \) sec.

(b) For the low-pass filter to be BIBO stable the impulse response \( h(t) \) must be absolutely integrable, i.e.,

\[
\int_{-\infty}^{\infty} |\frac{\sin(t)}{t}| dt < \infty
\]
To approximately compute this integral we use the following script. The integral is bounded, so the filter is BIBO stable.

```matlab
%% Pr 2.9
clear all; clf
syms x t x1
x=abs(sinc(t/pi));
for k=1:30,
x1=int(x,t,0,k);
xx(k)=subs(2*x1);
end
n=1:30;
figure(1)
subplot(211)
ezplot(x,[-30,30]); axis([-30 30 -0.1 1.2]);grid
subplot(212)
stem(n,xx); axis([1 30 0 8]);grid
```
Chapter 3

The Laplace Transform

3.1 Basic Problems

3.1 Find the Laplace transform of the following

(a) finite support signals, and indicate their region of convergence:

\begin{align*}
(i) & \quad x(t) = \delta(t - 1), \\
(ii) & \quad y(t) = \delta(t + 1) - \delta(t - 1) \\
(iii) & \quad z(t) = u(t + 1) - u(t - 1), \\
(iv) & \quad w(t) = \cos(2\pi t)[u(t + 1) - u(t - 1)]
\end{align*}

(b) causal signals, and indicate their region of convergence:

\begin{align*}
(i) & \quad x_1(t) = e^{-t}u(t), \\
(ii) & \quad y_1(t) = e^{-t}u(t - 1) \\
(iii) & \quad z_1(t) = e^{-t+1}u(t - 1), \\
(iv) & \quad w_1(t) = e^{-t}(u(t) - u(t - 1))
\end{align*}

Answers: (a) \(Y(s) = 2 \sinh(s)\), ROC the whole s-plane; \(W(s) = 2s \sinh(s)/(s^2 + 4\pi^2)\), ROC the whole s-plane; (b) \(Y_1(s) = e^{-(s+1)}/(s + 1)\), ROC \(\sigma > -1\).

Solution

(a) Finite–support signals.

i. \(\mathcal{L}[x(t)] = \int_{-\infty}^{\infty} \delta(t - 1)e^{-st}dt = e^{-s}\), ROC the whole s-plane

ii. \(\mathcal{L}[y(t)] = e^{s} - e^{-s} = 2\sinh(s)\), ROC the whole s-plane

iii. We have

\[\mathcal{L}[z(t)] = \mathcal{L}[u(t + 1)] - \mathcal{L}[u(t - 1)] = \frac{e^{s} - e^{-s}}{s} = \frac{2\sinh(s)}{s}\]

ROC whole s-plane, pole-zero cancellation at \(s = 0\).
iv. Notice that because \( \cos(2\pi(t \pm 1)) = \cos(2\pi t \pm 2\pi) = \cos(2\pi t) \) then

\[
\begin{align*}
w(t) &= \cos(2\pi t)[u(t + 1) - u(t - 1)] = \cos(2\pi(t + 1))u(t + 1) - \cos(2\pi(t - 1))u(t - 1) \\
\end{align*}
\]

so that

\[
W(s) = \frac{e^s}{s^2 + 4\pi^2} - \frac{e^{-s}}{s^2 + 4\pi^2} = \frac{(e^s - e^{-s})s}{s^2 + 4\pi^2} = \frac{2s \sinh(s)}{s^2 + 4\pi^2}
\]

The poles \( p_{1,2} = \pm j2\pi \) are cancelled by zeros \( z_{1,2} = \pm j2\pi \) (indeed, \( e^s - e^{-s}|_{s=\pm j2\pi} = 1 - 1 = 0 \)) so the ROC is the whole s-plane.

**b) Causal signals.**

i. \( X_1(s) = \mathcal{L}[x_1(t)] = 1/(s + 1), \) ROC \( \sigma > -1 \)

ii. Notice the exponential is not delayed, so the shifting property cannot be applied. Instead,

\[
Y_1(s) = \mathcal{L}[y_1(t)] = \mathcal{L}[e^{-1}e^{-(t-1)}u(t-1)] = \frac{e^{-(s+1)}}{s+1}, \quad \text{ROC } \sigma > -1
\]

iii. \( \mathcal{L}[z_1(t)] = \mathcal{L}[e^{-(t-1)}u(t-1)] = e^{-s}/(s + 1), \) ROC \( \sigma > -1 \)

iv. Since \( w_1(t) = x_1(t) - y_1(t), \) where \( x_1(t) \) and \( y_1(t) \) are given before then

\[
\mathcal{L}[w_1(t)] = X_1(s) - Y_1(s) = \frac{1 - e^{-(s+1)}}{s + 1},
\]

ROC whole s-plane because pole \( s = -1 \) is cancelled by zero, i.e., \( 1 - e^{-(s+1)}|_{s=-1} = 0. \)
3.2 Consider the pulse \( x(t) = u(t) - u(t - 1) \). Find the zeros and poles of \( X(s) \) and plot them.

(a) Suppose \( x(t) \) is the input of a LTI system with a transfer function \( H(s) = Y(s)/X(s) = 1/(s^2 + 4\pi^2) \), find and plot the poles and zeros of \( Y(s) = \mathcal{L}[y(t)] \), where \( y(t) \) is the output of the system.

(b) If the transfer function of the LTI system is

\[
G(s) = \frac{Z(s)}{X(s)} = \prod_{k=1}^{\infty} \frac{1}{s^2 + (2k\pi)^2}
\]

and the input is the above signal \( x(t) \), compute the output \( z(t) \).

Answers: \( X(s) \) has infinite number of zeros on \( j\Omega \)-axis; \( z(t) = \delta(t) \).

Solution

The transform is \( X(s) = (1 - e^{-s})/s \). The pole of \( X(s) \) is \( s = 0 \), while the zeros of \( X(s) \) are values of \( s \) that make \( 1 - e^{-s} = 0 \) or \( s_k = j2k\pi \) for \( -\infty < k < \infty \) and integer. The pole is cancelled with the \( k = 0 \) zero, so \( X(s) \) has an infinite number of zeros on the \( j\Omega \)-axis.

(a) The output of the LTI system has Laplace transform

\[
Y(s) = X(s)H(s) = \frac{(s + j2\pi)(s - j2\pi)}{s^2 + 4\pi^2} \prod_{k=2}^{\infty} (s + j2\pi k)(s - j2\pi k)
\]

\[
= \prod_{k=2}^{\infty} (s^2 + (2\pi k)^2)
\]

There are no poles, but an infinite number of zeros at \( \pm j2\pi k \) for integer \( k \geq 2 \).

(b) This is similar to the above problem, but here

\[
Z(s) = X(s)G(s) = 1
\]

which gives

\[
z(t) = \delta(t)
\]
3.3 Find the Laplace transform of the following signals and their region of convergence

(a) the reflection of the unit step signal, i.e., \( u(-t) \). And then use the result together with the Laplace transform of \( u(t) \) to see if you can obtain the Laplace transform of a constant or \( x(t) = u(t) + u(-t) \) (assume \( u(0) = 0.5 \) so there is no discontinuity at \( t = 0 \)).

(b) the non-causal signal \( y(t) = e^{-|t|}u(t+1) \). Carefully plot it, and find its Laplace transform \( Y(s) \) by separating \( y(t) \) into a causal and a non-causal signals, \( y_c(t) \) and \( y_{ac}(t) \), and plot them separately. Find the ROC of \( Y(s) \), \( Y_c(s) \) and \( Y_{ac}(s) \).

**Answers:** (a) \( x(t) \) has no Laplace transform; (b) \( Y(s) = (e^{s-1}(s+1) - 2)/(s^2 - 1) \).

**Solution**

(a) The Laplace transform of \( u(-t) \) is \(-1/s\), indeed

\[
\mathcal{L}[u(-t)] = \int_{-\infty}^{0} e^{-st} dt = \int_{0}^{\infty} e^{s\tau} d\tau = \frac{e^{s\tau}}{s} \bigg|_{\tau=0}^{\tau=\infty} = -\frac{1}{s}
\]

provided that \( \sigma < 0 \) so that the limit when \( \tau = \infty \) is zero. Thus

\[
\mathcal{L}[u(-t)] = -\frac{1}{s} \quad \text{ROC: } \sigma < 0
\]

Thus \( x(t) = u(t) + u(-t) = 1 \) for all \( t \) would have as Laplace transform

\[
X(s) = \frac{1}{s} - \frac{1}{s} = 0
\]

and as ROC the intersection of \( \sigma > 0 \) and \( \sigma < 0 \) which is null. So we cannot find the Laplace of \( x(t) = 1, -\infty < t < \infty \).

(b) Let \( y(t) = y_c(t) + y_{ac}(t) \) where the causal component \( y_c(t) = y(t)u(t) = e^{-t}u(t) \) and the anticausal component \( y_{ac}(t) = y(t)u(-t) = e^{t}[u(t+1) - u(t)] \). The Laplace transform of \( y_{ac}(t) \) is

\[
Y_{ac}(s) = \int_{-1}^{0} e^{-(s-1)t} dt = \int_{0}^{1} e^{(s-1)\tau} d\tau = \frac{e^{(s-1)}}{s-1} - \frac{1}{s-1}
\]

with ROC the whole plane (pole at \( s = 1 \) is cancelled by zero at 1).

The Laplace transform of \( y_c(t) \) is \( Y_c(s) = 1/(s+1) \), so that

\[
Y(s) = Y_c(s) + Y_{ac}(s) = \frac{1}{s+1} + \frac{e^{s-1} - 1}{s-1} = \frac{se^{s-1} + e^{s-1} - 2}{s^2 - 1} \quad \text{ROC } \sigma > -1
\]

The poles of \( Y(s) \) are \( s = \pm 1 \), and a zero is \( s = 1 \). Thus the zero cancels the pole at \( s = 1 \) leaving the pole at \( s = -1 \), and so the ROC is \( \sigma > -1 \).
3.4 You are given the following Laplace transform of the output $y(t)$ of a system with input $x(t)$ and Laplace transform $X(s)$:

$$Y(s) = \frac{(s-1)X(s)}{(s+2)^2 + 1} + \frac{1}{(s+2)^2 + 1}$$

(a) If $x(t) = u(t)$, find the zero-state response $y_{zs}(t)$.

(b) Find the zero-input response $y_{zi}(t)$.

(c) Determine the steady-state solution $y_{ss}(t)$.

Answers: $y_{zs}(t) = -(1/5)u(t) + (1/5)e^{-2t}\cos(t)u(t) + (7/5)e^{-2t}\sin(t)u(t)$.

Solution

(a) The following term corresponds to the zero-state

$$\frac{(s-1)X(s)}{(s+2)^2 + 1}$$

and the following term corresponds to zero-input

$$\frac{1}{(s+2)^2 + 1}$$

Zero-state response for $x(t) = u(t)$

$$Y_{zs}(s) = \frac{s-1}{s((s+2)^2 + 1)} = \frac{A}{s} + \frac{B+Cs}{(s+2)^2 + 1}$$

$$= -\frac{1}{5}s + \frac{9/5+s/5}{(s+2)^2 + 1} = -\frac{1}{5}s + \frac{1}{5}(s+2) + \frac{7}{5}$$

$$= -\frac{1}{5}s + \frac{1}{5}(s+2) + \frac{7}{5}$$

From the table of Laplace transforms

$$y_{zs}(t) = -(1/5)u(t) + (1/5)e^{-2t}\cos(t)u(t) + (7/5)e^{-2t}\sin(t)u(t)$$

(b) Zero-input response

$$Y_{zi}(s) = \frac{1}{(s+2)^2 + 1} \Rightarrow y_{zi}(t) = e^{-2t} \sin(t)u(t)$$

(c) As $t \to \infty$ $y(t) \to -1/5$. 

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3.5 The transfer function of a LTI system is
\[ H(s) = \frac{s}{(s + 1)^2 + 1} \]

(a) Use the Laplace transform to find the unit-step response \( s(t) = (h \ast x)(t) \).
(b) Find the response due to the following inputs

(i) \( x_1(t) = u(t) - u(t - 1) \),  
(ii) \( x_2(t) = \delta(t) - \delta(t - 1) \),  
(iii) \( x_3(t) = r(t) \)

**Answer:** \( s(t) = e^{-t} \sin(t)u(t) \).

**Solution**

(a) The Laplace transform of \( s(t) = (h \ast x)(t) \) is \( S(s) = H(s)X(s) \) which gives
\[ S(s) = \frac{1}{(s + 1)^2 + 1} \Rightarrow s(t) = e^{-t} \sin(t)u(t) \]

(b) By LTI

\[ y_1(t) = s(t) - s(t - 1) \]
\[ y_2(t) = ds(t)/dt - ds(t - 1)/dt \]
\[ y_3(t) = \int_{-\infty}^{t} s(t')dt' \]
3.6 Consider the impulse response of a LTI system \( h(t) = e^{-at}[u(t) - u(t - 1)] \) \( a > 0 \).

(a) Obtain the transfer function \( H(s) \).

(b) Find the poles and zeros of \( H(s) \).

(c) Is \( \lim_{a \to 0} H(s) \) equal to \( \frac{1 - e^{-s}}{s} \)?

(d) Indicate how the poles and zeros of \( H(s) \) move in the s-plane as \( a \to 0 \).

**Answers:** \( H(s) = (1 - e^{-(s+a)})/(s + a) \), ROC: whole s plane, pole cancelled by zero.

**Solution**

(a) The transfer function is
\[
H(s) = \frac{1}{s + a} - \frac{e^{-a}e^{-s}}{s + a} = \frac{1 - e^{-(s+a)}}{s + a}
\]
because pole/zero cancellation at \( s = -a \).

(b) Pole \( s = -a \); zeros are values of \( s \) such that \( e^{-(s+a)} = 1 = e^{j2\pi k} \) for \( k = 0, \pm 1, \pm 2, \cdots \), or \( s_k = -a - j2\pi k \) (the zero \( s_0 = -a \) cancels the pole at \( s = -a \)).

(c) As \( a \to 0 \)
\[
H(s) = \frac{1 - e^{-s}}{s}
\]

(d) Pole \( s = -a \) and zeros \( s_k = -a - j2\pi k \); pole/zero cancellation when \( k = 0 \). As \( a \to 0 \) the remaining zeros move towards the \( j\Omega \) axis.
3.7 A causal LTI system has a transfer function

\[ H(s) = \frac{Y(s)}{X(s)} = \frac{1}{1 - e^{-s}} \]

(a) Find the poles and zeros of \( H(s) \), and from this determine if the filter is BIBO stable or not.

(b) Draw a block diagram for such a system.

(c) Find the impulse response \( h(t) \) of the system and use it to verify your stability result.

**Answer:** \( h(t) \) is not absolutely integrable.

**Solution**

(a) Poles satisfy the identity \( e^{-s} = 1 = e^{j2\pi k} \) for \( k = 0, \pm 1, \ldots \), so there are an infinite number of poles, \( s_k = -j2\pi k \). No zeros. The system is unstable because of poles on \( j\Omega \)-axis.

(b) \( Y(s)(1 - e^{-s}) = X(s) \) gives \( y(t) = y(t-1) + x(t) \). A positive feedback system with a delay in the feedback represents the system. See Fig. 3.1.

(c) The transfer function is

\[ H(s) = \frac{1}{1 - e^{-s}} = \sum_{n=0}^{\infty} (e^{-s})^n \]

with inverse

\[ h(t) = \sum_{n=0}^{\infty} \delta(t - n) \]

which is not absolutely integrable

\[ \int_{-\infty}^{\infty} |h(t)| dt = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \delta(t - n) dt = \sum_{n=0}^{\infty} 1 \to \infty \]
3.8 Consider the following problems related to the convolution integral.

(a) The impulse response of a LTI system is \( h(t) = e^{-2t}u(t) \) and the system input is a pulse \( x(t) = u(t) - u(t-3) \). Find the output of the system \( y(t) \) by means of the convolution integral graphically and by means of the Laplace transform.

(b) It is known that the impulse response of an analog averager is \( h_1(t) = u(t) - u(t-1) \), consider the input to the averager \( x_1(t) = u(t) - u(t-1) \), determine graphically as well as by means of the Laplace transform the corresponding output of the averager \( y_1(t) = [h_1 * x_1](t) \). Is \( y_1(t) \) smoother than the input signal \( x_1(t) \)? provide an argument for your answer.

(c) Suppose we cascade 3 analog averagers each with the same impulse response \( h_i(t) = u(t) - u(t-1), i = 1, 2, 3 \), determine the transfer function of this system. If the duration of the input to the first averager is \( M \) sec., what would be the duration of the output of the 3rd averager?

Answers: (a) \( Y(s) = (1 - e^{-3s})/(s(s + 2)) \); (c) the support of the output will be \( 3M \).

Solution

(a) Using the Laplace transform

\[
H(s) = \frac{1}{s + 2} \\
X(s) = \frac{1 - e^{-3s}}{s} \\
Y(s) = H(s)X(s) = \frac{1 - e^{-3s}}{s(s + 2)}
\]

Letting

\[
F(s) = \frac{1}{s(s + 2)} = 0.5 \frac{1}{s} - 0.5 \frac{1}{s + 2} \Rightarrow f(t) = 0.5(1 - e^{-2t})u(t)
\]

then

\[
y(t) = f(t) - f(t-3) = 0.5(1 - e^{-2t})u(t) - 0.5(1 - e^{-2(t-3)})u(t-3)
\]

Graphically, we plot \( h(\tau) \) and \( x(t - \tau) \) as functions of \( \tau \) and shift \( x(t - \tau) \) from the left to the right and integrate the overlapping areas to get

\[
y(t) = \begin{cases} 
0 & t < 0 \\
\int_0^t e^{-2\tau} d\tau = 0.5(1 - e^{-2t}) & 0 \leq t < 3 \\
\int_{t-3}^t e^{-2\tau} d\tau = 0.5(e^{-2(t-3)} - e^{-2t}) & t \geq 3
\end{cases}
\]
which can be written as
\[
y(t) = 0.5(1 - e^{-2t})[u(t) - u(t - 3)] + 0.5[e^{-2(t-3)} - e^{-2t}]u(t - 3) \\
= 0.5(1 - e^{-2t})u(t) - 0.5(1 - e^{-2(t-3)})u(t - 3)
\]
coinciding with the result obtained using the Laplace transform.

(b) Using Laplace transform

\[
H_1(s) = \frac{1 - e^{-s}}{s} \quad X_1(s) = H(s) \\
Y_1(s) = \frac{1 - 2e^{-s} + e^{-2s}}{s^2}
\]

so that
\[
y_1(t) = r(t) - 2r(t - 1) + r(t - 2)
\]
a triangular pulse.

Graphically, letting \(x_1(\tau)\) and \(h_1(t - \tau)\) (moving from left to right) be functions of \(\tau\), and integrating their overlap gives

\[
y_1(t) = \begin{cases} 
0 & t < 0 \\
\int_0^t d\tau = t & 0 \leq t < 1 \\
\int_{t-1}^1 d\tau = 1 - t + 1 = 2 - t & 1 \leq t < 2 \\
\int_{t-1}^1 d\tau = 1 - t + 1 = 2 - t & t \geq 2 
\end{cases}
\]

coinciding with the result using the Laplace transform. The signal \(y_1(t)\) is smoother than \(x_1(t)\) since it is the output of an averager filter.

(c) Using the Laplace transform, letting \(h(t) = h_i(t), \: i = 1, 2, 3\)

\[
H(s) = \frac{1 - e^{-s}}{s} \\
(H(s))^3 = \frac{(1 - e^{-s})^3}{s^3} = \frac{1 - 3e^{-s} + 3e^{-2s} - e^{-3s}}{s^3}
\]
The support of the output will be \(3M\).
3.9 To see the effect of the zeros on the complete response of a system, suppose you have a system with a transfer function

\[ H(s) = \frac{Y(s)}{X(s)} = \frac{s^2 + 4}{s((s + 1)^2 + 1)} \]

(a) Find and plot the poles and zeros of \( H(s) \). Is this BIBO stable?

(b) Find the frequency \( \Omega_0 \) of the input \( x(t) = 2 \cos(\Omega_0 t)u(t) \) such that the output of the given system is zero in the steady state. Why do you think this happens?

(c) If the input is \( x(t) = 2 \sin(\Omega_0 t)u(t) \) would you get the same result as above? Explain why or why not.

**Answers:** \( h(t) \) not absolutely integrable; cosine as input gives \( Y(s) = 2/((s + 1)^2 + 1) \).

**Solution**

(a) The poles of \( H(s) \) are at \( s = 0 \) and \( s = -1 \pm j1 \), and the zeros are at \( s = \pm j2 \). This system is not BIBO stable because of the pole at the origin, the impulse response will not be absolutely integrable.

(b) If \( x(t) = 2 \cos(2t)u(t) \) then

\[ X(s) = \frac{2s}{s^2 + 4} \]

and thus

\[ Y(s) = \frac{2}{(s + 1)^2 + 1} \]

i.e., the pole at \( s = 0 \) and the zeros are cancelled by the transform of the cosine. The remaining poles are in the left-hand s-plane and so the steady state is zero.

This is a very special case because of the cancellation of the pole at \( s = 0 \) and the numerator due to the used input. The system is not stable, and the frequency response does not exist for all frequencies. Indeed it is infinite whenever the input frequency is 0.

(c) The input cannot be a sine of the same frequency because its Laplace transform does not cancel the pole at zero and so the steady state is not zero.
3.2 Problem using MATLAB

3.10 Inverse Laplace transform— Consider the following inverse Laplace transform problems.

(a) Given the Laplace transform

\[ Y(s) = \frac{s^4 + 2s + 1}{s^3 + 4s^2 + 5s + 2} \]

which is not proper, determine the amplitude of the \( \delta(t) \) and \( d\delta(t)/dt \) terms in the inverse signal \( y(t) \).

(b) Find the inverse Laplace transform of

\[ Z(s) = \frac{s^2 - 3}{(s + 1)(s + 2)} \]

Can you use the initial-value theorem to check your result? Explain.

(c) The inverse Laplace transform of

\[ X(s) = \frac{3s - 4}{(s + 1)(s + 2)} \]

should be of the form \( x(t) = [Ae^{-t} + B + Ce^{-2t}]u(t) \). Find the values of \( A \), \( B \) and \( C \). Use the MATLAB function \texttt{ilaplace} to get the inverse and to plot it.

\textbf{Answers:} \( y(t) = -4\delta(t) + d\delta(t)/dt + \cdots \); \( z(t) = \delta(t) - 2e^{-t}u(t) - e^{-2t}u(t) \).

\textbf{Solution} (a) By long division we have that

\[ Y(s) = -4 + s + \frac{11s^2 + 20s + 9}{s^3 + 4s^2 + 5s + 2} \]

The inverse of the first two terms are \(-4\delta(t)\) and \(d\delta(t)/dt\). The third term is proper rational and its partial fraction expansion can be found.

(b) \( Z(s) \) is not proper rational so we need to divide to get

\[ Z(s) = 1 - \frac{3s + 5}{(s + 1)(s + 2)} = 1 - \frac{2}{s + 1} - \frac{1}{s + 2} \]

so that

\[ z(t) = \delta(t) - 2e^{-t}u(t) - e^{-2t}u(t) \]

Applying the initial-value result we get

\[ z(0) = \lim_{s \to \infty} sX(s) \to \infty \]

given the presence of \( \delta(t) \) which is not defined at \( t = 0 \).

(c)

\[ X(s) = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 2} \]
where $A = -2$, $B = 7$ and $C = -5$ so that

$$x(t) = -2u(t) + 7e^{-t}u(t) - 5e^{-2t}u(t)$$

% Pr. 3_10
clear all; clf
syms s t w
disp('>>>>> Inverse Laplace <<<<<')
x=ilaplace((3*s-4)/(s^3+3*s^2+2*s))
figure(1)
ezplot(x,[0,10])
axis([0 10 -2.5 .5])
grid

>>>>> Inverse Laplace <<<<<
x = -5*exp(-2*t)+7*exp(-t)-2

Figure 3.2: Problem 10
### 3.11 Solving differential equations—

One of the uses of the Laplace transform is the solution of differential equations.

(a) Suppose you are given the ordinary differential equation that represents a LTI system,

\[ y^{(2)}(t) + 0.5y^{(1)}(t) + 0.15y(t) = x(t), \quad t \geq 0 \]

where \( y(t) \) is the output and \( x(t) \) is the input of the system, \( y^{(1)}(t) \) and \( y^{(2)}(t) \) are first and second order derivatives with respect to \( t \). The input is causal, i.e., \( x(t) = 0 \quad t < 0 \).

What should the initial conditions be for the system to be LTI? Find \( Y(s) \) for those initial conditions.

(b) If \( y^{(1)}(0) = 1 \) and \( y(0) = 1 \) are the initial conditions for the above ordinary differential equation, find \( Y(s) \). If the input to the system is doubled, i.e., the input is \( 2x(t) \) is \( Y(s) \) doubled so that its inverse Laplace transform \( y(t) \) is doubled? Is the system linear?

(c) Use MATLAB to find the solutions of the ordinary differential equation when the input is \( u(t) \) and \( 2u(t) \) with the initial conditions given above. Compare the solutions and verify your response in (b).

#### Answers:

(b) \( Y(s) = X(s)/(s^2 + 0.5s + 0.15) + (s + 1.5)/(s^2 + 0.5s + 0.15) \), not LTI.

#### Solution

(a) If the initial conditions are zero the system is LTI. The Laplace transform of the differential equations is then

\[ Y(s)[s^2 + 0.5s + 0.15] = X(s) \]

and so

\[ Y(s) = \frac{X(s)}{s^2 + 0.5s + 0.15} \]

(b) If the initial conditions are \( dy(0)/dt = 1 \) and \( y(0) = 1 \) the Laplace transform of the differential equation is

\[ [Y(s)s^2 - s - 1] + 0.5sY(s) - 1] + 0.15Y(s) = X(s) \]

which gives

\[ Y(s) = \frac{X(s)}{s^2 + 0.5s + 0.15} + \frac{s + 1.5}{s^2 + 0.5s + 0.15} \]

which can be written, with the appropriate definitions, as \( Y(s) = X(s)/A(s) + I(s)/A(s) \), so that when we double the input the Laplace transform of the corresponding output is

\[ \frac{2X(s)}{A(s)} + \frac{I(s)}{A(s)} \neq 2Y(s) \]

thus the system is not LTI.

(c) If \( x(t) = u(t) \) then the Laplace transform of the output (including the initial conditions is)
after replacing \( X(s) = 1/s \):

\[
Y(s) = \frac{s^2 + 1.5s + 1}{s^3 + 0.5s^2 + 0.15s}
\]

when we double the input the Laplace transform of the output (again including the initial conditions) is

\[
Y_1(s) = \frac{s^2 + 1.5s + 2}{s^3 + 0.5s^2 + 0.15s}.
\]

The following script is used to obtain the two responses. The second one is not the double of the first, so the system is not LTI — due to the initial conditions not being zero.

```matlab
%% Pr. 3.11
clear all; clf
% response to u(t)
den=[1 0.5 0.15 0];
um=[0 1 1.5 1];
syms s t y y1
[p,r]=pfeLaplace(num,den)
disp('>>>>> Inverse Laplace <<<<<')
y=ilaplace((num(2)*s^2+num(3)*s+num(4))/(den(1)*s^3 +den(2)*s^2+den(3)*s))
% response to 2u(t)
um=[0 1 1.5 2];
pfeLaplace(num,den)
disp('>>>>> Inverse Laplace <<<<<')
y1=ilaplace((num(2)*s^2+num(3)*s+num(4))/(den(1)*s^3 +den(2)*s^2+den(3)*s))
figure(1)
subplot(311)
ezplot(y,[0,40])
axis([0 40 -0.01 11]);grid;title('Response to u(t)')
subplot(312)
ezplot(y1,[0,40])
axis([0 40 -0.01 15]);grid;title('Response to 2 u(t)')
subplot(313)
ezplot(y1-2*y,[0,40]);grid;title('Difference of two responses')
axis([0 40 -10 10])
```

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Figure 3.3: Problem 11: Responses $y(t)$ and $y_1(t)$ corresponding to inputs $u(t)$ and $2u(t)$, the difference $y_1(t) - 2y(t) \neq 0$ indicates the system is not linear.
3.12 Zero steady-state response of analog averager— An analog averager can be represented by the differential equation
\[
\frac{dy(t)}{dt} = \frac{1}{T} [x(t) - x(t - T)]
\]
where \(y(t)\) is its output and \(x(t)\) the input.

(a) If the input-output equation of the averager is
\[
y(t) = \frac{1}{T} \int_{t-T}^t x(\tau)d\tau
\]
Show how to obtain the above differential equation and that \(y(t)\) is the solution of the differential equation.

(b) If \(x(t) = \cos(\pi t)u(t)\), choose the value of \(T\) in the averager so that the output \(y(t) = 0\) in the steady state. Graphically show how this is possible for your choice of \(T\). Is there a unique value for \(T\) that makes this possible? How does it relate to the frequency \(\Omega_0 = \pi\) of the sinusoid?

(c) Use the impulse response \(h(t)\) of the averager found before, to show using Laplace that the steady state is zero when \(x(t) = \cos(\pi t)u(t)\) and \(T\) is the above chosen value. Use MATLAB to solve the differential equation and to plot the response for the value of \(T\) you chose. [HINT: consider \(x(t)/T\) the input and use superposition and time-invariance to find \(y(t)\) due to \((x(t) - x(t - T))/T\).]

Answers:
(c) \(y(t) = (0.5/\pi)(\sin(\pi t)u(t) - \sin(\pi(t-2))u(t-2))\) for input \(\cos(\pi t)u(t)\)

Solution
(a) The input/output equation for the analog averager is
\[
y(t) = \frac{1}{T} \int_{t-T}^t x(\tau)d\tau
\]
and its derivative is
\[
\frac{dy(t)}{dt} = \frac{1}{T} [x(t) - x(t - T)]
\]
Likewise if we integrate this from \(-\infty\) to \(t\) it gives
\[
y(t) = \frac{1}{T} \left[ \int_{-\infty}^t x(\tau)d\tau - \int_{-\infty}^{t-T} x(\tau)d\tau \right]
\]
and changing the variable in the second integral to \(\sigma = \tau - T\) we get
\[
y(t) = \frac{1}{T} \left[ \int_{-\infty}^t x(\tau)d\tau - \int_{-\infty}^{t-T} x(\sigma)d\sigma \right] = \frac{1}{T} \int_{t-T}^t x(\tau)d\tau
\]
the equation for the analog averager.

(b) Graphically the convolution gives zero for \( t < 0 \); for values of \( 0 \leq t < T \) the output is a transient; and for \( t > T \) it is a periodic signal. If we choose \( T = 2 \), the period of the sinusoid, (or multiples of 2) the periodic output signal becomes zero—the averager computes the area under the input of a period, which is zero. It can also be shown analytically, for \( T = 2 \)

\[
y(t) = \frac{1}{2} \int_{t-2}^{t} \cos(\pi \tau) d\tau \\
= \frac{1}{2} \sin(\pi t) \bigg|_{t-2}^{t} \\
= \frac{1}{2\pi} [\sin(\pi t) - \sin(\pi (t-2))] = 0 \quad t \geq 2
\]

(c) The impulse response of the averager is \( h(t) = 0.5[u(t) - u(t-2)] \) for the value \( T = 2 \) found above. If the input is \( \cos(\pi t)u(t) \), then

\[
Y(s) = H(s)X(s) = \frac{1}{2s} [1 - e^{-2s}]X(s) = \frac{X(s)}{2s} - \frac{X(s)}{2s} e^{-2s}
\]

Replacing \( X(s) = s/(s^2 + \pi^2) \) above we have

\[
Y(s) = 0.5 \frac{1 - e^{-2s}}{s^2 + \pi^2} = Y_1(s) - Y_1(s)e^{-2s}
\]

where \( Y_1(s) = 0.5/(s^2 + \pi^2) \) so that

\[
y(t) = \frac{0.5}{\pi} (\sin(\pi t)u(t) - \sin(\pi (t-2))u(t-2)) \]

%% Prob 3_12
clear all; clf
T=2;
num=[0 0 1/T]
den=[1 0 pi^2]
syms s t y
figure(1)
subplot(221)
[r,p]=pfeLaplace(num,den);
disp(’>>>>> Inverse Laplace <<<<<’)
y1=ilaplace(0.5/(s^2+pi^2))
subplot(222)
ezplot(y1,[0,8])
axis([0 8 -0.2 0.2]); title(’y_1(t)’)
grid
y=y1-y1*heaviside(t-2)
subplot(223)
ezplot(y,[0,8])

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Figure 3.4: Problem 12: Poles/zeros of $Y_1(s)$ (top–left), $y_1(t)$ and $y(t)$ (bottom).
3.13 Polynomial multiplication— When the numerator or denominator are given in a factorized form, we need to multiply polynomials. Although this can be done by hand, MATLAB provides the function \texttt{conv} that computes the coefficients of the polynomial resulting from the product of two polynomials.

(a) Use \texttt{help} in MATLAB to find how \texttt{conv} can be used, and then consider two polynomials

\[ P(s) = s^2 + s + 1 \quad \text{and} \quad Q(s) = 2s^3 + 3s^2 + s + 1 \]

do the multiplication of these polynomials by hand to find \( Z(s) = P(s)Q(s) \) and use \texttt{conv} to verify your results.

(b) The output of a system has a Laplace transform

\[ Y(s) = \frac{N(s)}{D(s)} = \frac{(s + 2)}{s^2(s + 1)((s + 4)^2 + 9)} \]

use \texttt{conv} to find the denominator polynomial and then find the inverse Laplace transform using \texttt{ilaplace}.

\textbf{Answers:} \( Z(s) = P(s)Q(s) = 2s^5 + 5s^4 + 6s^3 + 5s^2 + 2s + 1 \)

\textbf{Solution}

(a)(b) The following script shows how to do polynomial multiplication using \texttt{conv} function.

The multiplication of the polynomials

\[ P(s) = s^2 + s + 1 \quad \text{and} \quad Q(s) = 2s^3 + 3s^2 + s + 1 \]

gives

\[ Z(s) = P(s)Q(s) = 2s^5 + 5s^4 + 6s^3 + 5s^2 + 2s + 1 \]

The script shows how to obtain this result in the Poly Multiplication part of the script.

In the Application part of the script we show how to find the numerator \( N_1(s) \) and the denominator \( D_1(s) \) of \( Y(s) \) by multiplying \( X(s) \) and \( H(s) \) to give

\[ Y(s) = \frac{X(s)N(s)}{D(s)} = \frac{(s + 2)}{s^2(s + 1)((s + 4)^2 + 9)} \]

so that the poles of \( Y(s) \) are \( s = 0 \) (double), and \( s = -4 \pm j3 \), and no zeros. The double pole gives a ramp and the other poles a modulated sinusoid by a decaying exponential.

\begin{verbatim}
% Pr. 3_13
% Poly multiplication
% coefficients of higher to lower orders of s
P=[1 1 1];Q=[2 3 1 1];
\end{verbatim}
\[ Z = \text{conv}(P, Q); \]
\[
\% Application
\]
\[
d1 = [1 1]; \quad d2 = [1 8 25]; \quad D = \text{conv}(d1, d2);
\]
\[
d3 = [1 0 0]; \quad \text{den} = \text{conv}(D, d3)
\]
\[
N = \text{length}(\text{den})
\]
\[
\text{num} = [\text{zeros}(1, N-2) \text{ 1 2}]
\]
\[
s\text{ym} \quad \text{syms} \quad s \quad t \quad y
\]
\[
\text{figure}(1)
\]
\[
\text{subplot}(121)
\]
\[
[r, p] = \text{pfeLaplace}(\text{num}, \text{den});
\]
\[
\text{disp}('\\text{\\texttt{\\textbackslashggggg Inverse Laplace <<<<<}}')
\]
\[
y = \text{ilaplace}\left(\frac{\text{num}(N-1)s + \text{num}(N)}{\text{den}(1)s^5 + \text{den}(2)s^4 + \text{den}(3)s^3 + \text{den}(4)s^2 + \text{den}(5)s + \text{den}(6)}\right)
\]
\[
\text{subplot}(122)
\]
\[
\text{ezplot}(y, [0, 50])
\]
\[
\text{axis}([0 50 -0.1 1.5]); \quad \text{grid}
\]

\[
\begin{array}{ccccccc}
\text{den} & = & 1 & 9 & 33 & 25 & 0 & 0 \\
\text{num} & = & 0 & 0 & 0 & 0 & 1 & 2 \\
\end{array}
\]

\textbf{>>>>> Zeros <<<<<}

\textbf{z} = -2

\textbf{>>>>> Poles <<<<<}

\textbf{p} = -4.0000 + 3.0000i
-4.0000 - 3.0000i
-1.0000
0

\textbf{>>>>> Residues <<<<<}

\textbf{r} = 0.0050 - 0.0026i
0.0050 + 0.0026i
0.0556
-0.0656
0.0800

\textbf{>>>>> Inverse Laplace <<<<<}

\textbf{y} = -41/625 + 1/18 \text{exp}(-t) + 2/25t + 113/11250 \text{exp}(-4t) \text{cos}(3t)
+ 59/11250 \text{exp}(-4t) \text{sin}(3t)
Figure 3.5: Problem 13: Poles/zeros of $Y(s)$ and $y(t)$. 
Chapter 4

Frequency Analysis: the Fourier Series

4.1 Basic Problems

4.1 Consider the following problems related to periodicity and Fourier series:

(a) Consider the signals

\[ x_1(t) = 1 + \cos(2\pi t) - \cos(6\pi t), \quad x_2(t) = 1 + \cos(2\pi t) - \cos(6t) \quad -\infty < t < \infty \]

i. Determine which of these signals is periodic and the corresponding fundamental period.

ii. For the periodic signal, find a trigonometric Fourier series representation.

iii. Indicate why it is not possible to obtain a trigonometric Fourier series representation for the non-periodic signal.

(b) Consider the signal

\[ x(t) = \frac{\sin(2t + \pi)}{\sin(t)} \]

i. Is this signal periodic? If so, what is its fundamental period \( T_0 \)?

ii. Find the Fourier series of \( x(t) \).

Answers: (a) \( x_1(t) \) periodic of fundamental period \( T = 1 \); \( x_2(t) \) does not have Fourier series;
(b) periodic of fundamental period \( T_0 = 2\pi \); \( x(t) = -2\cos(t) \).

Solution
(a) i. The sinusoidal components of $x_1(t)$ have periods $T_1 = 1$ and $T_2 = 1/3$, with ratio $T_1/T_2 = 3$ so that $x_1(t)$ is periodic of fundamental period $T_1 = 3T_2 = 1$.

The periods of the sinusoidal components of $x_2(t)$ are $T_1 = 1$ and $T_2 = 2\pi/6$, their ratio is $T_1/T_2 = 3/\pi$, not a rational number so $x_2(t)$ is not periodic.

ii. Trigonometric Fourier series of $x_1(t)$, with fundamental period $T_0 = 1$ and fundamental frequency $\Omega_0 = 2\pi$, is

$$x_1(t) = 1 + \cos(\Omega_0 t) - \cos(3\Omega_0 t)$$

iii. For $x_2(t)$, since $\Omega_2 = 6$ cannot be expressed as an integer multiple of $\Omega_1 = 2\pi$ or vice versa, there is no Fourier series for $x_2(t)$.

(b) i. For the signal to be periodic of period $T_0$ we must have

$$x(t + T_0) = \frac{\sin(2(t + T_0) + \pi)}{\sin(t + T_0)} = x(t)$$

this is possible if $T_0 = 2\pi$

ii. It can be shown that $x(t) = -2\cos(t)$, indeed $\sin(2t + \pi) = -\sin(2t)$ and

$$-\sin(2t) = -2\cos(t)\sin(t)$$

so

$$x(t) = -2\cos(t) = -e^{jt} - e^{-jt}$$

thus $X_1 = X_{-1} = -1$. 
4.2 Find the complex exponential Fourier series for the following signals. In each case plot the magnitude and phase line spectra for $k \geq 0$.

(i) $x_1(t) = \cos(5t + 45^\circ)$,  
(ii) $x_2(t) = \sin^2(t)$,  
(iii) $x_3(t) = \cos(3t) + \cos(5t)$

**Answers:** The Fourier series coefficients for $x_1(t)$ are $X_1 = X_{-1} = 0.5e^{j\pi/4}$.

**Solution**

(i) Fundamental frequency $\Omega_0 = 5$ so

$$x_1(t) = (0.5e^{j\pi/4})e^{j5t} + (0.5e^{-j\pi/4})e^{-j5t}$$

and $X_1 = X_{-1} = 0.5e^{j\pi/4}$.

(ii) We have

$$x_2(t) = \left(\frac{1}{2j}(e^{jt} - e^{-jt})\right)^2 = 0.5 - 0.25e^{j2t} - 0.25e^{-j2t}$$

and so $X_0 = 0.5$, $X_1 = X_{-1} = -0.25$ and $\Omega_0 = 2$ is the fundamental frequency.

(iii) Fundamental frequency of $x_3(t)$ is $\Omega_0 = 1$ (ratio of fundamental periods of sinusoidal components $T_1/T_2 = (2\pi/3)/(2\pi/5) = 5/3$ and $\Omega_0 = 2\pi/3T_1 = 1$).

$$x_3(t) = 0.5e^{j3\Omega_0 t} + 0.5e^{-j3\Omega_0 t} + 0.5e^{j5\Omega_0 t} + 0.5e^{-j5\Omega_0 t}$$

so that $X_3 = X_{-3} = 0.5$ and $X_5 = X_{-5} = 0.5$.

See Fig. 4.1 for magnitude and phase line spectra.

![Figure 4.1: Problem 2](image-url)
4.3 Let the Fourier series coefficients of a periodic signal \( x(t) \) of fundamental frequency \( \Omega_0 = 2\pi/T_0 \) be \( \{X_k\} \). Consider the following functions of \( x(t) \).

\[
y(t) = 2x(t) - 3, \quad z(t) = x(t - 2) + x(t), \quad w(t) = x(2t)
\]

Determine if \( y(t) \), \( z(t) \) and \( w(t) \) are periodic, and if so give the corresponding Fourier coefficients in terms of those for \( x(t) \).

**Answers:** \( Z_k = X_k(1 + e^{-2j\Omega_0 k}) \); \( W_k = X_k/2 \) for \( k \) even and 0 otherwise.

**Solution**

(a) If the periodic signal

\[
x(t) = \sum_k X_ke^{j\Omega_0 kt}
\]

then

\[
y(t) = 2x(t) - 3 = (2X_0 - 3) + \sum_{k \neq 0} 2X_ke^{j\Omega_0 kt}
\]

is also periodic of period \( T_0 \) with Fourier coefficients

\[
Y_k = \begin{cases} 
2X_0 - 3 & k = 0 \\
2X_k & k \neq 0
\end{cases}
\]

The signal

\[
z(t) = x(t - 2) + x(t) = \sum_k X_ke^{j\Omega_0 k(t-2)} + \sum_k X_ke^{j\Omega_0 kt}
\]

\[
= \sum_k [X_k(1 + e^{-2j\Omega_0 k})]e^{j\Omega_0 kt}
\]

is periodic of period \( T_0 \) and with Fourier series coefficients \( Z_k = X_k(1 + e^{-2j\Omega_0 k}) \).

The signal

\[
w(t) = x(2t) = \sum_k X_ke^{j\Omega_0 k2t} = \sum_{m \text{ even}} X_{m/2}e^{j\Omega_0 mt}
\]

is periodic of period \( T_0/2 \), with Fourier series coefficients

\[
W_k = \begin{cases} 
X_{k/2} & k \text{ even} \\
0 & \text{otherwise}
\end{cases}
\]
4.4 The period of a periodic signal \( x(t) \) with fundamental period \( T_0 = 2 \) is \( x_1(t) = \cos(t)[u(t) - u(t - 2)] \).

(a) Plot the signal \( x(t) \) and find the Fourier series coefficients \( X_k \) for \( x(t) \) using the integral equation.

(b) Use the Laplace transform to find the Fourier series coefficients \( \{X_k\} \).

(c) After considering the symmetry of the zero-mean signal \( x(t) - X_0 \), can you determine whether the \( X_k \) should be real, purely imaginary or complex? Indicate how you make this determination.

(d) To simplify computing the Laplace transform of \( x_1(t) \) consider the following expression:

\[
\cos(t)u(t - 2) = [A \cos(t - 2) + B \sin(t - 2)]u(t - 2).
\]

Find the values of \( A \) and \( B \) and show how to compute \( \mathcal{L}[\cos(t)u(t - 2)] \).

**Answers:** \( X_k = (jk\pi(1 - \cos(2)) + \sin(2))/(2(1 - k^2\pi^2)) \); \( A = \cos(2), B = -\sin(2) \).

**Solution**

(a) See Fig. 4.2. Fourier series coefficients (\( \Omega_0 = \pi \))

\[
X_k = \frac{1}{2} \int_0^2 \frac{1}{2} (e^{jt} + e^{-jt})e^{-jk\pi t} dt = \frac{1}{4} \left[ \frac{e^{jt(1-k\pi)}}{j(1-k\pi)} + \frac{e^{-jt(1+k\pi)}}{-j(1+k\pi)} \right]^2_0
\]

\[
= \frac{1}{4} \left[ \frac{e^{2} - 1}{j(1-k\pi)} - \frac{e^{-2} - 1}{j(1+k\pi)} \right]
\]

\[
= \frac{1}{4(k^2\pi^2 - 1)} \left[ j(e^{2} - e^{-2}) + j\pi k(e^{2} + e^{-2}) - j(1+k\pi) + j(1-k\pi) \right]
\]

\[
= \frac{jk\pi(1 - \cos(2)) + \sin(2)}{2(1 - k^2\pi^2)}
\]
(b) \( X_1(s) = \mathcal{L}[\cos(t)(u(t) - u(t - 2))] = \mathcal{L}[\cos(t)u(t) - \cos(t)u(t - 2)] \)

\[
\mathcal{L}[\cos(t)u(t - 2)] = \int_2^\infty \cos(t)e^{-st}dt = \int_0^\infty \cos(\rho + 2)e^{-s(\rho+2)}d\rho
\]
\[
= e^{-2s} \left[ \int_0^\infty \cos(2)e^{-s\rho}d\rho - \int_0^\infty \sin(2)e^{-s\rho}d\rho \right]
\]
\[
= e^{-2s} \left[ \cos(2)\mathcal{L}[\cos(t)u(t)] - \sin(2)\mathcal{L}[\sin(t)u(t)] \right]
\]
\[
= e^{-2s} \left( \frac{\cos(2)s}{s^2 + 1} - \frac{\sin(2)}{s^2 + 1} \right)
\]

We then have

\[
X_k = 0.5 \left[ \frac{s}{s^2 + 1} - \frac{e^{-2s}(\cos(2)s - \sin(2))}{s^2 + 1} \right]_{s=jk\pi}
\]
\[
= \frac{jk\pi(1 - \cos(2)) + \sin(2)}{2(1 - k^2\pi^2)}
\]

which coincides with the expression obtained using the integral.

(c) Setting \( k = 0 \) in the above expression, we find the mean is \( X_0 = \sin(2)/2 \). Since the signal \( x(t) - X_0 \) is neither even nor odd the Fourier coefficients \( X_k \) should be complex, as they are.

(d) Letting

\[
\cos(t) = A\cos(t - 2) + B\sin(t - 2)
\]
\[
= \sqrt{A^2 + B^2}\cos(t - 2 - \tan^{-1}(B/A))
\]

which gives

\[
\sqrt{A^2 + B^2} = 1
\]
\[
\tan^{-1}(B/A) = -2
\]

which are satisfied by \( A = \cos(2) \) and \( B = -\sin(2) \), indeed \( A^2 + B^2 = (\cos(2))^2 + (\sin(2))^2 = 1 \) and \( \tan^{-1}(B/A) = \tan^{-1}[-\tan(2)] = -2 \). So that the Laplace transform

\[
\mathcal{L}[\cos(t)u(t - 2)] = \mathcal{L}[(A\cos(t - 2) + B\sin(t - 2))u(t - 2)]
\]
\[
= A\mathcal{L}[\cos(t - 2)u(t - 2)] + B\mathcal{L}[\sin(t - 2)u(t - 2)]
\]
\[
= \frac{e^{-2s}(\cos(2)s - \sin(2))}{s^2 + 1}
\]

which coincides with the result using the Laplace transform obtained before.
4.5 Consider the following problems related to steady state and frequency responses.

(a) The input $x(t)$ and the output $y(t)$, $-\infty < t < \infty$, of a filter with transfer function $H(s)$ are

$$x(t) = 4 \cos(2\pi t) + 8 \sin(3\pi t), \quad y(t) = -2 \cos(2\pi t).$$

Determine as much as possible about the frequency response $H(j\Omega)$ of the filter using the above input and output.

(b) Given a LTI system with frequency response

$$|H(j\Omega)| = u(\Omega + 2) - u(\Omega - 2), \quad \angle H(j\Omega) = \begin{cases} -\pi/2 & \Omega \geq 0 \\ \pi/2 & \Omega < 0 \end{cases}$$

If the input signal $x(t)$ is periodic with Fourier series

$$x(t) = \sum_{k=1}^{\infty} \frac{2}{k^2} \cos \left( \frac{3kt}{2} \right)$$

calculate the steady-state response $y_{ss}(t)$ of the system.

**Answers:** (a) $H(j2\pi) = -0.5$, $H(j3\pi) = 0$; (b) $y_{ss}(t) = 2 \cos(3t/2 - \pi/2)$.

**Solution**

(a) The steady state is

$$y(t) = 4|H(j2\pi)| \cos(2\pi t + \angle H(j2\pi)) + 8|H(j3\pi)| \cos(3\pi t - \pi/2 + \angle H(j3\pi))$$

so $H(j2\pi) = 0.5e^{j\pi} = -0.5$, $H(j3\pi) = 0$. Nothing else can be learned about the filter from the input/output.

(b) $y_{ss}(t) = \sum_{k=1}^{\infty} \frac{2}{k^2} |H(j3k/2)| \cos \left( \frac{3kt}{2} + \angle H(j3k/2) \right)$

$$= 2|H(j3/2)| \cos(3t/2 + \angle H(j3/2))$$

$$= 2 \cos(3t/2 - \pi/2)$$

since for frequencies bigger than 2 the magnitude response is zero.
4.6 Consider the following problems related to filtering of periodic signals:

(a) A periodic signal \( x(t) \) of fundamental frequency \( \Omega_0 = \pi/4 \) is the input of an ideal band-pass filter with the following frequency response

\[
|H(j\Omega)| = \begin{cases} 
1 & \pi/4 \leq \Omega \leq 3\pi/2 \\
1 & -3\pi/2 \leq \Omega \leq -\pi \\
0 & \text{otherwise}
\end{cases}
\]

\[
\angle[H(j\Omega)] = \begin{cases} 
\pi/2 & \Omega \leq 0 \\
-\pi/2 & \Omega > 0 \\
0 & \text{otherwise}
\end{cases}
\]

The non-zero Fourier series coefficients of \( x(t) \) are \( X_1 = X_{-1} = j \), \( X_5 = X_{-5} = 2 \).

i. Express \( x(t) \) in the form

\[
x(t) = \sum_{k=0}^{\infty} A_k \cos(\Omega_k + \theta_k).
\]

ii. Find the output \( y(t) \) of the ideal band-pass filter.

(b) The Fourier series of a periodic signal \( x(t) \) is

\[
x(t) = \sum_{k=0}^{\infty} (1 - (-1)^k) \cos(k\pi t)
\]

If \( x(t) \) is filtered with a filter having the following frequency response

\[
|H(j\Omega)| = \begin{cases} 
1 & -4.5\pi \leq \Omega \leq -1.5\pi \\
1 & 1.5\pi \leq \Omega \leq 4.5\pi \\
0 & \text{otherwise}
\end{cases}
\]

\[
\angle[H(j\Omega)] = \begin{cases} 
\pi/2 & \Omega \leq 0 \\
-\pi/2 & \Omega > 0 \\
0 & \text{otherwise}
\end{cases}
\]

i. Carefully plot the frequency response, magnitude and phase, of the filter and determine the type of filter it is.

ii. Calculate the steady-state response \( y_{ss}(t) \) of the filter for the given input.

Answers: (a) \( y(t) = 4 \cos(5\pi(t - 1)/4) \); (b) \( y_{ss}(t) = 2 \cos(3\pi t - \pi/2) \).

Solution

(a)

\[
x(t) = (je^{j\pi t/4} - je^{-j\pi t/4}) + 2(e^{j5\pi t/4} + e^{-j5\pi t/4})
\]

\[
= -2 \sin(\pi t/4) + 4 \cos(5\pi t/4) = 2 \cos(\pi t/4 + \pi/2) + 4 \cos(5\pi t/4)
\]

\[
y(t) = 2|H(j\pi/4)| \cos(\pi t/4 + \pi/2 + \angle[H(j\pi/4)]) + 4|H(j5\pi/4)| \cos(5\pi t/4 + \angle[H(j5\pi/4)])
\]

\[
= 4 \cos(5\pi t/4 - 5\pi/4) = 4 \cos(5\pi(t - 1)/4)
\]

(b) i. The filter is band-pass:
ii. According to the frequency response of the band-pass filter $y(t)$ can have a second, third and fourth harmonics, but since the magnitude of the second and fourth Fourier coefficients are $X_2 = X_4 = 0$, the output has only the third harmonic giving

$$y(t) = 2|H(j3\pi)|\cos(3\pi t + \angle H(j3\pi)) = 2\cos(3\pi t - \pi/2).$$
4.7 A periodic signal $x(t)$, of fundamental frequency $\Omega_0 = \pi$, has a period

$$x_1(t) = \begin{cases} 
1 + t & -1 \leq t \leq 0 \\
1 - t & 0 < t \leq 1 
\end{cases},$$

The signal $x(t)$ is the input of an ideal low-pass filter with the frequency response $H(j\Omega)$ shown in Fig. 4.4. Let $y(t)$ be the output of the system.

![Figure 4.4: Problem 7](image)

(a) Determine the Fourier series coefficients needed to find the output $y(t)$ of the filter.

(b) Is the output signal $y(t)$ periodic? If so, determine its fundamental period $T_0$, and its dc value.

(c) Provide the constants $A$, $B$ and $C$ in the output: $y(t) = A + B \cos(\pi t + C)$.

**Answers:** Need $X_0$ and $X_1$; $A = 1, B = 8/\pi^2, C = -\pi/2$.

**Solution**

(a) The fundamental frequency is $\Omega_0 = \pi (T_0 = 2)$, so only the Fourier coefficients corresponding to $k = 0$ and $k = \pm 1$, i.e., frequencies $\Omega = 0, \pm \Omega_0$, are needed as the filter deletes any other frequency component.

$$x_1(t) = r(t + 1) - 2r(t) + r(t - 1) \quad \Rightarrow \quad X_1(s) = \frac{e^s - 2 + e^{-s}}{s^2}$$

$$X_k = \frac{1}{2} \left( \frac{-1}{k^2 \pi^2} \right) (2 \cos(\pi k) - 2) = \frac{1 - \cos(\pi k)}{k^2 \pi^2} \quad k \neq 0$$

$$X_0 = 0.5 \text{ (by observation)}, \quad X_1 = X_{-1} = \frac{2}{\pi^2}$$

(b) Yes, $y(t)$ is periodic of period $2\pi$ and dc value $2X_0 = 1$.

(c) The steady state output is

$$y(t) = H(j0)X_0 + H(j\pi)X_1e^{j\pi t} + H(-j\pi)X_{-1}e^{-j\pi t}$$

$$= 2 \times 0.5 + \frac{8}{\pi^2} \left[ \frac{e^{j(\pi t - \pi/2)} + e^{-j(\pi t - \pi/2)}}{2} \right]$$

$$= \frac{1}{A} + \frac{8}{B} \cos(\pi t - \pi/2)$$

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4.8 Given the Fourier series representation for a periodic signal \( x(t) \), we can compute derivatives of it just like with any other signal.

(a) Consider the periodic train of pulses shown in Fig. 4.5, compute its derivative \( y(t) = \frac{dx(t)}{dt} \) and carefully plot it. Find the Fourier series of \( y(t) \).

(b) Use the Fourier series representation of \( x(t) \) and find its derivative to obtain the Fourier series of \( y(t) \). How does it compare to the Fourier series obtained above.

![Figure 4.5: Problem 8 and 9.](image)

Answer: \( y(t) = \sum_k 4j \sin(\pi k/2)e^{2\pi kt} \).

**Solution**

(a) The derivative of a period of \( x(t) \) in \(-0.5 \leq t \leq 0.5\) gives a period

\[
y_1(t) = \frac{dx_1(t)}{dt} = 2\delta(t + 0.25) - 2\delta(t - 0.25) \quad -0.5 \leq t \leq 0.5
\]

of the periodic signal \( y(t) \). The Laplace transform of \( y_1(t) \) is \( Y_1(s) = 2e^{0.25s} - 2e^{-0.25s} \) so that the Fourier series of \( y(t) \) of period \( T_0 = 1 \), and \( \Omega_0 = 2\pi \) (just like \( x(t) \)) is

\[
Y_k = \frac{1}{T_0} Y_1(s) |_{s=j2\pi k} = 4j \sin(\pi k/2)
\]

giving a Fourier series expansion

\[
y(t) = \sum_k 4j \sin(\pi k/2)e^{j2\pi kt}
\]

(b) A period of \( x(t) \) is

\[
x_1(t) = 2u(t + 0.25) - 2u(t - 0.25)
\]

\[
X_1(s) = \frac{2}{s}(e^{0.25s} - e^{-0.25s})
\]

\[
X_k = \frac{1}{T_0} X_1(s) |_{s=j2\pi k} = \frac{2\sin(\pi k/2)}{\pi k}
\]

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so that
\[ x(t) = \sum_{k} X_k e^{j2\pi kt} \]

The Fourier series
\[ y(t) = \frac{dx(t)}{dt} = \sum_{k} j2\pi kX_k e^{j2\pi kt} = \sum_{k} Y_k e^{j2\pi kt} \]

where
\[ Y_k = j2\pi kX_k = 4jk\pi \frac{\sin(\pi k/2)}{\pi k} = 4j \sin(\pi k/2) \]

which coincides with the result in part (a).
4.9 Consider the integral of the Fourier series of the pulse signal \( p(t) = x(t) - 1 \) of period \( T_0 = 1 \), where \( x(t) \) is given in Fig. 4.5.

(a) Given that an integral of \( p(t) \) is the area under the curve, find and plot the function
\[
s(t) = \int_{-\infty}^{t} p(t) \, dt \quad t \leq 1
\]
indicate the values of \( s(t) \) for \( t = 0, 0.25, 0.5, 0.75 \) and 1.

(b) Find the Fourier series of \( p(t) \) and \( s(t) \) and relate their Fourier series coefficients.

(c) Suppose you want to compute the integral
\[
\int_{-T_0/2}^{T_0/2} p(t) \, dt
\]
using the Fourier series of \( p(t) \). What is the integral equal to?

(d) You can also compute the integral from the plot of \( p(t) \)
\[
\int_{-T_0/2}^{T_0/2} p(t) \, dt
\]
What is it? Does it coincide with the result obtained using the Fourier series? Explain

**Answer:** \( p(t) = \sum_{k, \text{odd}} 4e^{j2\pi kt}/(k\pi) \).

**Solution**

(a) The pulse \( p(t) \) has an average of zero, so its integral is zero in each period, for \( kT_0 \leq t \leq (k + 1)T_0 \), \( k \) an integer, and \( T_0 = 1 \). If \( s_1(t) \) is the integral of \( p_1(t) \) for \( 0 \leq t \leq 1 \) we have that
\[
s_1(t) = \int_{0}^{t} p_1(\tau) \, d\tau = \int_{0}^{t} [x_1(\tau) - 1] \, d\tau = \int_{0}^{t} x_1(\tau) \, d\tau - t \quad 0 \leq t \leq 1
\]
Notice that \( x_1(t) = 2 \) for \( 0 \leq t \leq 0.25 \) and for \( 0.75 \leq t \leq 1 \), and zero for \( 0.25 \leq t \leq 0.75 \), so for the given times,

\[
s(0) = \int_{0}^{0} 2d\tau - 0 = 0
\]
\[
s(0.25) = \int_{0}^{0.25} 2d\tau - 0.25 = 0.25
\]
\[
s(0.5) = \int_{0}^{0.25} 2d\tau - 0.5 = 0
\]
\[
s(0.75) = \int_{0}^{0.25} 2d\tau - 0.75 = -0.25
\]
\[
s(1) = \left[ \int_{0}^{0.25} 2d\tau + \int_{0.75}^{1} 2d\tau \right] - 1 = 0
\]
Letting the period of \( p(t) \) between 0 and 1 be
\[
p_1(t) = x_1(t) - [u(t) - u(t-1)] = [2u(t) - 2u(t - 0.25) + 2u(t - 0.75) - 2u(t-1)] - [u(t) - u(t-1)]
\[
= u(t) - 2u(t - 0.25) + 2u(t - 0.75) - u(t-1)
\]
its integral gives the period
\[
s_1(t) = 0.25r(t) - 0.5r(t - 0.25) + 0.5r(t - 0.75) - 0.25r(t-1)
\]
of \( s(t) \), \( 0 \leq t \leq 1 \), which can be used to verify the above results.

(b) The pulse \( p(t) \) has a period between 0 and 1 equal to
\[
p_1(t) = u(t) - 2u(t - 0.25) + 2u(t - 0.75) - u(t-1)
\]
so that the Fourier series coefficients of \( p(t) \) are given by \( (T_0 = 1, \Omega_0 = 2\pi) \)
\[
P_k = \left. \frac{1}{T_0 \pi} e^{-0.5s} (e^{0.5s} - 2e^{0.25s} + 2e^{-0.25s} - e^{-0.5s}) \right|_{s=j2\pi k} = \frac{1}{\pi k} e^{-j\pi k/2} (\sin(\pi k) - 2\sin(\pi k/2)) = \frac{4}{\pi k} (-1)^{(k+1)} \sin(\pi k/2)
\]
where we have replaced \( \sin(\pi k) = 0 \). The value of \( P_k = 0 \) when \( k \) is even since \( \sin(\pi k/2) = 0 \) for \( k \) even, and for odd \( k \) since \( \sin(\pi k/2) = (-1)^{(k+1)} \) then \( P_k = 4/\pi k \). Since the average of \( p(t) \) is zero, we have that
\[
p(t) = \sum_{k, \text{ odd}} \frac{4}{k\pi} e^{j\pi k t}
\]
Since
\[
S(s) = \frac{0.25}{s^2} e^{-0.5s} (e^{0.5s} - 2e^{0.25s} + 2e^{-0.25s} - e^{-0.5s}) = \frac{0.25}{s} P(s)
\]
then \( (T_1 = 0.5 \text{ and } \Omega_1 = 4\pi) \):
\[
S_k = \left. \frac{1}{T_1} S(s) \right|_{s=j\Omega_1} = \left. \frac{2}{s} \frac{0.25}{s} P(s) \right|_{s=j4\pi k}
\]
(c) The integral over a period \([-T_0/2, T_0/2] = [-0.5, 0.5]\) is
\[
\int_{-0.5}^{0.5} p(t) dt = \sum_{k, \text{ odd}} \frac{4}{k\pi} \int_{-0.5}^{0.5} e^{j\pi k t} dt = \sum_{k, \text{ odd}} \frac{4}{k\pi} j^{2\pi k} \left|_{-0.5}^{0.5} \right| = \sum_{k, \text{ odd}} \frac{4}{k\pi} \sin(\pi k) \pi k = 0
\]
since \( \sin(\pi k) = 0 \) for \( k \) odd. Since \( p(t) \) has zero mean, the integral over a period is zero, coinciding with the above.
4.2 Problems using MATLAB

4.10 Addition of Periodic Signals — Consider a saw-tooth signal $x(t)$ with period $T_0 = 2$ and period

$$x_1(t) = \begin{cases} t & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the Fourier coefficients $X_k$ using the Laplace transform. Consider the cases when $k$ is odd and even ($k \neq 0$). You need to compute $X_0$ directly from the signal.

(b) Let $y(t) = x(-t)$, find the Fourier coefficients $Y_k$.

(c) The sum $z(t) = x(t) + y(t)$ is a triangular function. Find the Fourier coefficients $Z_k$ and compare them to $X_k + Y_k$.

(d) Use MATLAB to plot $x(t)$, $y(t)$, $z(t)$ and their corresponding magnitude line spectra.

Answers: $X_k = \frac{j}{2\pi k}$, $k \neq 0$ even, $X_0 = 0.25$

Solution

(a) The period $x_1(t) = r(t) - r(t-1) - u(t-1)$, and $T_0 = 2$ so that $\Omega_0 = \pi$. The Fourier coefficients are

$$X_k = \frac{1}{2} \left[ \frac{1 - e^{-s}}{s^2} - \frac{e^{-s}}{s} \right]_{s=j\pi k} = \frac{1 - e^{-s}(1 + s)}{2s^2} \bigg|_{s=j\pi k}$$

$$= \frac{1}{2} \left( \frac{1}{\pi^2 k^2} \left[ (-1)^k(1 + j\pi k) - 1 \right] \right) = \begin{cases} \frac{j}{2\pi k} & k \neq 0, \text{ even} \\ 0.25 & k = 0 \\ -\frac{2 - j\pi k}{(2\pi^2 k^2)} & k \text{ odd} \end{cases}$$

The average value which cannot be obtained from above is

$$X_0 = \frac{1}{2} \int_0^1 t \, dt = 0.25$$

(b) The Fourier series of $y(t)$ is

$$y(t) = x(-t) = \sum_k X_{-k} e^{jk\Omega_0 t}$$

so that

$$Y_k = X_{-k} = \begin{cases} -\frac{j}{2\pi k} & k \neq 0, \text{ even} \\ 0.25 & k = 0 \\ -\frac{2 - j\pi k}{(2\pi^2 k^2)} & k \text{ odd} \end{cases}$$

(c) Adding the above signals $z(t) = x(t) + y(t)$ we get triangular pulses. The Fourier coefficients of $z(t)$ are

$$Z_k = X_k + Y_k = \begin{cases} 0 & k \neq 0, \text{ even} \\ 0.5 & k = 0 \\ -\frac{2}{(\pi^2 k^2)} & k \text{ odd} \end{cases}$$
Notice that the Fourier coefficients of $x(t)$ and $y(t)$ which are neither even nor odd are complex, while the Fourier coefficients of $z(t)$ are real given that it is an even signal.

A period of $z(t)$ is

$$z_1(t) = r(t) - 2r(t-1) + r(t-2)$$

and $T_0 = 2$, $\Omega_0 = \pi$. The Fourier coefficients of $z(t)$ are

$$Z_k = \frac{1}{2} Z_1(s) \big|_{s=j\pi k} = \frac{(-1)^k}{\pi^2 k^2} (1 - \cos(\pi k)) = \begin{cases} 0 & k \neq 0, \text{ even} \\ -2/(\pi^2 k^2) & k \text{ odd} \end{cases}$$

and by inspection $Z_0 = 0.5$, which coincide with the ones obtained before. The following script gives an approximation of $z(t)$ using 41 harmonics, and shown in Fig. 4.6.

```matlab
% Pr. 4_10
clear all; clf
% magnitude line spectra
X1(1)=0.25;
for k=1:2:40,
    X1(k+1)=abs(-(2+j*pi*k)/(2*pi^2*k^2));
end
for k=2:2:40,
    X1(k+1)=abs(1/(2*pi*k));
end
kk=-40:40;
X=[fliplr(X1) X1(2:41)];
figure(1)
subplot(311)
stem(kk,X); grid; axis([-40 40 0 1]); ylabel('|X_k|')

Y1(1)=0.25;
for k=1:2:40,
    Y1(k+1)=abs((2+j*pi*k)/(2*pi^2*k^2));
end
for k=2:2:40,
    Y1(k+1)=abs(-1/(2*pi*k));
end
Y=[fliplr(Y1) Y1(2:41)];
subplot(312)
stem(kk,Y); grid; axis([-40 40 0 1]); ylabel('|Y_k|')

Z=[fliplr(Y1+X1) Y1(2:41)+X1(2:41)];
subplot(313)
stem(kk,Z); grid; axis([-40 40 0 1])
```

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```matlab
xlabel('k'); ylabel('|Z_k|')

% approximate of z(t)
t=0:0.001:10;
z=0.5*ones(1,length(t));
for k=1:2:10,
    z=z-4*cos(k*pi*t)/(pi*k)^2;
end
figure(2)
plot(t,z); grid
xlabel('t (sec)'); ylabel('z_1(t)')
```

Figure 4.6: Problem 10: Line spectra for $x(t)$, $y(t)$ and $z(t)$, (right) approximate $\hat{z}(t)$. 
4.11 Fourier series coefficients via Laplace — The computation of the Fourier series coefficients is helped by the relation between the formula for these coefficients and the Laplace transform of a period of the periodic signal.

(a) A periodic signal $x(t)$ of period $T_0 = 2$ sec., has as a period the signal $x_1(t) = u(t) - u(t - 1)$. Use the Laplace transform of $x_1(t)$ to obtain the Fourier coefficients of $x(t)$.

(b) Use MATLAB to approximate $x(t)$ with 40 harmonics, plot the approximate signal $\hat{x}(t)$ and its magnitude line spectrum.

Answers: $X_k = \sin(\pi k/2) e^{-jk\pi/2} / (\pi k/2)$, $X_0 = 0.5$.

Solution

(a) $x(t)$ is a train of rectangular pulses. The Laplace transform of the period $x_1(t)$ is

$$X_1(s) = \frac{1}{s}(1 - e^{-s}) = \frac{e^{-s/2}(e^{s/2} - e^{-s/2})}{s}$$

and $T_0 = 2$, so $\Omega_0 = \pi$. The Fourier series coefficients are then

$$X_k = \frac{1}{2} X_1(s) \big|_{s = j\pi k} = \frac{\sin(\pi k/2)}{\pi k/2} e^{-j\pi k/2} \quad X_0 = 0.5$$

(b) The following script is used to find and plot the magnitude line spectrum of $x(t)$, and to approximate the signal with 40 harmonics.

```matlab
% Pr. 4_11
clear all; clf
% magnitude line spectra
X(1)=0.5;
for k=1:40,
    X(k+1)=abs(0.5*sin(pi*k/2)/(pi*k/2));
end
kk=-40:40;
X=[fliplr(X) X(2:41)];
```

Figure 4.7: Problem 11: Magnitude line spectrum and approximate of train of pulses.
figure(1)
stem(kk,X); grid; axis([-40 40 0 0.6]); ylabel('|X_k|')
% approximate of z(t)
t=0:0.001:10;
x=0.5*ones(1,length(t));
for k=1:40,
    x=x+(sin(pi*k/2)/(pi*k/2))*cos(pi*k*t-k*pi/2);
end
figure(2)
plot(t,x); grid
xlabel('t (sec)'); ylabel('x_a(t)')
4.12 Time-support and frequency content — The support of a period of a periodic signal relates inversely to
the support of the line spectrum. Consider two periodic signals: $x(t)$ of period $T_0 = 2$ and $y(t)$ of period
$T_1 = 1$ and with periods

$$
x_1(t) = u(t) - u(t - 1) \quad 0 \leq t \leq 2
$$

$$
y_1(t) = u(t) - u(t - 0.5) \quad 0 \leq t \leq 1
$$

(a) Find the Fourier series coefficients for $x(t)$ and $y(t)$.

(b) Use MATLAB to plot the magnitude line spectra of the two signals from 0 to $20\pi$ rad/sec. Plot them
on the same figure so you can determine which has a broader support. Indicate which signal is
smoother and explain how it relates to its line spectrum.

Answers: $y_1(t) = x_1(2t)$, line spectrum of $x(t)$ is compressed when compared to that of $y(t)$.

Solution
The periods of the two signals are:

$$
x_1(t) = u(t) - u(t - 1) \quad 0 \leq t \leq T_0 = 2
$$

$$
y_1(t) = x_1(2t) = u(2t) - u(2t - 1) = u(t) - u(t - 0.5) \quad 0 \leq t \leq T_1 = 1
$$

the Fourier series coefficients of these signals are

$$
X_k = \frac{1}{2\pi} e^{-s/2} (e^{s/2} - e^{-s/2}) |_{s=j\pi k} = \frac{\sin(\pi k/2)}{\pi k} e^{-j\pi k/2}
$$

$$
Y_k = \frac{1}{s} e^{-s/4} (e^{s/4} - e^{-s/4}) |_{s=j2\pi k} = \frac{\sin(\pi k/2)}{\pi k} e^{-j\pi k/2}
$$

We see that the coefficients for the two signals are identical, but the fundamental frequency for $x(t)$ is
$\Omega_0 = 2\pi/T_0 = \pi$ while the one for $y(t)$ is $\Omega_1 = 2\pi/T_1 = 2\pi = 2\Omega_0$ so that the line spectrum of $x(t)$ is
compressed when compared to that of $y(t)$. Thus compression in a period $y(t) = x(2t)$ corresponds to
expansion in the frequency domain. Thus, $x(t)$ is smoother than $y(t)$ as this displays higher frequencies.

% Pr. 4_12
clear all; clf
% magnitude line spectra
X(1)=0.5; Y(1)=X(1);
for k=1:40,
    X(k+1)=abs(0.5 *sin(pi*k/2)/(pi*k*0.5));
end
Y=X;
kk=-40:40;
X=[flipl(X) X(2:41)];
Y=[flipl(Y) Y(2:41)];
figure(1)
subplot(211)
stem(pi*kk,X); grid;axis([-40*pi 40*pi 0 0.6]); ylabel(‘$|X_k|$’)

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\begin{verbatim}
subplot(212)
stem(2*pi*kk,Y); grid;axis([-40*2*pi 40*2*pi 0 0.6]); ylabel('|Y_k|')
\end{verbatim}

Figure 4.8: Problem 12: The line spectrum of $Y_k$ is an expansion of that of $X_k$ (notice the frequencies for each).
4.13 Windowing and music sounds — In the computer generation of musical sounds, pure tones need to be windowed to make them more interesting. Windowing mimics the way a musician would approach the generation of a certain sound. Increasing the richness of the harmonic frequencies is the result of the windowing as we will see in this problem. Consider the generation of a musical note with frequencies around 880 Hz. Assume our “musician” while playing this note uses three strokes corresponding to a window \( w_1(t) = r(t) - r(t - T_1) - r(t - T_2) + r(t - T_0) \), so that the resulting sound would be the multiplication, or windowing, of a pure sinusoid \( \cos(2\pi f_A t) \) by a periodic signal \( w(t) \) with \( w_1(t) \) a period that repeats every \( T_0 = 5T \) where \( T \) is the period of the sinusoid. Let \( T_1 = T_0/4 \) and \( T_2 = 3T_0/4 \).

(a) Analytically determine the Fourier series of the window \( w(t) \) and plot its line spectrum using MATLAB. Indicate how you would choose the number of harmonics needed to obtain a good approximation to \( w(t) \).

(b) Use the modulation or the convolution properties of the Fourier series to obtain the coefficients of the product \( s(t) = \cos(2\pi f_A t)w(t) \). Use MATLAB to plot the line spectrum of this periodic signal and again determine how many harmonic frequencies you would need to obtain a good approximation to \( s(t) \).

(c) The line spectrum of the pure tone \( p(t) = \cos(2\pi f_A t) \) only displays one harmonic, the one corresponding to the \( f_A = 880 \) Hz frequency, how many more harmonics does \( s(t) \) have? To listen to the richness in harmonics use the function \texttt{sound} to play the sinusoid \( p(t) \) and \( s(t) \) (use \( F_s = 2 \times 880 \) Hz to play both).

(d) Consider a combination of notes in a certain scale, for instance let

\[
p(t) = \sin(2\pi \times 440t) + \sin(2\pi \times 550t) + \sin(2\pi \times 660t).
\]

Use the same windowing \( w(t) \), and let \( s(t) = p(t)w(t) \). Use to plot \( p(t) \) and \( s(t) \) and to compute and plot their corresponding line spectra. Use \texttt{sound} to play \( p(nT_s) \) and \( s(nT_s) \) using \( F_s = 1000 \).

**Answers:** Since the fundamental period of the sinusoid is \( T = 1/880 \) let \( T_0 = 5T = 5/880 \) sec., \( T_1 = T_0/4 \) and \( T_2 = 3T_0/4 \).

**Solution**

(a) The fundamental period of the sinusoid is \( T = 1/880 \) so that \( T_0 = 5T = 5/880 \) sec., \( T_1 = T_0/4 \) and \( T_2 = 3T_0/4 \). The Fourier series coefficients of \( \omega(t) \) are

\[
W_k = \frac{1}{T_0} \left. W_1(s) \right|_{s=j\Omega_0} = \frac{1}{s^2} (1 - e^{-sT_0/4} - e^{-s3T_0/4} + e^{-sT_0}) \left| s = jk\Omega_0 \right.
\]

\[
= \frac{e^{-sT_0/2}}{s^2} (e^{sT_0/2} - e^{sT_0/4} - e^{-s3T_0/4} + e^{-sT_0/2}) \left| s = jk\Omega_0 \right.
\]

\[
= \frac{2e^{-jk\pi}}{-\Omega_0^2 k^2} (\cos(\pi k) - \cos(\pi k/2)) \quad \Omega_0 = \frac{2\pi}{T_0} = 352\pi \text{ rad/sec}
\]
(b) The signal \( s(t) = w(t) \cos(2\pi f_A t) \) shifts the harmonics of \( w(t) \) to a central frequency \( 2\pi f_A \). The number of harmonic frequencies is increased by windowing the sinusoid or sum of sinusoids.

%Pr. 4_13
clear all; clc
% parameters
Fs = 880; % frequency
T = 1/Fs; % sample period
T0 = 5*T; T1 = T0/4; T2 = T0*3/4;
% a)----------------------------------------------------------
% define one period of \( w \) signal -> \( w_1 \)
Ts=1/17600; % overall sample period
for k=1:H,
a(k)=2*(cos(k*w0*T0/4)-cos(k*w0*T0/2))/T0/kˆ2/w0ˆ2;
end
a=[DC 2*a];
figure(2)
for N1=2:H+1,
x=a(1)*ones(1,length(t));
for k=2:N1,
x=x+a(k)*cos((k-1)*w0.*(t+T0/2));
end
plot(t,x);
axis([0 max(t) 1.1*min(x) 1.1*max(x)]);
hold on;
plot(t,w1,'r')
ylabel('Amplitude');xlabel('t (sec)');
legend('aprox \( w_1(t) \) ','\( w_1(t) \)');grid;
hold off
pause(0.1);
clear x;
end

clear H a x1

% b)----------------------------------------------------------

s = cos(2*pi*Fs*t).*w1; % modulated signal s

figure(3);
FSeries(s,Ts,T0,N); % line spectrum of s

% how many harmonics H are necessary to approximate s?
w0=2*pi/T0;
DC=0.001065;H=3;
for k=1:H,
    a(k)=2*(cos(k*w0*T0/4)-cos(k*w0*T0/2))/T0/kˆ2/w0ˆ2;
end
a=[DC 2*a];

figure(4)

for N1=2:H+1,
x=a(1)*ones(1,length(t));
for k=2:N1,
x=x+a(k)*cos((k-1)*w0.*(t+T0/2));
x1=x.*cos(5*w0.*t);
end
plot(t,x1);
axis([0 max(t) 1.1*min(s) 1.1*max(s)]);
hold on;
plot(t,s,'r')
ylabel('Amplitude'); xlabel('t (sec)');
legend('aprox s(t)','s(t)'); grid;
hold off;
clear x;
end

% c)----------------------------------------------------------

p = cos(2*pi*Fs*t); % signal p

figure(5);
FSeries(p,Ts,T0,N); % line spectrum of p

% sound(p); pause(1); sound(w0*s);pause(2);

% d)----------------------------------------------------------

p = sin(2*pi*440*t)+sin(2*pi*550*t)+sin(2*pi*660*t); % signal p
s = p.*w1; % modulated signal s

figure(6);
FSeries(s,Ts,T0,N); % line spectrum of s

% sound(100*p); pause(1); sound(100*w0*s);
Figure 4.9: Problem 13. Right to left: window and line spectra, approximation of window, windowed sinusoid and its line spectra.

Figure 4.10: Problem 13: from right to left: approximation of windowed signal, line spectra of sinusoid, windowed sum of sinusoids and line spectra.
4.14 Computation of π — As you know, π is an irrational number that can only be approximated by a number with a finite number of decimals. How to compute this value recursively is a problem of theoretical interest. In this problem we show that the Fourier series can provide that formulation.

(a) Consider a train of rectangular pulses \( x(t) \), with a period

\[
x_1(t) = 2[u(t + 0.25) - u(t - 0.25)] - 1, \quad -0.5 \leq t \leq 0.5
\]

and period \( T_0 = 1 \). Plot the periodic signal and find its trigonometric Fourier series.

(b) Use the above Fourier series to find an infinite sum for π.

(c) If \( \pi_N \) is an approximation of the infinite sum with \( N \) coefficients, and \( \pi \) is the value given by \( \), find the value of \( N \) so that \( \pi_N \) is 95% of the value of \( \pi \) given by MATLAB.

**Answer:** \( \pi = 4 \sum_{k=1}^{\infty} \sin(\pi k/2)/k \)

**Solution**

(a) The signal \( x(t) \) is zero-mean. The Fourier series coefficients are obtained from \( (\Omega_0 = 2\pi/T_0 = 2\pi) \)

\[
X_1(s) = \frac{2}{s} (e^{0.25s} - e^{-0.25s})|_{s=j2\pi} = \frac{\sin(\pi k/2)}{\pi k/2}
\]

so that

\[
x(t) = \sum_{k=-\infty,\neq 0}^{\infty} \frac{\sin(\pi k/2)}{\pi k/2} e^{j2\pi k t}
\]

\[
= 2 \sum_{k=1}^{\infty} \frac{\sin(\pi k/2)}{\pi k/2} \cos(2\pi k t)
\]

Since \( x(0) = 1 \), letting \( t = 0 \) in both sides and then multiplying by \( \pi \) we get that

\[
\pi = 4 \sum_{k=1}^{\infty} \frac{\sin(\pi k/2)}{k}
\]

If we let

\[
\pi_N = 4 \sum_{k=1}^{N} \frac{\sin(\pi k/2)}{k} = 0.95\pi
\]

we use MATLAB to find the value of \( N \).

```matlab
% Pr. 4_14
clear all; clc
N=100; x = 0; %DC value
for k=1:N
```
x = x + 4*sin(k*pi/2)/k;
xx(k) = x;
end
format long
display('Error of approximation for')
N
e = pi-x
ppi=pi*ones(1,length(xx)); n=1:N;
figure(1)
subplot(211); plot(n,xx); title('Approximation of PI'); axis([1 N 0 4.5])
hold on; plot(n,ppi,'r'); grid; hold off; legend('approximate', 'pi')
subplot(212); plot(n,xx-pi); title('Error of approximation'); grid
% N for 95 percent of pi
sum = 0; %initial approximation
for i = 1:N,
    sum = sum + 4*cos(i*pi/2-pi/2)/i;
    if (sum >=(0.95*pi))&&(sum<=pi) break;
end
display([num2str(i) ' coeficients needed of Fourier series for 95\% approx. of PI '])

Error of approximation for
N = 100
e = 0.019998000998784
15 coeficients needed of Fourier series for 95\% approx. of PI
Figure 4.11: Problem 14: Approximation of $\pi$ by series, error of approximation
4.15 Square error approximation of periodic signals — To understand the Fourier series consider a more general problem, where a periodic signal \( x(t) \), of period \( T_0 \), is approximated as a finite sum of terms

\[
\hat{x}(t) = \sum_{k=-N}^{N} \hat{X}_k \phi_k(t)
\]

where \( \{\phi_k(t)\} \) are ortho-normal functions. To pose the problem as an optimization problem, consider the square error

\[
\varepsilon = \int_{T_0} |x(t) - \hat{x}(t)|^2 dt
\]

and we look for the coefficients \( \{\hat{X}(k)\} \) that minimize \( \varepsilon \).

(a) Assume that \( x(t) \) as well as \( \hat{x}(t) \) are real valued, and that \( x(t) \) is even so that the Fourier series coefficients \( X_k \) are real. Show that the error can be expressed as

\[
\varepsilon = \int_{T_0} x^2(t) dt - 2 \sum_{k=-N}^{N} \hat{X}_k \int_{T_0} x(t) \phi_k(t) dt + \sum_{\ell=-N}^{N} |\hat{X}_\ell|^2 T_0.
\]

(b) Compute the derivative of \( \varepsilon \) with respect to \( \hat{X}_n \) and set it to zero to minimize the error. Find \( \hat{X}_n \).

(c) In the Fourier series the \( \{\phi_k(t)\} \) are the complex exponentials and the \( \{\hat{X}(k)\} \) coincide with the Fourier series coefficients. To illustrate the above procedure consider the case of the pulse signal \( x(t) \), of period \( T_0 = 1 \) and a period

\[
x_1(t) = 2[u(t + 0.25) - u(t - 0.25)]
\]

Use MATLAB to compute and plot the approximation \( \hat{x}(t) \) and the error \( \varepsilon \) for increasing values of \( N \) from 1 to 100.

(d) Concentrate your plot of \( \hat{x}(t) \) around one of the discontinuities, and observe the Gibb’s phenomenon. Does it disappear when \( N \) is very large. Plot \( \hat{x}(t) \) around the discontinuity for \( N = 1000 \).

**Answer:** \( d\varepsilon/d\hat{X}_n = -2 \int_{T_0} x(t) \phi_n(t) dt + 2T_0 \hat{X}_n = 0 \).

**Solution**

(a) Assuming \( x(t) \) and \( \hat{x}(t) \) being real and even, so that \( \hat{X}_k \) are real we have

\[
\varepsilon = \int_{T_0} (x(t) - \hat{x}(t))^2 dt = \int_{T_0} x^2(t) - 2x(t)\hat{x}(t) + \hat{x}^2(t) dt
\]

\[
\varepsilon = \int_{T_0} x^2(t) dt - 2 \sum_k \hat{X}_k \int_{T_0} x(t) \phi_k(t) dt + \sum_k \sum_\ell \hat{X}_k \hat{X}_\ell^* \phi_k(t) \phi_\ell^*(t)
\]

\[
\varepsilon = \int_{T_0} x^2(t) dt - 2 \sum_k \int_{T_0} x(t) \hat{X}_k \phi_k(t) dt + \sum_k \sum_\ell \hat{X}_k \hat{X}_\ell \int_{T_0} \phi_k(t) \phi_\ell^*(t)
\]
To minimize the error we set its derivative w.r.t. $\hat{X}_n$ to zero, i.e.,

$$\frac{dc}{d\hat{X}_n} = -2 \int_{T_0} x(t)\phi_n(t)dt + 2T_0\hat{X}_n = 0$$

so that $\hat{X}_n = \frac{1}{T_0} \int_{T_0} x(t)\phi_n(t)dt$

% Pr. 4_15
clear all; clc
% c)d)--------------------------------------------------------
% Fourier Series of a train of pulses of period T0=1 and dc value 1
w0 = 2*pi; T0 = 1; DC=1; N=100;
for k=1:N,
a(k)=sin(k*pi/2)/(k*pi/2);
end
a=[DC 2*a];

Ts = 0.001; %sample period
t=0:Ts:1-Ts; L=length(t); %number of samples
x1=[ones(1,L/4) zeros(1,L/2) ones(1,L/4)]; %train signal
x1=2*DC*x1; %DC gain adaptation
for N1=2:N+1,
x=a(1)*ones(1,length(t));
for k=2:N1,
x=x+a(k)*cos(2*pi*(k-1).*t);
end
%plot figures
figure(1)
plot(t,x); axis([0 max(t) 1.1*min(x) 1.1*max(x)]); hold on
plot(t,x1,'r'); ylabel('x(t), x_a(t)'); xlabel('t (sec)'); hold off
figure(3)
plot(t,x); axis([0.175 0.275 1.75 2.25]); hold on
plot(t,x1,'r'); title('Gibbs Phenomenon'); ylabel('x(t), x_a(t)'); xlabel('t (sec)');
pause(0.05)

%compute error
e=(x1-x).^2;
error(N1-1)=0;
for k = 1:length(e)
    error(N1-1) = error(N1-1) + e(k);
end
clear x
end

figure(2)
axis([1 N 1.1*max(error)]); hold on
plot(error,'linewidth',2);
title('e = int((x-x1)^2)'); ylabel('Amplitude of error'); xlabel('t (samples)')
hold off

Figure 4.12: Problem 15: Gibb’s phenomenon: (top–left) square pulse and approximate, (bottom) detail of approximation around the discontinuity; (top-right) approximation error.
Chapter 5

Frequency Analysis: the Fourier Transform

5.1 Basic Problems

5.1 The Fourier transform of a signal \( x(t) \) is

\[
X(\Omega) = \frac{2}{1 + \Omega^2}.
\]

Use properties of the Fourier transform to

(a) find the integral

\[
\int_{-\infty}^{\infty} x(t)dt,
\]

(b) find the value \( x(0) \).

(c) Let \( s = j\Omega \) or \( \Omega = (s/j) \), find \( X(s) \) and from it obtain \( x(t) \).

Answers: \( x(0) = 1; x(t) = e^{-|t|} \).

Solution

(a) The Fourier pair is

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega \quad \Leftrightarrow \quad X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt
\]

For \( \Omega = 0 \) the Fourier transform gives

\[
\int_{-\infty}^{\infty} x(t)dt = X(0) = 2
\]
(b) From the inverse Fourier transform

\[
x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) d\Omega = \frac{2}{2\pi} \int_{0}^{\infty} \frac{2}{1 + \Omega^2} d\Omega \\
= \frac{2}{\pi} \tan^{-1}(\Omega)|_{0}^{\infty} = \frac{2}{\pi} (\pi/2) = 1
\]

(c) \( \Omega = s/j \)

\[
X(s) = \frac{2}{1 + (s/j)^2} = \frac{-2}{(s - 1)(s + 1)} = \frac{1}{1 - s} + \frac{1}{s + 1} \quad \text{then}
\]

\[
x(t) = e^{t}u(-t) + e^{-t}u(t)
\]
5.2 Find the Fourier transform of $\delta(t - \tau)$ and use it to find the Fourier transforms of

(a) $\delta(t - 1) + \delta(t + 1)$
(b) $\cos(\Omega_0 t)$
(c) $\sin(\Omega_0 t)$

**Answers:** $\mathcal{F}[\delta(t - 1) + \delta(t + 1)] = 2 \cos(\Omega)$ and apply duality for the others.

**Solution**

$\mathcal{F}[\delta(t - \tau)] = \mathcal{L}[\delta(t - \tau)]|_{s = j\Omega} = e^{-j\Omega\tau}$ so

(a) By linearity and time–shift

$$\mathcal{F}[\delta(t - 1) + \delta(t + 1)] = 2 \cos(\Omega)$$

(b) By duality

$$0.5[\delta(t - \tau) + \delta(t + \tau)] \leftrightarrow \cos(\Omega\tau)$$

$$\cos(\Omega_0 t) \leftrightarrow \pi[\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)]$$

by letting $\tau = \Omega_0$ in the second equation.

(c) Considering

$$\mathcal{F}[\delta(t - 1) - \delta(t + 1)] = 2j\sin(\Omega),$$

by duality

$$-0.5j[\delta(t - \tau) + \delta(t + \tau)] \leftrightarrow \sin(\Omega\tau)$$

$$\sin(\Omega_0 t) \leftrightarrow -j\pi[\delta(\Omega + \Omega_0) + \delta(\Omega + \Omega_0)]$$

by letting $\tau = \Omega_0$ in the second equation.
5.3 A periodic signal \( x(t) \) has a period
\[
x_1(t) = r(t) - 2r(t - 1) + r(t - 2), \quad T_0 = 2.
\]

(a) Find the Fourier series of \( z(t) = d^2x(t)/dt^2 \) using the Laplace transform. Then use the derivative property to find the Fourier transform of \( x(t) \).

(b) To check your results figure out what the dc value of \( x(t) \) and \( z(t) \) should be, and whether the Fourier coefficients should be real, purely imaginary or complex.

(c) Show that the following equation can be used to compute the Fourier transform of the periodic signal \( z(t) \):
\[
Z(\Omega) = Z_1(\Omega) \frac{2\pi}{T_0} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_0)
\]
where \( Z_1(\Omega) \) is the Fourier transform of \( z_1(t) = d^2x_1(t)/dt^2 \). Find its inverse \( z(t) \).

Answer: \( z(t) = \sum_m 2e^{j(2m+1)\pi t} \).

Solution

(a) A period of \( z(t) \) is \( z_1(t) = d^2x_1(t)/dt^2 = \delta(t) - 2\delta(t - 1) + \delta(t - 2), \) with period \( T_0 = 2 \) \((\Omega_0 = \pi)\), its Fourier series coefficients are
\[
Z_k = \frac{1}{2} Z_1(s)|_{s=j\pi k} = \frac{1}{2} 2e^{-s} (\cosh(s) - 1)|_{s=j\pi k}
= (-1)^{k+1} (1 - (-1)^k) = \begin{cases} 2 & k \text{ odd} \\ 0 & k \text{ even} \end{cases}
\]

then
\[
z(t) = \sum_{k=-\infty}^{\infty} 2e^{j\pi t} = \sum_{m=-\infty}^{\infty} 2e^{j(2m+1)\pi t}
\]
by letting \( k = 2m + 1 \). By the derivative property
\[
X_k = \frac{Z_k}{(jk\Omega_0)^2} = \frac{-2}{k^2\Omega_0^2}, \quad k \text{ odd}
\]
and
\[
x(t) = \sum_{k=-\infty, \text{odd}}^{\infty} \frac{2(-1)^k}{k^2\Omega_0^2} e^{jk\pi t} = \sum_{m=-\infty}^{\infty} \frac{2(-1)^{2m+1}}{(2m + 1)^2\Omega_0^2} e^{j(2m+1)\pi t}
\]

(b) The average \( X_0 = 0.5 \), and \( x(t) - 0.5 \) is even so \( X_k, k \neq 0 \) are even and \( z(t) \) is even and zero mean, so \( Z_k \) are real and \( Z_0 = 0 \)
(c) The Fourier transform is

\[
Z(\Omega) = Z_1(\Omega) \frac{2\pi}{T_0} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_0) = \sum_{k=-\infty}^{\infty} 2\pi \frac{Z_1(\Omega_0)}{T_0} \delta(\Omega - k\Omega_0)
\]

and the inverse

\[
z(t) = \sum_k \mathcal{F}^{-1}[2\pi Z_k \delta(\Omega - k\Omega_0)] = \sum_k Z_k e^{jk\Omega_0}
\]
5.4 A sinc signal \( x(t) = \frac{\sin(0.5t)}{\pi t} \) is passed through an ideal low-pass filter with a frequency response \( H(j\Omega) = u(\Omega + 0.5) - u(\Omega - 0.5) \)

(a) Find the Fourier transform \( X(\Omega) \) and carefully plot it.
(b) Find the output \( y(t) \) of the filter by calculating first \( Y(\Omega) \). What is the convolution integral of \( x(t) \) with \( x(t) \)? Explain.

**Answers:**
\[ X(\Omega) = u(\Omega + 0.5) - u(\Omega - 0.5); \]
\[ y(t) = (x * x)(t) = x(t). \]

**Solution**

(a) Let \( X(\Omega) = A[u(\Omega + \Omega_0) - u(\Omega - \Omega_0)] \) its inverse Fourier transform is
\[ x(t) = \frac{1}{2\pi} \int_{-\Omega_0}^{\Omega_0} A e^{j\Omega t} d\Omega = \frac{A \sin(\Omega_0 t)}{\pi t} \]
so \( A = 1, \Omega_0 = 0.5 \) and \( X(\Omega) = u(\Omega + 0.5) - u(\Omega - 0.5) \).

(b) \( Y(\Omega) = H(\Omega)X(\Omega) = X(\Omega) \) so that \( y(t) = (x * x)(t) = x(t) \), or convolution of a sinc function with itself is a sinc.
5.5 The Fourier transform of a signal $x(t)$ is

$$X(\Omega) = \frac{\Omega}{\pi} [u(\Omega + \pi) - u(\Omega - \pi)]$$

(a) Carefully plot $X(\Omega)$ as function of $\Omega$.
(b) Determine the value of $x(0)$

**Answer:** $x(0) = 0.5$.

**Solution**

(a) Plot of $X(\Omega)$ as function of $\Omega$:

(b)

$$x(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\Omega}{\pi} d\Omega = \frac{2}{2\pi} \int_{0}^{\pi} \frac{\Omega}{\pi} d\Omega = \frac{1}{2}$$
5.6 The Fourier series coefficients of a periodic signal \( x(t) \), with fundamental frequency \( \Omega_0 = \pi/4 \) are \( X_1 = X_{-1} = j \), \( X_5 = X_{-5} = 2 \) and the rest are zero. Suppose \( x(t) \) is the input to a band-pass filter with the following magnitude and phase responses

\[
|H(j\Omega)| = \begin{cases} 
1 & \pi \leq \Omega \leq 1.5\pi \\
1 & -1.5\pi \leq \Omega \leq -\pi \\
0 & \text{otherwise}
\end{cases} \\
\angle H(j\Omega) = -\Omega.
\]

Let \( y(t) \) be the output of the filter.

(a) Determine the Fourier series of \( x(t) \) in the trigonometric form

\[
x(t) = \sum_{k=0}^{\infty} X_k \cos(k\Omega_0 t + \angle X_k)
\]

and then find the steady-state response of \( y(t) \).

(b) Find the Fourier transform of \( x(t) \) and \( y(t) \) and carefully plot their magnitudes and phases.

Answers: \( x(t) = -2 \sin(\pi t/4) + 4 \cos(5\pi t/4) \); \( y(t) = 4 \cos(5\pi t/4 - 5\pi/4) \).

Solution

(a) We have

\[
x(t) = j(e^{j\pi t/4} - e^{-j\pi t/4}) + 2(e^{5j\pi t/4} + e^{-5j\pi t/4})
= -2 \sin(\pi t/4) + 4 \cos(5\pi t/4)
\]

The steady state response of the output is

\[
y(t) = 4|H(j5\pi/4)| \cos(5\pi t/4 + \angle H(j5\pi/4))
= 4 \cos(5\pi t/4 - 5\pi/4)
\]

because the frequency component at \( \pi/4 \) is filtered out.

(b) The Fourier transforms are

\[
X(\Omega) = 2\pi [X_1 \delta(\Omega - \pi/4) + X_{-1} \delta(\Omega + \pi/4) + X_5 \delta(\Omega - 5\pi/4) + X_{-5} \delta(\Omega + 5\pi/4)]
= 2\pi [j \delta(\Omega - \pi/4) - j \delta(\Omega + \pi/4) + 2 \delta(\Omega - 5\pi/4) + 2 \delta(\Omega + 5\pi/4)]
\]

\[
Y(\Omega) = 4\pi e^{-j5\pi/4} \delta(\Omega - 5\pi/4) + 4\pi e^{j5\pi/4} \delta(\Omega + 5\pi/4)
\]
5.7 An analog averager is characterized by the following relationship

\[ \frac{dy(t)}{dt} = 0.5[x(t) - x(t - 2)] \]

where \( x(t) \) is the input and \( y(t) \) the output. If \( x(t) = u(t) - 2u(t - 1) + u(t - 2) \)

(a) Find the Fourier transform of the output \( Y(\Omega) \).

(b) Find \( y(t) \) from \( Y(\Omega) \).

**Answer:** \( y(t) = 0.5r(t) - r(t - 1) + r(t - 3) - 0.5r(t - 4) \).

**Solution**

(a) The FT of the equation characterizing the averager is

\[ j\Omega Y(\Omega) = 0.5[X(\Omega) - X(\Omega)e^{-j2\Omega}] \]

so that

\[ Y(\Omega) = X(\Omega) \frac{\sin(\Omega)e^{-j\Omega}}{\Omega} \]

where the FT of \( x(t) \) is

\[ X(\Omega) = \frac{e^{-j\Omega}}{j\Omega} [2(\cos(\Omega) - 1)] = -4 \frac{\sin^2(\Omega/2)}{j\Omega} e^{-j\Omega} \]

(b) Calling \( Z(\Omega) = X(\Omega)/(j\Omega) \) we have that \( y(t) = 0.5z(t) - 0.5z(t - 2) \) and

\[ Z(\Omega) = \frac{X(\Omega)}{j\Omega} = \left( \frac{\sin(\Omega/2)}{\Omega/2} \right)^2 e^{-j\Omega} \]

so that \( z(t) \) is the integral of \( x(t) = u(t) - 2u(t - 1) + u(t - 2) \), or \( z(t) = r(t) - 2r(t - 1) + r(t - 2) \), replacing it in the above expression for \( y(t) \) we get

\[
\begin{align*}
y(t) & = 0.5r(t) - r(t - 1) + 0.5r(t - 2) - 0.5r(t - 2) + r(t - 3) - 0.5r(t - 4) \\
& = 0.5r(t) - r(t - 1) + r(t - 3) - 0.5r(t - 4)
\end{align*}
\]
5.2 Problems using MATLAB

5.8 Fourier transform from Laplace transform of infinite support signals — For signals with infinite support, their Fourier transforms cannot be derived from the Laplace transform unless they are absolutely integrable or the region of convergence of the Laplace transform contains the \( j\Omega \) axis. Consider the signal \( x(t) = 2e^{-2|t|} \)

(a) Plot the signal \( x(t) \) for \(-\infty < t < \infty\).

(b) Use the evenness of the signal to find the integral

\[
\int_{-\infty}^{\infty} |x(t)| dt
\]

and determine whether this signal is absolutely integrable or not.

(c) Use the integral definition of the Fourier transform to find \( X(\Omega) \).

(d) Use the Laplace transform of \( x(t) \) to verify the above found Fourier transform.

(e) Use MATLAB’s symbolic function \texttt{fourier} to compute the Fourier transform of \( x(t) \). Plot the magnitude spectrum corresponding to \( x(t) \).

**Answers:** \( X(\Omega) = 8/(4 + \Omega^2) \).

**Solution**

(a) This is a decaying exponential for both negative and positive times.

(b) The signal is absolutely integrable,

\[
\int_{-\infty}^{\infty} |x(t)| dt = 2 \int_{0}^{\infty} 2e^{-2t} dt = 4 \frac{e^{-2t}}{-2} \bigg|_{t=0}^{t=\infty} = 2
\]

(c) Using the integral definition of the Fourier transform

\[
X(\Omega) = 2 \int_{-\infty}^{0} e^{2t} e^{-j\Omega t} dt + 2 \int_{0}^{\infty} e^{-2t} e^{-j\Omega t} dt
\]

\[
= \frac{2e^{(2-j\Omega)t}}{2 - j\Omega} \bigg|_{t=-\infty}^{t=0} - \frac{2e^{-(2+j\Omega)t}}{2 + j\Omega} \bigg|_{t=0}^{t=\infty}
\]

\[
= \frac{2}{2 - j\Omega} + \frac{2}{2 + j\Omega}
\]

\[
= \frac{4}{4 + \Omega^2}
\]

(d) The Laplace transform of \( X(s) \) is

\[
X(s) = \frac{2}{s+2} + \frac{2}{s+2} = \frac{8}{4 - s^2}
\]

with ROC: \(-2 \leq \sigma \leq 2\), which includes the \( j\Omega \)-axis. so that

\[
X(\Omega) = X(s) \big|_{s=j\Omega} = \frac{8}{4 + \Omega^2}
\]
%% Pr. 5_8
syms t w x
x=2*exp(-2*abs(t)); X=fourier(x)
figure(1)
subplot(211)
ezplot(x,[-2,2]); grid; axis([-2 2 0 2.2])
subplot(212)
ezplot(abs(X),[-30,30]); grid; axis([-20 20 0 2.2]);
xlabel('\Omega (rad/sec)'); ylabel('|X(\Omega)|')

Figure 5.2: Problem 8
5.9 Smoothness and frequency — Let the signal \( x(t) = r(t+1) - 2r(t) + r(t-1) \) and \( y(t) = d x(t)/d t \).

(a) Plot \( x(t) \) and \( y(t) \)

(b) Find \( X(\Omega) \) and carefully plot its magnitude spectrum. Is \( X(\Omega) \) real? Explain.

(c) Find \( Y(\Omega) \) (use properties of Fourier transform) and carefully plot its magnitude spectrum. Is \( Y(\Omega) \) real? Explain.

(d) Determine from the above spectra which of these two signals is smoother. Use MATLAB integration function \textit{int} to find the Fourier transforms. Plot \( 20 \log_{10} |Y(\Omega)| \) and \( 20 \log_{10} |X(\Omega)| \) and decide. Would you say in general that computing the derivative of a signal generates high frequencies, or possible discontinuities?

\textbf{Answers:} \( |Y(\Omega)| = |X(\Omega)||\Omega| \).

\textbf{Solution}

(a) The triangular pulse \( x(t) \) is even

\[ x(t) = r(t+1) - 2r(t) + r(t-1) \]

and its derivative is

\[ y(t) = u(t+1) - 2u(t) + u(t-1) \]

which is an odd signal composed of two pulses of duration 1.

(b) The Fourier transform of \( x(t) \) is computed from its Laplace transform

\[ X(s) = \frac{1}{s^2} |e^s - 2 + e^{-s}| = \frac{e^s}{s^2} [1 - 2e^{-s} + e^{-2s}] = \frac{e^s}{s^2} (1 - e^{-s})^2 \]

which cancels out the poles at \( s = 0 \) so that the region of convergence is the whole \( s \)-plane. By letting \( s = j\Omega \) we have that

\[ X(\Omega) = \frac{2}{\Omega^2} \left[ \cos(\Omega) - 1 \right] = \frac{\sin^2(\Omega/2)}{(\Omega/2)^2} \]

which is real given that \( x(t) \) is even.

(c) Because of the derivative

\[ Y(\Omega) = j\Omega X(\Omega) = j\Omega \frac{\sin^2(\Omega/2)}{(\Omega/2)^2} \]

which is imaginary given that \( y(t) \) is odd. The magnitude

\[ |Y(\Omega)| = |X(\Omega)||\Omega| \]

which is the product of the sinc square and the absolute value of \( \Omega \). Thus \( x(t) \) is smoother than its derivative \( y(t) \).
%% Pr. 5_9
syms x t w y X Y
x=(t+1)*(heaviside(t+1)-heaviside(t))+(1-t)*(heaviside(t)-heaviside(t-1));
y=heaviside(t+1)-2*heaviside(t)+heaviside(t-1);
figure(1)
subplot(211)
ezplot(x,[-1.5,1.5]);grid
subplot(212)
ezplot(y,[-1.5,1.5]);grid
X=int(x*exp(-j*w*t),t,-Inf,-Inf);
Y=int(y*exp(-j*w*t),t,-Inf,-Inf);
figure(2)
subplot(211)
XX=X*conj(X);
ezplot(10*log10(XX),[-100,100]);grid;axis([-100 100 -80 1]);ylabel('X(\Omega)')
subplot(212)
YY=Y*conj(Y);
ezplot(10*log10(YY),[-100,100]);grid;axis([-100 100 -80 1]);ylabel('Y(\Omega)')

Figure 5.3: Problem 9: signals and their magnitude spectra (in dB).
Chapter 6

Application of Laplace Analysis to Control

6.1 Basic Problems

6.1 Consider a series RC circuit with input a voltage source $v_i(t)$ and output the voltage across the capacitor $v_o(t)$.

(a) Draw a negative feedback system for the circuit using an integrator, a constant multiplier and an adder.

(b) Let the input be a battery, i.e., $v_i(t) = Au(t)$, find the steady state error $e(t) = v_i(t) - v_o(t)$.

Answer: $V_o(s)/V_i(s) = G(s)/(1 + G(s))$ with $G(s) = 1/(RCs)$.

Solution

(a) The transfer function of the RC circuit is

$$\frac{V_o(s)}{V_i(s)} = \frac{1/RC}{s + 1/(RC)}$$

An equivalent negative feedback with a feed–forward transfer function $G(s)$, and feedback transfer function $H(s) = 1$ gives

$$\frac{V_o(s)}{V_i(s)} = \frac{G(s)}{1 + G(s)}$$

Comparing the two we get that $G(s) = 1/(RCs)$. Considering an integrator has a transfer function $1/s$ then we have the equivalent representations for the RC circuit shown in Fig. 6.1.

(b) If $v_i(t) = Au(t)$, $V_i(s) = A/s$, then

$$V_o(s) = \frac{A/RC}{s(s + 1/(RC))} = \frac{B}{s} + \frac{D}{s + 1/RC}$$
Since $1/RC > 0$, i.e., the pole of $V_o(s)$ is in the left-hand $s$-plane, then in the steady state $v_o(t) = B$, where

$$B = V_o(s)|_{s=0} = \frac{A/RC}{1/(RC)} = A$$

so that in the steady-state the error signal $e(t) = v_i(t) - v_o(t)$ will be zero.
6.2 Suppose the feed–forward transfer function of a negative feedback system is \( G(s) = \frac{N(s)}{D(s)} \), and the feedback transfer function is unity. Let \( X(s) \) be the Laplace transform of the input \( x(t) \) of the feedback system.

(a) Given that the Laplace transform of the error is \( E(s) = X(s)[1 - F(s)] \) where \( F(s) = \frac{G(s)}{1 + G(s)} \) is the overall transfer function of the feedback system, find an expression for the error in terms of \( X(s), N(s) \) and \( D(s) \). Use this equation to determine the conditions under which the steady state error is zero for \( x(t) = u(t) \).

(b) If the input is \( x(t) = u(t) \), \( N(s) = 1 \), and \( D(s) = (s + 1)(s + 2) \) find an expression for \( E(s) \) and from it determine the initial value \( e(0) \) and the final value \( \lim_{t \to \infty} e(t) \) of the error.

**Answers:** \( X(s) = 1/s \), zero steady–state error: roots of \( N(s) + D(s) \) in left–hand s-plane, \( D(s) \) of the form \( sD_1(s) \).

**Solution**

(a) Replacing \( F(s) \) we have

\[
E(s) = X(s) \left[ 1 - \frac{G(s)}{1 + G(s)} \right] = X(s) \frac{1}{1 + \frac{N(s)}{D(s)}}
\]

\[
= \frac{X(s)D(s)}{D(s) + N(s)}
\]

If \( X(s) = 1/s \) then for the error to go to zero in the steady state, the roots of \( D(s) + N(s) \) should be in the left–hand s-plane and \( D(s) \) must cancel the pole of \( E(s) \) at zero contributed by \( X(s) \), i.e., \( D(s) \) must be of the form \( D(s) = sD_1(s) \).

(b) For \( G(s) = 1/(s + 1)(s + 2) \) and \( X(s) = 1/s \) the Laplace transform of the error is

\[
E(s) = \frac{D(s)}{s(1 + D(s))} = \frac{(s + 1)(s + 2)}{s(s^2 + 3s + 3)}
\]

According to the initial value theorem we have that

\[
e(0) = \lim_{s \to \infty} sE(s) = \lim_{s \to \infty} \frac{(1 + 1/s)(1 + 2/s)}{1 + 3/s + 3/s^2} = 1
\]

The poles of \( E(s) \) are \( s = 0 \) and \( s_{1,2} = -3/2 \pm j\sqrt{3}/4 \) since \( s^2 + 3s + 3 = (s + 3/2)^2 + 3/4 \). A partial fraction expansion for \( E(s) \) is

\[
E(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 3s + 3}
\]

where

\[
A = sE(s)|_{s=0} = \frac{D(0)}{1 + D(0)} = \frac{2}{3}
\]

in this case the steady state error is \( 2/3 \). The error is not zero in the steady–state because \( D(s) \) does not cancel the pole due to \( X(s) \).
6.3 Let \( H(s) = Y(s)/X(s) \) be the transfer function of the feedback system in Fig. 6.2. The impulse response of the plant (with transfer function \( H_p(s) \)) is \( h_p(t) = \sin(t)u(t) \).

(a) If we want the impulse response of the feedback system (with input \( x(t) \) and output \( y(t) \)) to be \( h(t) = h_p(t)e^{-t}u(t) \), what should be the transfer function \( H_c(s) \) of the controller?

(b) Find the unit-step response \( s(t) \) of the feedback system.

---

### Answers:

\[ H_c(s) = (s^2 + 1)/(s + 1)^2; \quad s(t) = [0.5 + 0.707 \cos(t + 3\pi/4)]u(t) \]

### Solution

(a) \( h(t) = h_p(t)e^{-t} = e^{-t}\sin(t)u(t) \) so

\[ H(s) = \frac{1}{(s + 1)^2 + 1} \]

For feedback system

\[ H(s) = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)} = \frac{H_c(s)}{s^2 + 1 + H_c(s)} \]

after replacing \( H_p(s) = 1/(s^2 + 1) \). Then

\[ \frac{H_c(s)}{s^2 + 1 + H_c(s)} = \frac{1}{(s + 1)^2 + 1} \Rightarrow H_c(s) = \frac{s^2 + 1}{(s + 1)^2} \]

(b) Unit-step response \( s(t) \)

\[ \begin{align*}
S(s) &= \frac{H(s)}{s} = \frac{1}{s(s + 1)^2 + 1} \\
&= \frac{A}{s} + \frac{B}{s - (-1 + j)} + \frac{B^*}{s - (-1 - j)}
\end{align*} \]

where

\[ A = sS(s)\big|_{s=0} = 0.5 \]

\[ B = \frac{1}{s(s + 1 + j)}\big|_{s=-1+j} = \frac{1}{2\sqrt{2}}e^{3\pi/4} \]

thus

\[ s(t) = [0.5 + 0.707 \cos(t + 3\pi/4)]u(t) \]
6.4 Consider the cascade connection of two continuous-time systems shown in Fig. 6.3 where

\[
\frac{dw(t)}{dt} + w(t) = x(t), \quad \frac{dy(t)}{dt} + 2y(t) = w(t)
\]

![Diagram of cascade connection](Image)

Figure 6.3: Problem 4.

(a) Determine the input/output differential equation for the overall cascade connection.

(b) Suppose that \(w(0) = 1\), \(y(0) = 0\) and \(x(t) = 0\) for \(t \geq 0\),

i. Compute \(w(t)\) for \(t \geq 0\).

ii. Compute then \(y(t)\) for \(t \geq 0\).

Answers: (a) \(d^2y(t)/dt^2 + 3dy(t)/dt + 2y(t) = x(t)\); (b) \(y(t) = (e^{-t} - e^{-2t})u(t)\)

Solution

(a) Laplace transforms

\[
(s + 1)W(s) = X(s) \\
(s + 2)Y(s) = W(s)
\]

replacing \(W(s)\) in the second equation gives

\[
(s + 2)Y(s) = \frac{X(s)}{s + 1} \Rightarrow Y(s) = \frac{1}{(s + 1)(s + 2)} \Rightarrow \frac{1}{s^2 + 3s + 2}
\]

so that

\[
\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)
\]

(b) i. For the first system, the Laplace transform of its differential equation is

\[
sW(s) - w(0) + W(s) = 0 \Rightarrow W(s) = \frac{1}{s + 1} \Rightarrow w(t) = e^{-t}u(t)
\]

ii. For the second system with input \(w(t) = e^{-t}u(t)\)

\[
sY(s) - y(0) + 2Y(s) = \frac{1}{s + 1} \Rightarrow Y(s) = \frac{1}{(s + 1)(s + 2)}
\]

so that

\[
Y(s) = \frac{1}{s + 1} - \frac{1}{s + 2} \Rightarrow y(t) = (e^{-t} - e^{-2t})u(t)
\]
6.5 Consider a second–order system with transfer function

\[ H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + s/Q + 1} \]

where \( Y(s) \) and \( X(s) \) are the Laplace transforms of output \( y(t) \) and the input \( x(t) \) of the system. \( Q \) is called the quality factor.

(a) If the feedback gain is unity, determine the feedforward transfer function \( G(s) \).

(b) Find the poles of \( H(s) \), and plot them expressing the poles as \( p_{1,2} = re^{\pm j\phi} \), where \( \phi \) is an angle measured with respect to the negative axis, and \( r \) is a positive radius, give the values of \( r \) and \( \phi \). Show that \( Q = 1/(2 \cos(\phi)) \).

(c) Consider the cases when \( Q = 0.5, Q = \sqrt{2}/2 \) and \( Q \rightarrow \infty \), and determine the corresponding poles. Find the impulse response \( h(t) \) for these cases. What happens as \( Q \) increases? Explain.

**Answers:**

\[ G(s) = \frac{1}{s(s + 1/Q)}; Q \rightarrow \infty \rightarrow H(s) = \frac{1}{s^2 + 1}. \]

**Solution:**

(a) Let \( G(s) = \frac{1}{s(s + 1/Q)} \) be the Laplace transform in the feedforward loop and with unit gain in the feedback then

\[ H(s) = \frac{G(s)}{1 + G(s)} = \frac{1/(s(s + 1/Q))}{1 + 1/(s(s + 1/Q))} = \frac{1}{s^2 + s/Q + 1} \]

(b) Writing \( s^2 + s/Q + 1 = (s + 1/(2Q))^2 + (1 - 1/(4Q^2)) \) the poles are

\[ p_{1,2} = -\frac{1}{2Q} \pm j\sqrt{1 - 1/(4Q^2)} \]

which have a magnitude of one and the angle \( \phi \) measured with respect to the negative real–axis is

\( \phi = \tan^{-1} \frac{\sqrt{1 - 1/(4Q^2)}}{1/2Q} = \tan^{-1}(\sqrt{4Q^2 - 1}) \)

The real part

\[ \frac{1}{2Q} = 1 \cos(\phi) \quad \Rightarrow \quad Q = \frac{1}{2 \cos(\phi)} \]

(c) When \( Q = 0.5, 0.707, \infty \) the angle \( \phi = 0, \pi/4, \pi/2 \). For

\[ H(s) = \frac{1}{(s + 1/(2Q))^2 + (1 - 1/(4Q^2))} \quad \text{when} \]

\( Q = 0.5 \)

\[ H(s) = \frac{1}{(s + 1)^2} \]

\( Q = 0.707 \)

\[ H(s) = \frac{1}{(s + 1/\sqrt{2})^2 + 1/2} \]

\( Q \rightarrow \infty \)

\[ H(s) = \frac{1}{s^2 + 1} \]
and the corresponding impulse responses are

\[ h(t) = te^{-t}u(t) \]
\[ h(t) = \frac{1}{\sqrt{2}} e^{-t/\sqrt{2}} \sin(t/\sqrt{2})u(t) \]
\[ h(t) = \sin(t)u(t) \]

As \( Q \) increases the impulse response becomes more oscillatory.
6.6 Given the two realizations in Fig. 6 obtain the corresponding transfer functions

\[ H_1(s) = \frac{Y_1(s)}{X_1(s)} \quad \text{and} \quad H_2(s) = \frac{Y_2(s)}{X_2(s)}. \]

**Answer:** \( H_1(s) = \frac{Y_1(s)}{X_1(s)} = \frac{(s - 2)}{(s^2 - s - 2)} \)

---

**Solution:** For the realization (1) we have the following equations

\[ y_1(t) = v_1(t) - 2v_2(t) \]
\[ \dot{v}_1(t) = x_1(t) + v_1(t) + 2v_2(t) \]
\[ \dot{v}_2(t) = v_1(t) \]

which in the Laplace transform are

1. \( Y_1(s) = V_1(s) - 2V_2(s) \)
2. \( V_1(s)(s - 1) = 2V_2(s) + X_1(s) \)
3. \( sV_2(s) = V_1(s) \)

Replacing \( V_2(s) = \frac{V_1(s)}{s} \) from equation (iii) in equation (ii) we have that

\[ V_1(s)(s - 1 - 2/s) = X_1(s) \]
so that \( V_1(s) = \frac{X_1(s)}{(s - 1)} \) and and using (iii) \( 2V_2(s) = 2V_1(s)/s = \frac{X_1(s)(2/s)}{(s - 1 - 2/s)} \) so that equation (i) becomes

\[
Y_1(s) = \frac{1 - 2/s}{s - 1 - 2/s} \frac{X_1(s)}{s}
\]

so that

\[
H_1(s) = \frac{Y_1(s)}{X_1(s)} = \frac{s - 2}{s^2 - s - 2}
\]

For the realization (2) we have the following equations

\[
y_2(t) = w_1(t)
\]

\[
\dot{w}_1(t) = x_2(t) + y_2(t) + w_2(t)
\]

\[
\dot{w}_2(t) = -2x_2(t) + 2y_2(t)
\]

or in the Laplace domain

\[
(i) \quad Y_2(s) = W_1(s)
\]

\[
(ii) \quad sW_1(s) = X_2(s) + Y_2(s) + W_2(s)
\]

\[
(iii) \quad sW_2(s) = -2X_2(s) + 2W_2(s)
\]

Replacing \( W_2(s) \) from equation (iii) into equation (ii) we get

\[
W_1(s) = \frac{(1/s - 2/s^2)X_2(s) + (1/s + 2/s^2)Y_2(s)}{}
\]

which replaced in equation (i) gives

\[
(1 - 1/s - 2/s^2)Y_2(s) = (1/s - 2/s^2)X_2(s)
\]

so that the system transfer function is

\[
H_2(s) = \frac{Y_2(s)}{X_2(s)} = \frac{s - 2}{s^2 - s - 2}
\]

Indicating that the two realizations are different realizations of the same system.
6.2 Problems using MATLAB

6.7 Diagonalization of state variables — Suppose you are given the observer space representation with matrix and vectors

\[ A_o = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}, \quad b_o = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad c_o^T = [ 1 \ 0 ] \]

To find a transformation that diagonalizes \( A_o \) use MATLAB function eigs which calculates the eigenvalues and eigenvectors corresponding to the matrix and allows us to express

\[ V^{-1}A_oV = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \]

where \( V \) is a matrix created with the eigenvectors and \( \{ \lambda_i \}, i = 1, 2 \), are the eigenvalues.

(a) Find the characteristic equation

\[ \text{det}(sI - A_o) \]

corresponding to the state variable representation, and show it is the same as the denominator of the transfer function (use function ss2tf to obtain the transfer function from the state variable representation).

(b) Use the matrix \( V \) as an invertible transform to obtain a new set of state variables with a diagonal matrix \( A \) and vectors \( b \) and \( c^T \). Obtain these matrix and vectors.

(c) Suppose that you find the controller form by letting \( A_c = A_o^T \), \( b_c = c_o^T \) and \( c_c^T = b_o^T \), repeat the above diagonalization and comment on your results.

Answers: \( H(s) = (s - 1)/((s - 1)(s + 2)) = 1/(s + 2) \)

Solution

(a)(b) To find the transfer function let the input \( v(t) = \delta(t), \ y(t) = h(t) \) and initial conditions equal zero, then in the Laplace domain

\[ H(s) = c_o^T(sI - A_o)^{-1}b = \frac{1}{s^2 + s - 2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} s + 1 \\ s + 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{s - 1}{s^2 + s - 2} \]

The denominator of \( H(s) \) is \( D(s) = s^2 + s - 2 = (s - 1)(s + 2) \) is the determinant of \( sI - A_o \), or the characteristic equation. Notice the pole-zero cancellation so that the simplified transfer function is \( H(s) = 1/(s + 2) \).

% Pr. 6.7
% Observer
Ao=[-1 1; 2 0]
b0=[1 ;-1]
c0=[1 0]'
% Transfer function
[num,den]=ss2tf(Ao,b0,c0',0)
roots(den)
% Diagonalization
[V,D]=eigs(Ao)
A = \( (\text{inv}(V)) \times Ao \times V \)
b = \( (\text{inv}(V)) \times bo \)
c = \( V' \times co \)
\[\text{[num, den]} = \text{ss2tf}(A, b, c', 0)\]
\[\text{roots}(\text{den})\]

\begin{align*}
Ao &= \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} \\
bo &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
co &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\text{num} &= \begin{bmatrix} 0 & 1.0000 & -1.0000 \end{bmatrix} \\
\text{den} &= \begin{bmatrix} 1 & 1 & -2 \end{bmatrix} \\
\text{ans} &= -2 \end{align*}

\begin{align*}
V &= \begin{bmatrix} -0.7071 & -0.4472 \\ 0.7071 & -0.8944 \end{bmatrix} \\
D &= \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \\
A &= \begin{bmatrix} -2.0000 & 0.0000 \\ 0 & 1.0000 \end{bmatrix} \\
b &= \begin{bmatrix} -1.4142 \\ 0 \end{bmatrix} \\
c &= \begin{bmatrix} -0.7071 \\ -0.4472 \end{bmatrix} \\
\text{num} &= \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \\
\text{den} &= \begin{bmatrix} 1 & 1 & -2 \end{bmatrix} \\
\text{ans} &= -2 \end{align*}

Showing that the two realizations correspond to the same transfer function.

(c)

%Controller
Ac=Ao’
bc=co’
cc=bo’
[V, D] = eigs(Ao)
A = \( (\text{inv}(V)) \times Ao \times V \)
b = \( (\text{inv}(V)) \times bo \)
c = \( V' \times co \)
\[\text{[num, den]} = \text{ss2tf}(A, b, c', 0)\]
\[\text{roots}(\text{den})\]

\begin{align*}
Ac &= \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} \\
bc &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
cc &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
V &= \begin{bmatrix} -0.7071 & -0.4472 \\ 0.7071 & -0.8944 \end{bmatrix} \\
D &= \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \\
\end{align*}
\[
A = \begin{bmatrix}
-2.0000 & 0.0000 \\
0 & 1.0000
\end{bmatrix}
\]
\[
b = \begin{bmatrix}
-1.4142 \\
0
\end{bmatrix}
\]
\[
c = \begin{bmatrix}
-0.7071 \\
-0.4472
\end{bmatrix}
\]
\[
\text{num} = \begin{bmatrix}
0 & 1 & -1
\end{bmatrix}
\]
\[
\text{den} = \begin{bmatrix}
1 & 1 & -2
\end{bmatrix}
\]
\[
\text{ans} = -2
\]

Again showing that the two realizations correspond to the same transfer function and that the observer and the controller forms represent the same system.
Chapter 7

Fourier Analysis in Communications and Filtering

7.1 Basic Problems

7.1 Consider a first–order system with transfer function

\[ H(s) = \frac{Ks + z_1}{s + p_1} \]

where \( K > 0 \) is a gain, \( z_1 \) is a zero and \( p_1 \) is a pole.

(a) If we want unity dc gain, i.e., \(|H(j0)| = 1\), what should be the value of \( K \).

(b) What types of filters (lowpass, bandpass, band-eliminating, all-pass and high-pass) can you obtain with this first-order system?

(c) Let \( H_1(s) = \frac{1}{s + 1} \), corresponding to a low-pass filter, determine the dc gain of this filter and use it to find the half-power frequency of the filter.

(d) Determine the magnitude response of a filter with transfer function \( H_2(s) = \frac{s - 1}{s + 1} \). Plot the poles and zeros of this filter. What type of filter is it?

(e) What type of filter has the transfer function \( H_3(s) = \frac{s}{s + 1} \) ?

Answers: Low-, high- and all-pass filters can be obtained with \( H(s) \); \( H_2(s) \) is all-pass filter.

Solution

(a) \(|H(j0)| = K|z_1|/|p_1| = 1\), so \( K = |p_1|/|z_1| \)

(b) Low-pass, high-pass and all-pass can be obtained with this first-order system. Band-pass and stop-band require second or higher-order systems, as poles must be complex conjugates.
(c) For the low-pass filter, \(|H_1(j0)| = 1\), the half-power frequency \(\Omega_{hp}\) is such that

\[
|H_1(j\Omega_{hp})|^2 = |H_1(j0)|^2 / 2 = 1 / 2
\]

\[
\frac{1}{\Omega_{hp}^2 + 1} = \frac{1}{2}
\]

so \(\Omega_{hp} = 1\).

(d) The filter with \(H_2(s) = (s - 1)/(s + 1)\) is an all-pass filter. Its pole is \(s = -1\) and zero \(s = 1\). Given this symmetry the magnitude response is unity for all values of \(\Omega\).

(e) The filter with \(H_3(s) = s/(s + 1)\) is a high-pass filter, \(|H(j0)| = 0\) and when \(\Omega \to \infty\) the magnitude response grows and becomes 1 at \(\infty\).
7.2 The loss at a frequency $\Omega = 2000$ (rad/sec) is $\alpha(2000) = 19.4$ dBs for a fifth-order lowpass Butterworth filter with unity dc gain. If we let $\alpha(\Omega_p) = \alpha_{max} = 0.35$ dBs, determine
- the half-power frequency $\Omega_{hp}$, and
- the passband frequency $\Omega_p$ of the filter.

**Answers:** $\Omega_{hp} = 1280.9$ rad/sec; $\Omega_p = 999.82$ (rad/sec)

**Solution**

Using the loss function

$$\alpha(\Omega) = 10 \log_{10} \left( 1 + \left(\frac{\Omega}{\Omega_{hp}}\right)^{2N} \right) \Rightarrow \Omega_{hp} = \frac{\Omega}{\left(10^{0.1\alpha(\Omega)} - 1\right)^{1/2N}}$$

which gives for $\Omega = 2000$

$$\Omega_{hp} = \frac{2000}{\left(10^{1.94} - 1\right)^{0.1}} = 1280.9 \text{ rad/sec.}$$

When $\alpha(\Omega_p) = \alpha_{max}$ we have

$$\Omega_p = \Omega_{hp}(10^{0.1\alpha_{max}} - 1)^{1/2N}$$

which gives after replacing the values on the right term $\Omega_p = 999.82$ (rad/sec).
7.2 Problems using MATLAB

7.3 Design of low-pass Butterworth/Chebyshev filters — The specifications for a low-pass filter are

\[
\alpha(0) = 20 \text{ dBs} \\
\Omega_p = 1500 \text{ rad/sec}, \quad \alpha_1 = 20.5 \text{ dBs} \\
\Omega_s = 3500 \text{ rad/sec}, \quad \alpha_2 = 50 \text{ dBs}
\]

(a) Determine the minimum order of the low-pass Butterworth and Chebyshev filters, and determine which is smaller.

(b) Give the transfer function of the designed low-pass Butterworth and Chebyshev filters (make sure the dc loss is as specified).

(c) Determine the half-power frequency of the designed filters by letting \(\alpha(\Omega_p) = \alpha_{max}\)

(d) Find the loss function values provided by the designed filters at \(\Omega_p\) and \(\Omega_s\). How are these values related to the \(\alpha_{max}\) and \(\alpha_{min}\) specifications? Explain. Which of the two filters provides more attenuation in the stop-band?

(e) If new specifications for the passband and stopband frequencies are \(\Omega_p = 750\) (rad/sec) and \(\Omega_s = 1750\) (rad/sec), are the minimum order

Answers: Minimum order: Butterworth \(N = 6\), \(\Omega_{hp} = 1.787 \times 10^3\).

Solution

Since the dc loss is not zero, the normalized loss specifications are

\[
\alpha_{max} = \alpha_1 - \alpha(0) = 0.5 \\
\alpha_{min} = \alpha_2 - \alpha(0) = 30
\]

with a dc loss of 20 dB. The following script is used to find the answers

```matlab
% Pr 7_3
clear all; clf
alphamax=0.5 ;alphamin=30; Wp=1500; Ws=3500;
alpha0=20;
% Butterworth
K=10ˆ(\alpha0/20)
D=(10ˆ(0.1*alphamin)-1)/(10ˆ(0.1*alphamax)-1);
E=Ws/Wp;
N=ceil(log10(D)/(2*log10(E)))
Whp=Wp/(10ˆ(\alpha0/20)-1)ˆ(1/(2*N))
alpha_p=10*log10(1+(Wp/Whp)ˆ(2*N))
alpha_s=10*log10(1+(Ws/Whp)ˆ(2*N))
% Chebyshev
N=ceil(acosh(Dˆ(0.5))/acosh(E))
eps=sqrt(10ˆ(0.1*alphamax)-1)
Whp1=Wp*cosh(acosh(1/eps)/N)
alpha_p=10*log10(1+(epsˆ2)*cos(N*acos(1)))ˆ2)
alpha_s=10*log10(1+(epsˆ2)*cosh(N*acosh(Ws/Whp))ˆ2)
alphal=alpha_p+10*log10(1+(epsˆ2)*cosh(N*acosh(0)))ˆ2);
Kc=10ˆ(\alpha0/20)
```
The following are the results

% Butterworth
K = 10 % dc gain
N = 6 % minimum order
Whp = 1.7874e+03 % half-power frequency
alpha_p = 0.5000 % loss at Wp
alpha_s = 35.0228 % loss at Ws
% Chebyshev
N = 4 % min order
eps = 0.3493 % ripple factor
Whp1 = 1.6397e+03 % half-power freq
alpha_p = 0.5000 % loss at Wp
alpha_s = 36.6472 % loss at Ws
Kc = 10.5925 % dc gain

Notice the computation of the dc gain in the Chebyshev filter. In this case the dc loss depends on the order of the filter and so it is not necessarily 0 dB, so to get the dc gain $K_c$ we use

$$\alpha(0) = 10 \log_{10} \left[ \frac{K^2}{1 + \epsilon^2 C_N^2(0)} \right] = 20 \log_{10} K - 10 \log_{10} (1 + \epsilon^2 C_N^2(0))$$

as indicated in the script.

(d) The minimum orders of the filters depend on the ratio of the two frequencies and since it remains the same these do not change.
7.4 Getting rid of 60 Hz hum with different filters — A desirable signal \( x(t) = \cos(100\pi t) - 2\cos(50\pi t) \) is recorded as \( y(t) = x(t) + \cos(120\pi t) \), i.e., as the desired signal but with a a 60 Hz hum. We would like to get rid of the hum and recover the desired signal. Use symbolic MATLAB to plot \( x(t) \) and \( y(t) \).

Consider the following three different alternatives (use symbolic MATLAB to implement the filtering and use any method to design the filters):

- Design a band-eliminating filter to get rid of the 60 Hz hum in the signal. Plot the output of the band-eliminating filter.
- Design a high-pass filter to get the hum signal and then subtract it from \( y(t) \). Plot the output of the high-pass filter.
- Design a band-pass filter to get rid of the hum. Plot the output of the band-pass filter.

Is any of these alternatives better than the others? Explain.

Solution
The following script is used to compute the different signals, filters, and to plot them.

```matlab
% Pr 7_4
clear all; clf
syms t w
x=cos(100*pi*t)-2*cos(50*pi*t);
y=x+cos(120*pi*t);
figure(1)
subplot(211)
ezplot(x,[0:1]) % desired signal
subplot(212)
ezplot(y,[0:1]) % signal with hum
X=fourier(x);
% butterworth filter
N=5;
%wn=[2*pi*59.9 2*pi*60.1]; %b,a=butter(N,wn,’stop’,’s’);C=1 % band-eliminating
% [b,a]=butter(2*N,110*pi,’high’,’s’);C=2 % high-pass
% [b,a]=butter(N,[20*pi 90*pi],’s’);C=2 % band-pass
% frequency responses
W=0:1:150*pi;
Hm=abs(freqs(b,a,W));
figure(2)
subplot(211)
plot(W/(2*pi),Hm); axis([0 75 0 1.1]);xlabel(’f (Hz)’); ylabel(’|H|’)
M=2*N
% generation of frequency response from coefficients
n=M:-1:0;
U=(j*w).^n;
num=b*conj(U’); den=a*conj(U’);
H=num/den; % Butterworth LPF
% output of filter
Yl=X*H;
yl=real(ifourier(Yl,t));
subplot(212)
if C==1,
```

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ezplot(y1, [0:1])
else
    ezplot(x-y1, [0:1])
end
title('denoised signal')

The most natural of the three approaches is the one using the stopband filter, the other depend on information on the desired signal that might or might not be available. The only needed information for the stopband filter is the frequency of the hum.

Figure 7.1: Problem 4: top: original and original with hum signals; bottom left to right: notch filter and filtered signal, high-pass filter and denoised signal, an bandpass filter and denoised signal.
Chapter 8

Sampling Theory

8.1 Basic Problems

8.1 Consider the sampling of a sinc signal and related signals.

(a) For the signal \( x(t) = \sin(t)/t \), find its magnitude spectrum \( |X(\Omega)| \) and determine if this signal is band-limited or not.

(b) What would be the sampling period \( T_s \) you would use for sampling \( x(t) \) without aliasing?

(c) For a signal \( y(t) = x^2(t) \) what sampling frequency \( f_s \) would you use to sample it without aliasing? How does this frequency relate to the sampling frequency used to sample \( x(t) \)?

(d) Find the sampling period \( T_s \) to sample \( x(t) \) so that the sampled signal \( x_s(0) = 1 \) otherwise \( x(nT_s) = 0 \) for \( n \neq 0 \).

Answers: \( x(t) \) band-limited with \( \Omega_{max} = 1 \) (rad/sec); for \( x^2(t) \), \( T_s \leq \pi/2 \).

Solution
(a) To find \( X(\Omega) \), we use duality or find the inverse Fourier transform of a pulse of amplitude \( A \) and bandwidth \( \Omega_0 \), that is

\[
X(\Omega) = A[u(\Omega + \Omega_0) - u(\Omega - \Omega_0)]
\]

so that

\[
x(t) = \frac{1}{2\pi} \int_{-\Omega_0}^{\Omega_0} A e^{j\Omega t} d\Omega = \frac{A}{2\pi} \frac{e^{j\Omega t}}{jt} \bigg|_{\Omega_0}^{\Omega_0} \]

\[
= \frac{A}{\pi t} \sin(\Omega_0 t)
\]

which when compared with the given \( x(t) = \sin(t)/t \) gives that \( A = \pi \) and \( \Omega_0 = 1 \) or

\[
X(\Omega) = \pi[u(\Omega + 1) - u(\Omega - 1)]
\]
indicating that $x(t)$ is band-limited with a maximum frequency $\Omega_{\text{max}} = 1$ (rad/sec).

(b) To sample without aliasing the sampling frequency should be chosen to be

$$f_s = \frac{1}{T_s} \geq 2\frac{\Omega_{\text{max}}}{2\pi}$$

which gives a sampling period

$$T_s \leq \frac{\pi}{\Omega_{\text{max}}} = \pi \text{ sec/sample}$$

(c) The spectrum of $y(t) = x^2(t)$ is the convolution in the frequency

$$Y(\Omega) = \frac{1}{2\pi} (X(\Omega) * X(\Omega))$$

which would have a maximum frequency $\Omega_{\text{max}} = 2$, giving a sampling frequency which is double the one for $x(t)$. The sampling period for $y(t)$ should be

$$T_s \leq \frac{\pi}{2}$$

(d) The signal $x(t) = \sin(t)/t$ is zero whenever $t = \pm k\pi$, for $k = 1, 2, \cdots$ so that choosing $T_s = \pi$ (the Nyquist sampling period) we obtain the desired signal $x_s(0) = 1$ and $x(nT_s) = 0$. 

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8.2 Consider the signal $x(t) = \delta(t+1) + \delta(t-1)$.

(a) Find its Fourier transform $X(\Omega)$. Determine if $x(t)$ is band–limited or not. If band–limited, give its maximum frequency.

(b) Filtering $x(t)$ with a filter of magnitude $H(j\Omega) = u(\Omega + 1) - u(\Omega - 1)$ and zero phase the output is $y(t)$. Is $y(t)$ band–limited? If so, what is its maximum frequency.

(c) Obtain an expression for $y(t)$, is it time–limited?

Answers: $X(\Omega) = 2\cos(\Omega)$; $y(t)$ is not time–limited.

Solution

(a) $X(\Omega) = e^{j\Omega} + e^{-j\Omega} = 2\cos(\Omega)$ which has no maximum frequency, so $x(t)$ is not bandlimited.

(b) $Y(\Omega) = X(\Omega)H(j\Omega) = 2\cos(\Omega)[u(\Omega + 1) - u(\Omega - 1)]$ with maximum frequency of 1 and so $y(t)$ is band-limited.

(c) It is not time limited, as

$$y(t) = \frac{1}{2\pi} \int_{-1}^{1} 2\cos(\Omega)e^{j\Omega}d\Omega = \frac{1}{2\pi} \int_{-1}^{1} [e^{j\Omega(t+1)} + e^{j\Omega(t-1)}]d\Omega$$

$$= \frac{\sin(t + 1)}{\pi(t + 1)} + \frac{\sin(t - 1)}{\pi(t - 1)}$$
8.3 Suppose you wish to sample an amplitude modulated signal \( x(t) = m(t) \cos(\Omega_c t) \) where \( m(t) \) is the message signal and \( \Omega_c = 2\pi \times 10^4 \) (rad/sec) is the carrier frequency.

(a) If the message is an acoustic signal with frequencies in a band of \([0, 22]\) kHz, what would be maximum frequency present in \( x(t) \)?

(b) Determine the range of possible values of the sampling period \( T_s \) that would allow to sample \( x(t) \) satisfying the Nyquist sampling rate condition.

(c) Given that \( x(t) \) is a bandpass signal, compare the above sampling period with the one that can be used to sample bandpass signals.

**Answers:** Nyquist: \( \Omega_s \geq 128\pi \times 10^3 \).

**Solution**

(a) The Fourier transform of \( x(t) \) is

\[
X(\Omega) = 0.5M(\Omega - \Omega_c) + 0.5M(\Omega + \Omega_c)
\]

where \( M(\Omega) \) is the Fourier transform of \( m(t) \). The maximum frequency present in \( x(t) \) is

\[
\Omega_{max} = \Omega_c + 2\pi \times 22 \times 10^3 = 2\pi(10 + 22)10^3 = 64\pi \times 10^3
\]

(b) The sampling frequency is

\[
\Omega_s = \frac{2\pi}{T_s} \geq 2 \times 64\pi \times 10^3 = 128\pi \times 10^3
\]

so that

\[
T_s \leq \frac{1}{64} \times 10^{-3} \text{ sec/sample}
\]

(c) The bandwidth of the message is \( B = 2\pi \times 22 \times 10^3 = 44\pi \times 10^3 \) rad/sec, using this frequency \( x(t) \) can be sampled at \( 2B = 88\pi \times 10^3 \) rad/sec which is much smaller than the one found above.
8.4 The input/output relation of a non-linear system is \( y(t) = x^2(t) \), where \( x(t) \) is the input and \( y(t) \) is the output.

(a) The signal \( x(t) \) is band-limited with a maximum frequency \( \Omega_M = 2000\pi \text{ (rad/sec)} \), determine if \( y(t) \) is also band-limited and if so what is its maximum frequency \( \Omega_{\text{max}} \).

(b) Suppose that the signal \( y(t) \) is low-pass filtered. The magnitude of the low-pass filter is unity and the cut-off frequency is \( \Omega_c = 5000\pi \text{ (rad/sec)} \), determine the value of the sampling period \( T_s \) according to the given information.

(c) Is there a different value for \( T_s \) that would satisfy the Nyquist sampling rate condition for sampling both \( x(t) \) and \( y(t) \) and that is larger than the one obtained above? Explain.

Answers: \( y(t) \) is band-limited; \( T_s \leq 0.25 \times 10^{-3} \text{ sec/sample} \).

Solution
(a) If the signal \( x(t) \) is band-limited \( y(t) = x^2(t) \) has a Fourier transform \( Y(\Omega) = (1/2\pi)(X(\Omega) * X(\Omega)) \) having a bandwidth double that of \( x(t) \), or

\[
\Omega_{\text{max}} y = 2\Omega_M = 4000\pi
\]

so \( y(t) \) is band-limited.

(b) Filtering with a low-pass filter of cut-off frequency \( 5000\pi \) would not change the maximum frequency of \( Y(\Omega) \) so that

\[
T_s \leq \frac{\pi}{4000\pi} = \frac{1}{4000} = 0.25 \times 10^{-3}
\]

(c) No. For \( x(t) \),

\[
T_{s1} \leq \frac{\pi}{2000\pi} = 0.5 \times 10^{-3}
\]

and if \( T_s = 0.25 \times 10^{-3} \) then \( T_{s1} \leq 2T_s \), so that we need to use \( T_s \) to sample both \( x(t) \) and \( y(t) \).
8.5 A sinusoid \( x(t) = \cos(t) \) is a band-limited signal with maximum frequency \( \Omega_{max} = 1 \)

(a) Using Fourier transform properties determine the maximum frequency of \( x^2(t) \). What sampling period \( T_s \) can be used to sample \( x^2(t) \) without aliasing. Verify your results using trigonometric identities.

(b) Can you generalize the above results to \( x^N(t) \), for integer \( N > 1 \)? Is \( x^N(t) \) band-limited for all the given values of \( N \)? Consider the case \( N = 3 \) and use trigonometric identities to verify the results.

**Answers:** (a) \( T_s \leq \pi/2 \).

**Solution**

(a) If \( \mathcal{F}(\cos(t)) = X(\Omega) \), then \( \mathcal{F}(x^2(t)) = (X * X)(\Omega)/(2\pi) \), i.e., the convolution of \( X(\Omega) \) with itself, having twice the bandwidth of \( X(\Omega) \), so the maximum frequency of \( x(t) \) is \( \Omega_{max} = 2 \), and \( T_s \leq \pi/2 \). Thus \( x(t) \) is band-limited with twice the maximum frequency of \( \cos(t) \).

Using \( \cos^2(t) = 0.5(1 + \cos(2t)) \) we see that its maximum frequency is 2 and it is band-limited and \( T_s \leq \pi/2 \).

(b) In general \( \mathcal{F}(x^N(t)) \) would be \( N \) convolutions of \( X(\Omega) \) with itself, and the bandwidth being \( N \times \) bandwidth of \( \cos(t) \), i.e., \( N \), so the maximum frequency is \( \Omega_{max} = N \) and \( T_s \leq \pi/N \).

If \( N = 3 \), \( \cos^3(t) = 0.25(3\cos(t) + \cos(3t)) \) with \( \Omega_{max} = 3 \), and \( T_s \leq \pi/3 \).
8.6 Let \( x(t) = \cos(\pi t)[u(t) - u(t - 2)] \) be the input of a zero-order hold sampler. The sampler samples every \( T_s \) sec. starting at \( t = 0 \), and the impulse response of the zero-order hold is \( h(t) = u(t) - u(t - T_s) \). The quantizer and coder shown in Fig. 7.12 in the text is used to generate a digital signal.

(a) Let \( T_s = 0.5 \) sec/sample, find the values of the sampled signal using the ideal sampler. Determine the corresponding digital signal.

(b) Suppose now that we let \( T_s = 0.25 \) sec/sample. Carefully plot the sampled signal using the ideal sampler and then plot the output \( y(t) \) of the zero-order hold system. Determine the corresponding digital signal.

(c) For the above two cases find and plot the quantization error \( \epsilon(nT_s) = x(nT_s) - \hat{x}(nT_s) \), where \( \hat{x}(nT_s) \) is the output of the quantizer.

**Answer:** The binary code is \{01 00 10 00 01 00 \ldots\}.

**Solution**

(a) The sample values for \( T_s = 0.5 \) are

\[
\begin{array}{c|c}
 nT_s & x(nT_s) \\
\hline
 0 & \cos(0) = 1 \\
 0.5 & \cos(\pi/2) = 0 \\
 1.0 & \cos(\pi) = -1 \\
 1.5 & \cos(3\pi/2) = 0 \\
 2.0 & \cos(2\pi) = 1 \\
\end{array}
\]

\( x(nT_s) = 0 \) for \( nT_s > 2 \).

The binary code for \( nT_s = 0, 0.5, 1, 1.5, 2, 2.5 \ldots \) is

\[01 00 10 00 01 00 \ldots\]

corresponding to the quantized signal

\[ \hat{x}(nT_s) = \Delta \delta(nT_s) + 0\delta((n - 1)T_s) - 2\Delta \delta((n - 2)T_s) + 0\delta((n - 3)T_s) + \Delta \delta((n - 4)T_s) \]

where \( \Delta = 2/4 = 0.5 \), so that the quantization error is

\[
\epsilon(nT_s) = x(nT_s) - \hat{x}(nT_s) = 0.5\delta(nT_s) + 0\delta((n - 1)T_s) + 0\delta((n - 2)T_s) + 0\delta((n - 3)T_s) + 0.5\delta((n - 4)T_s)
\]

(b) The sample values for \( T_s = 0.25 \) are

\[
\begin{array}{c|c}
 nT_s & x(nT_s) \\
\hline
 0 & \cos(0) = 1 \\
 0.25 & \cos(\pi/4) = 0.707 \\
 0.5 & \cos(\pi/2) = 0 \\
 0.75 & \cos(3\pi/4) = -0.707 \\
 1.0 & \cos(\pi) = -1 \\
 1.25 & \cos(5\pi/4) = -0.707 \\
 1.5 & \cos(3\pi/2) = 0 \\
 1.75 & \cos(7\pi/4) = 0.707 \\
 2.0 & \cos(2\pi) = 1 \\
\end{array}
\]
The binary code for $nT_s = 0, 0.25, \ldots, 1.75, 2, 2.25, \ldots$

\[
01 \ 01 \ 00 \ 11 \ 11 \ 00 \ 01 \ 01 \ 00, \ \cdots
\]
corresponding to the quantized signal

\[
\hat{x}(nT_s) = \Delta \delta((n - 1)T_s) + \delta((n - 1)T_s) - \Delta \delta((n - 3)T_s) \\
-2\Delta \delta((n - 4)T_s) - \Delta \delta((n - 5)T_s) - \delta((n - 6)T_s) + \Delta \delta((n - 7)T_s) \\
+ \Delta \delta((n - 7)T_s)
\]

where as before $\Delta = 0.5$, so that the quantization error is

\[
\varepsilon(nT_s) = 0.5\delta((n - 1)T_s) + 0.207\delta((n - 1)T_s) + 0\delta((n - 2)T_s) - 0.207\delta((n - 3)T_s) \\
+ 0\delta((n - 4)T_s) - 0.207\delta((n - 5)T_s) + 0\delta((n - 6)T_s) + 0.207\delta((n - 7)T_s) \\
+ 0.5\delta((n - 8)T_s)
\]
8.7 Suppose \( x(t) \) has a Fourier transform \( X(\Omega) = u(\Omega + 1) - u(\Omega - 1) \)

(a) Determine possible values of the sampling frequency \( \Omega_s \) so that \( x(t) \) is sampled without aliasing.

(b) Let \( y(t) = x^2(t) \), find the Fourier transform \( Y(\Omega) \), indicating its maximum frequency. Is \( y(t) \) band-limited?

(c) If \( z(t) = x(t) + y(t) \), what is the maximum frequency of \( z(t) \)? Find values for the sampling period \( T_s \) that can be used to sample \( z(t) \) without aliasing.

**Answers:** For \( x(t) \): \( \Omega_s > 2\Omega_{\text{max}} = 2 \); \( y(t) \) is band-limited.

**Solution**

(a) For \( x(t) \) to be sampled without aliasing we need \( \Omega_s > 2\Omega_{\text{max}} = 2 \).

(b) If \( y(t) = x^2(t) \) then \( Y(\Omega) = |X * X|/(2\pi) \) which gives a triangular spectrum in \([-2 \ 2]\) frequency band, i.e., it is band-limited.

(c) The maximum frequency of \( z(t) \) is \( \Omega_{\text{max}} = 2 \) (rad/sec) so \( \Omega_s \geq 4 \) (rad/sec) and \( T_s \leq \pi/2 \) sec.
8.2 Problems using MATLAB

8.8 Uncertainty in time and frequency — Signals of finite time support have infinite support in the frequency domain, and a band-limited signal has infinite time support. A signal cannot have finite support in both domains.

(a) Consider the signals \( x(t) = e^{-t^2} \) (the Gaussian function) and \( x_1(t) = u(t+1) - u(t-1) \). Use MATLAB to find their Fourier transform \( X(\Omega) \), and \( X_1(\Omega) \), and to compute the energy of the signals in \( \Omega \in [-4, 4] \).

(b) The fact that a signal cannot be of finite support in both domains is expressed well by the uncertainty principle which says that

\[
2\Delta(t)\Delta(\Omega) \geq 1
\]

where

\[
\Delta(t) = \left( \frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 \, dt}{E_x} \right)^{0.5}
\]

measures the duration of the signal for which the signal is significant in time, and

\[
\Delta(\Omega) = \left( \frac{\int_{-\infty}^{\infty} \Omega^2 |X(\Omega)|^2 \, d\Omega}{2\pi E_x} \right)^{0.5}
\]

measures the frequency support for which the Fourier representation is significant. The energy of the signal is represented by \( E_x \). Use MATLAB to compute \( \Delta(t) \) and \( \Delta(\Omega) \) for the signals \( x(t) \) and \( x_1(t) \) and verify that the uncertainty principle is satisfied. Comment on the difference in results for the two signals; pay attention to the smoothness and the frequency content of the signals.

Solution
(a) For the Gaussian signal we notice that the signal and its spectrum look very similar, and that the spectrum is narrower that the signal. For \( x_1(t) \), we notice that it has discontinuities and its spectrum has high frequencies because of them. See Figs. 8.1 and 8.2.

(b) Computing \( \Delta(t) \) and \( \Delta(\Omega) \) we verify the uncertainty bound. It is 1 for the Gaussian signal (the only signal that will give this value) and 1.2197 for the square pulse \( x_1(t) \). You need to make some simple changes to the script for \( x_1(t) \), the script shown is for the Gaussian signal.

```matlab
% Pr. 8.8

clear all; clf
syms x t X X1 w z
sigma=1
%x=heaviside(t+sigma)-heaviside(t-sigma);
x=exp(-t^2/(2*sigma^2));
X=fourier(x,w);
XX=X*conj(X);

figure(1)
subplot(211)
ezplot(x,[-4*sigma,4*sigma]);grid;
subplot(212)
```

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ezplot(XX, [-4*sigma, 4*sigma]); grid;

Ex = int(x^2, t, -4*sigma, 4*sigma)

figure(2)
subplot(211)
 ezplot((x^2)*(t^2)/Ex, [-4*sigma, 4*sigma]); grid;
 subplot(212)
 ezplot(XX*(w^2)/Ex, [-4*sigma, 4*sigma]); grid;

z = int((x^2)*(t^2), t, -4*sigma, 4*sigma)/Ex
zz = subs(sqrt(z)) % Delta(t)

v = int((X^2)*(w^2)/(2*pi), w, -4*sigma, 4*sigma)/Ex
vv = subs(sqrt(v)) % Delta(W)

2*zz*vv

Figure 8.1: Problem 8: Gaussian signal \( x(t) \) and its spectrum (both are real) (left); integrands used to calculate \( \Delta(t) \) and \( \Delta(\Omega) \) (right)

Figure 8.2: Problem 8: Rectangular pulse \( x_1(t) \) and its spectrum (left); integrands used to calculate \( \Delta(t) \) and \( \Delta(\Omega) \) (right)
Chapter 9

Discrete–time Signals and Systems

9.1 Basic Problems

9.1 For a finite-support signal \( x[n] = r[n](u[n] - u[n - 11]) \) where \( r[n] \) is the discrete-time ramp function,

(a) Find the energy of \( x[n] \).

(b) Find the even \( x_e[n] \) and the odd \( x_o[n] \) components of \( x[n] \).

(c) Show that the energy of \( x[n] \) the sum of the energies of \( x_e[n] \) and \( x_o[n] \).

Answers: \( \varepsilon_x = 385 \).

Solution

(a) Energy of \( x[n] \)

\[
\varepsilon_x = \sum_{n=0}^{10} n^2 = \frac{10 \times 11 \times 21}{6} = 385
\]

Using the formula

\[
\sum_{k=1}^{N} k^2 = \frac{N(N + 1)(2N + 1)}{6}
\]

(b) Calling \( p[n] = u[n] - u[n - 11] \) so \( x[n] = r[n]p[n] \) we have

\[
x_e[n] = 0.5(r[n]p[n] + r[-n]p[-n])
\]

\[
x_o[n] = 0.5(r[n]p[n] - r[-n]p[-n])
\]
The energy

$$\varepsilon_x = \sum_{n=-\infty}^{\infty} (x_e[n] + x_o[n])^2$$

$$= \sum_{n=-\infty}^{\infty} x_e^2[n] + x_o^2[n] + 2x_e[n]x_o[n]$$

$$= \sum_{n=-\infty}^{\infty} x_e^2[n] + \sum_{n=-\infty}^{\infty} x_o^2[n]$$

since $x_e[n]x_o[n]$ is odd its sum is zero. Also from plots of $x_e[n]$ and $x_o[n]$ we have that

$$\varepsilon_x = 4 \sum_{n=0}^{10} \frac{n^2}{4} = \sum_{n=0}^{10} n^2 = 385$$
9.2 A two-bit quantizer has as input \( x(nT_s) \) and as output \( \hat{x}(nT_s) \), or

\[
k\Delta \leq x(nT_s) < (k + 1)\Delta \quad \rightarrow \quad \hat{x}(nT_s) = k\Delta, \quad k = -2, -1, 0, 1
\]

(a) Is this system time-invariant? Explain.

(b) Suppose that the value of \( \Delta \) in the quantizer is 0.25, and the sampled signal is \( x(nT_s) = nT_s, T_s = 0.1 \) and \(-5 \leq n \leq 5\). Use the sampled signal to determine whether the quantizer is a linear system or not. Explain.

(c) From the results in this problem and the sampling theory, would you say that the A/D converter is a LTI system? Explain.

**Answers:** Quantizer is non-linear, time-invariant; sampler is linear time-varying.

**Solution**

(a) Yes, the quantizer is time-invariant. Indeed, \( x(nT_s) \) and \( x(nT_s - MT_s) \) will give the same values just shifted in time. In this case the \( M \) must be an integer as the sampled signal is discrete in time.

(b) The samples are given by \( x(nT_s) = nT_s = 0.1n \). For \( \Delta = 0.25 \) the sample \( x[1] = 0.1 \) when quantized gives \( \hat{x}[1] = 0 \) since \( 0 < 0.1 < \Delta \). If we multiply the signal by 3 so that the value \( 3x[1] = 0.3 \) would give \( \hat{x}[1] = 1 \) since \( \Delta < 0.3 < 2\Delta \). Since the second output is not the first output multiplied by 3, the quantizer is not linear.

We saw before that the sampler is linear but time-varying, while the quantizer is time-invariant but non-linear, so the A/D converter which is composed of these systems is not LTI.
9.3 A continuous-time system is characterized by the ordinary differential equation

\[ \frac{dy(t)}{dt} + y(t) = 2x(t) + \frac{dx(t)}{dt} \]

This equation is discretized by approximating the derivatives for a signal ρ(t) as

\[ \frac{d\rho(t)}{dt} \approx \frac{\rho(nT + T) - \rho(nT)}{T} \]

around \( t = nT \), and for a small value of \( T \). Obtain a block diagram that represents the obtained difference equation from the differential equation, with input \( x(nT) \) and output \( y(nT) \) and \( T = 1 \).

**Answers:** \( y(nT) = (1 - T)y((n - 1)T) + (2T - 1)x((n - 1)T) + x(nT) \).

**Solution**

Discretization of differential equation:

\[ \frac{y(nT + T) - y(nT)}{T} + y(nT) = 2x(nT) + \frac{x(nT + T) - x(nT)}{T} \]

\[ \Rightarrow \quad y((n + 1)T) = (1 - T)y(nT) + (2T - 1)x(nT) + x((n + 1)T) \]

and letting \( m = n + 1 \) we get

\[ y(mT) = (1 - T)y((m - 1)T) + (2T - 1)x((m - 1)T) + x(mT) \]

Figure 9.1 shows the block diagram when \( T = 1 \).

![Block diagram](image)

**Figure 9.1:** Problem 3: block diagram of difference equation when \( T = 1 \).
9.4 An LTI discrete-time system has the impulse response \( h[n] = (-1)^n u[n] \). Use the convolution sum to compute the output response \( y[n] \), \( n \geq 0 \) when the input is \( x[n] = u[n] - u[n-3] \) and the initial conditions are zero. In particular, find the following values of the output \( y[n] \), \( 0 \leq n \leq 4 \).

**Answers:** \( y[0] = 1 \), \( y[1] = 0 \), \( y[2] = 1 \), \( y[3] = -1 \), \( y[4] = 1 \).

**Solution**

The convolution sum is

\[
y[n] = \sum_{m=0}^{n} x[m] h[n-m]
\]

\[
y[0] = x[0]h[0] = 1
\]

\[
y[1] = x[0]h[1] + x[1]h[0] = -1 + 1 = 0
\]

\[
\]

\[
\]

\[
\]

\[
y[n] = x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] = (-1)^n + (-1)^{n-1} + (-1)^{n-2} = (-1)^{n-2} \quad n \geq 5
\]
9.5 An LTI system represented by the difference equation $y[n] = -y[n-1] + x[n]$, $n \geq 0$, is initially at rest. The input of the system is $x[n]$ and the output $y[n]$.

(a) Using the difference equation find the system response $y[n]$ for $n \geq 0$ when the input is

$$x[n] = \begin{cases} 1 & n = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $y[n]$ for $n = -1, 0, 1 \ldots, 10$

(b) Using the above result and the linearity and time-invariance of the system find the output $y_1[n]$ corresponding to a new input $x_1[n]$ which is

$$x_1[n] = \begin{cases} 1 & n = 0, 1, 2, 3, 4, 5 \\ 0 & \text{otherwise} \end{cases}$$

\textbf{Answers:} $y[n] = 0$ for $n < 0$, $y[0] = 1$, $y[1] = 0$, $y[n] = (-1)^{n}$ for $n \geq 2$; $y_1[n] = y[n] + y[n - 3]$.

\textbf{Solution}

(a) According to the difference equation the output is

\[
\begin{align*}
y[0] &= -y[-1] + x[0] = 0 + 1 = 1 \\
y[1] &= -y[0] + x[1] = -1 + 1 = 0 \\
& \vdots
\end{align*}
\]

or

\[
\begin{align*}
y[n] &= 0 & n < 0 \\
y[0] &= 1 \\
y[1] &= 0 \\
y[n] &= (-1)^{n} & n \geq 2
\end{align*}
\]

(b) By linearity and time-invariance since $x_1[n] = x[n] + x[n - 3]$ then

\[
y_1[n] = y[n] + y[n - 3] = \begin{cases} y[n] & n \leq 2 \\ 0 & n = 3 \\ 1 & n = 4 \\ 0 & n \geq 5 \end{cases}
\]
9.6 The output of a discrete-time system is \( y[n] = w[n]x[n] \) where \( x[n] \) is the input, and \( w[n] = u[n] - u[n-5] \) is a rectangular window.

(a) The input is \( x[n] = 4 \sin(\pi n/2) \), \( -\infty < n < \infty \), determine if \( x[n] \) is periodic, and if so indicate its fundamental period \( N_0 \).

(b) Let \( z[n] = w[n]x[n-N] \) for what values of \( N \) is \( y[n] = z[n] \)? According to this, is the system time-invariant? Explain.

(c) Suppose that we let \( x_1[n] = x[n]u[n] \), i.e., \( x_1[n] \) is the causal component of \( x[n] \). Determine the outputs due to \( x_1[n] \) and to \( x_1[n-4] \). Again according to this, is the system time-invariant? Explain.

Answers: \( x[n] \) is periodic of fundamental period \( N_0 = 4 \); the system is time-varying.

Solution

(a) The discrete frequency of \( x[n] \) is \( \omega_0 = \pi/2 \) which can be written as \( 2\pi m/N_0 \) with \( m = 1 \), \( N_0 = 4 \). So \( x[n] \) is periodic of fundamental period \( N_0 = 4 \).

(b) Since \( x[n-kN_0] = x[n] \) for any integer \( k \) and the fundamental period \( N_0 = 4 \), then taking \( N = kN_0 \) we will have that

\[
z[n] = x[n - kN_0]w[n] = x[n]w[n] = y[n]
\]

so that delaying the input by a multiple of \( N_0 \) does not change the output. The output in this case is just the period of \( x[n] \) between 0 and 4. However, that is not so if the delay is different from a multiple of the fundamental period, thus the system is time-varying.

(c) The output corresponding to \( x_1[n] = x[n]u[n] \) is

\[
y_1[n] = x_1[n]w[n] = x[n]u[n]w[n] = x[n]w[n] = y[n]
\]

obtained before. However, the output corresponding to \( x_1[n-4] \) is

\[
\]

which is not the previous result shifted, so that the system is time-varying.
9.7 Express the signal $u_{12}[m, n]$ with support in the first and second quadrant in terms of one-dimensional unit-step functions and determine if it is separable.

**Solution**

We have

$$u_{12}[m, n] = \begin{cases} 
1 & -\infty < m < \infty, \ n \geq 0 \\
0 & \text{otherwise} 
\end{cases}$$

$$= u_1[m, n] + u_2[m, n] - u_1[0, n] = u[m]u[n] + u[-m]u[n] - u[n]$$

$$= (u[m] + u[-m] - 1)u[n]$$

i.e., it is separable.
9.8 Although few signals encountered in practice are separable, any signal \( x[m, n] \), \( 0 \leq m \leq M - 1 \), \( 0 \leq n \leq N - 1 \) and zero otherwise, can be written as the sum of finite number of separable sequences

\[
x[m, n] = \sum_{i=0}^{N-1} x_{i1}[m] x_{i2}[n].
\]

Choose \( x_{i1}[m] \) and \( x_{i2}[n] \) to obtain this representation and show it can be expressed in terms of \( \delta[m, n] \).

**Solution**

A simple way to obtain such a representation is by letting the signal be expressed as the sum of isolated columns. This is done by choosing for \( 0 \leq i \leq N - 1 \)

\[
x_{i1}[m] = x[m, i] \\
x_{i2}[n] = \delta[n - i]
\]

so that

\[
x[m, n] = \sum_{i=0}^{N-1} x[m, i] \delta[n - i]
\]

As functions of \( m \), the signal \( x[m, i] \) can be expressed as

\[
x[m, i] = \sum_{k=0}^{M-1} x[k, i] \delta[m - k]
\]

and thus

\[
x[m, n] = \sum_{i=1}^{N-1} x[m, i] \delta[n - i] = \sum_{i=0}^{N-1} \sum_{k=0}^{M-1} x[k, i] \delta[m - k] \delta[n - i]
\]

which is the representation sought.
9.9 Suppose both the input $x[m, n]$ and the impulse response $h[m, n]$ of a LSI system are separable.

(a) Obtain a simpler expression for the output $y[m, n]$ of the system.

(b) Suppose $x[m, n] = (u[m] - u[m - 2])(u[n] - u[n - 2])$ and $h[m, n] = (u[m] - u[m - 2])(u[n] - u[n - 2])$ where $u[n]$ is the one-dimensional unit-step signal. Find $y[m, n]$

Solution

(a) Letting $x[m, n] = x_1[m]x_2[n]$ and $h[m, n] = h_1[m]h_2[n]$ we have that

$$y[m, n] = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} x_1[k]x_2[\ell]h_1[m-k]h_2[n-\ell]$$

$$= \sum_{k=-\infty}^{\infty} x_1[k]h_1[m-k] \sum_{\ell=-\infty}^{\infty} x_2[\ell]h_2[n-\ell]$$

$$= (x_1 * h_1)[m] (x_2 * h_2)[n]$$

i.e., the product of two one-dimensional convolutions.

(b) Letting $x_1[m] = h_1[m] = u[m] - u[m - 2]$ we have that their convolution

$$(x_1 * h_1)[m] = \delta[m] + 2\delta[m - 1] + \delta[m - 2]$$

and also letting $x_2[n] = h_2[n] = u[n] - u[n - 2]$ we have

$$(x_2 * h_1)[n] = \delta[n] + 2\delta[n - 1] + \delta[n - 2]$$

and so the response is

$$y[m, n] = (\delta[m] + 2\delta[m - 1] + \delta[m - 2])(\delta[n] + 2\delta[n - 1] + \delta[n - 2])$$

$$= (\delta[m, n] + 2\delta[m, n - 1] + \delta[m, n - 2])$$

$$+ (2\delta[m - 1, n] + 4\delta[m - 1, n - 1] + 2\delta[m - 1, n - 2])$$

$$+ (\delta[m - 2, n] + 2\delta[m - 2, n - 1] + \delta[m - 2, n - 2])$$
9.2 Problems using MATLAB

9.10 Discrete sequence — Consider the formula

\[ x[n] = x[n-1] + x[n-3] \quad n \geq 3 \]

\[
\begin{align*}
x(0) &= 0 \\
x(1) &= 1 \\
x(2) &= 2 \\
\end{align*}
\]

Find the rest of the sequence for \(0 \leq n \leq 5\) and plot it using function `stem`.


**Solution**

The equation

\[ x[n] = x[n-1] + x[n-3] \quad n \geq 3 \]

with initial conditions \(x[0] = 0, x[1] = 1, x[2] = 2\) can be solved recursively for \(n \geq 3\) as follows

\[
\begin{align*}
& & & \vdots \ \\
\end{align*}
\]

The following script obtains the recursive solution for \(0 \leq n \leq 49\)

```matlab
% Pr 9_10
clear all; clf
x(1)=0; x(2)=1; x(3)=2; % initial conditions
for m=4:50,
x(m)=x(m-1)+x(m-3);
end
n=0:49;
figure(1)
stem(n,x); title('x[n]'); xlabel('n'); ylabel('x[n]'); grid
```

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Figure 9.2: Problem 10.
9.11 Generation of periodic discrete-time signal — Periodic signals can be generated by obtaining a period and adding shifted versions of this period. Suppose we wish to generate a train of triangular pulses. A period of the signal is \( x[n] = 0.5(r[n] - 2r[n-2] + r[n-4]) \) where \( r[n] \) is the discrete-time ramp signal. Obtain the values of \( x[n] \) for \( 0 \leq n \leq 4 \) and write a script to generate \( x[n] \) and the periodic signal \( y[n] \)

\[
y[n] = \sum_{k=-\infty}^{\infty} x[n - 4k]
\]

and plot 4 of the periods.

**Answers:** Period \( x[n] = 0.5\delta[n - 1] + \delta[n - 3]) + 0.5\delta[n - 2] \).

**Solution**
(a) A period of a train of pulses is

\[
x[n] = 0.5(r[n] - 2r[n-2] + r[n-4]) = \begin{cases} 
0 & n < 0 \\
r[n] = 0.5n & 0 \leq n \leq 2 \\
r[n] - 2r[n-2] = 0.5(4-n) & 2 < n \leq 4 \\
r[n] - 2r[n-2] + r[n-4] = 0 & n > 4
\end{cases}
\]

The following MATLAB script will generate \( x[n] \) for \( n \geq 0 \) (it is zero before) and the periodic signal \( y[n] \).

```matlab
% Pr 9_11
x=zeros(1,5);
for m=1:3,
    x(m)=m-1;
end
for m=3:5,
    x(m)=4-(m-1)
end
x=x/2;

n=0:4;
figure(1)
subplot(211)
stem(n,x); axis([0 4 -0.1 1.1])
hold on; plot(n,x,':r'); hold off; title('x[n]

x=[0 0.5 1 0.5 0] % another way to get the first period
y=[x x x x]; % four periods of y
N=length(y); n=0:N-1;
subplot(212)
stem(n,y); axis([0 N-1 -0.1 1.1])
hold on; plot(n,y,':r'); hold off; title('y[n]'); xlabel('n')
```

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Figure 9.3: Problem 11: One period, 4 periods of $y[n]$. 
9.12 **Envelope modulation** — In the generation of music by computer, the process of modulation is extremely important. When playing an instrument, the player typically does it in three stages: (1) rise time, (2) sustained time and (3) decay time. Suppose we model these three stages as an envelope continuous-time signal given by

\[ e(t) = \frac{1}{0.025} [r(t) - r(t - 0.025)] - \frac{1}{0.1} [r(t - 200) + r(t - 300)] \]

where \( r(t) \) is the ramp signal.

(a) For a simple tone \( x(t) = \cos(2\pi/T_0 t) \), the modulated signal is \( y(t) = x(t)e(t) \). Find the period \( T_0 \) so that 100 cycles of the sinusoid occur for the duration of the envelope signal.

(b) Simulate in the modulated signal using the value of \( T_0 \) and a simulation sampling time of \( T_s = 0.1 \). Plot \( y(t) \) and \( e(t) \) (discretized with the sampling period \( T_s \)) and listen to the modulated signal using sound.

**Answers:** If \( e(t) \) has duration \( T_e = 300T_s \) then \( T_0 = T_e/100 = 3T_s \).

**Solution**

(a) If the duration of \( e(t) \) is \( T_e = 300T_s \) and for 100 cycles of the sinusoid \( 100T_0 \) then \( T_0 = T_e/100 = 3T_s \).

(b) If we let \( t = nT_s \), then the sampled envelope is

\[
\begin{align*}
   e(nT_s) & = (1/3)(r(nT_s) - r((n - 30)T_s)) - (1/10)(r((n - 200)T_s) - r((n - 300)T_s)) \\
   & = \begin{cases} 
   nT_s/3 & 0 \leq nT_s \leq 30T_s \\
   10T_s & 30T_s \leq nT_s \leq 200T_s \\
   3 - nT_s/10 & 200T_s \leq nT_s \leq 300T_s 
   \end{cases}
\end{align*}
\]

The following script generates the window \( e[n] \) and the modulated signal \( y[n] = e[n]x[n] \).

```matlab
% Pr 9.12
clear all; clf
Ts=0.1;
t1=0:Ts:3;
t2=20:Ts:30;
e=[t1/3 ones(1,200-30) 3-t2/10]
t=0:Ts:30+Ts;
figure(1)
plot(t,e,’r’); grid
n=0:length(e)-1;
x=cos(2*pi*n/3);
hold on
plot(t,x.*e,’b’)
hold off; axis([0 30 -1.2 1.2]);
xlabel(’nT_s’);ylabel(’x(nT_s)e(nT_s)’)
```
Figure 9.4: Problem 12: window $e(nT_s)$ and modulated cosine $y(n) = x(nT_s)e(nT_s)$ for $T_s = 0.1$. 
9.13 LTI and convolution sum — The impulse response of a discrete-time system is 
\[ h[n] = (-0.5)^n u[n] \].

(a) If the input of the system is 
\[ x[n] = \delta[n] + \delta[n-1] + \delta[n-2] \], use the linearity and time-
invariance of the system to find the corresponding output \( y[n] \).

(b) Find the convolution sum corresponding to the above input, and show that your solution coincides with the output \( y[n] \) obtained above.

(c) Use the function \( \text{conv} \) to find the output \( y[n] \) due to the given input \( x[n] \). Plot \( x[n], h[n] \) and \( y[n] \) using.

\[ y[n] = (-0.5)^n u[n] + (-0.5)^{n-1} u[n-1] + (-0.5)^{n-2} u[n-2] \].

Solution

(a) Using the superposition property of LTI systems and that \( h[n] \) is the response to \( \delta[n] \) gives

\[
y[n] = h[n] + h[n - 1] + h[n - 2] = (-0.5)^n u[n] + (-0.5)^{n-1} u[n-1] + (-0.5)^{n-2} u[n-2]
\]

\[
y[n] = \begin{cases} 
1 & n = 0 \\
0.5 & n = 1 \\
0.75 & n = 2 \\
3(-0.5)^n & n \geq 3
\end{cases}
\]
as the output for the given input \( x[n] \).

(b) The convolution sum is given by

\[
y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]
\]

\[
= \sum_{k=-\infty}^{\infty} (\delta[k] + \delta[k-1] + \delta[k-2]) h[n-k]
\]

\[
= \sum_{k=-\infty}^{\infty} \delta[k] h[n-k] + \sum_{k=-\infty}^{\infty} \delta[k-1] h[n-k] + \sum_{k=-\infty}^{\infty} \delta[k-2] h[n-k]
\]

\[
= h[n] + h[n - 1] + h[n - 2]
\]
since the multiplication by the delta functions gives zero except when their arguments are zero, or \( k = 0, k = 1 \) and \( k = 2 \). This result coincides with the one in part (a).

(c) The following script is used to compute the convolution sum of the given \( x[n] \) and \( h[n] \):

```matlab
% Pr 9_13
clear all; clf
x=[1 1 1 zeros(1,98)];
n=0:100;
h=(-0.5).^n;
y=conv(h,x);
n1=0:20;
figure(1)
subplot(311)
stem(n1,x(1:21)); ylabel(’x[n]’);grid
```
Figure 9.5: Problem 13: Input $x[n]$, impulse response $h[n]$ and convolution sum $y[n]$. 
9.14 Periodicity — Consider a $2 \times 3$ array

$$p[m, n] = \begin{cases} 1 & \text{for } m = n = 0 \\ 0 & \text{for } 0 < m \leq 2, 0 < n \leq 1 \end{cases}$$

(a) Generate a signal $x[m, n]$ with period $p[n, m]$ and rectangular periods $N_1 = 3$ and $N_2 = 2$, i.e., $x[m, n] = x[m + 3, n] = x[m, n + 2]$. Use the functions $imshow$ and $stem3$ to display the signals.

(b) Use $p[m, n]$ to create an array $y[m, n]$ with periodic matrix

$$N = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

i.e.,

$$y[m, n] = y[m + 3, n + 1] = y[m, n + 2],$$

and plot several periods as an image. Is this signal rectangular periodic? If so indicate the periods.

Solution

(a) The array will be a stacking of $p[m, n]$ horizontally and vertically to generate $x[m, n]$ which can be visualized for $m \geq 0, n \geq 0$ for 9 periods in first quadrant

$$p[m, n - 4] \quad p[m - 3, n - 4] \quad p[m - 6, n - 4] \quad \cdots$$

$$p[m, n - 2] \quad p[m - 3, n - 2] \quad p[m - 6, n - 2] \quad \cdots$$

$$p[m, n] \quad p[m - 3, n] \quad p[m - 6, n] \quad \cdots$$

(b) In this case the $[0, 0]$ value appears in $[3, 1]$ and in $[0, 2]$ as indicated by the periodicity matrix. The overall array can be seen as in the first quadrant

$$p[m, n - 4] \quad p[m - 3, n - 5] \quad p[m - 6, n - 6] \quad \cdots$$

$$p[m, n - 2] \quad p[m - 3, n - 3] \quad p[m - 6, n - 4] \quad \cdots$$

$$p(m, n) \quad p[m - 3, n - 1] \quad p[m - 6, n - 2] \quad \cdots$$

The generated $y[m, n]$ is also periodic of periods $N_1 = 6$ and $N_2 = 2$. The period for these values contains 12 points instead of the 9 for the given $p[m, n]$.

```matlab
% Pr_9_14
clear all;clf
p=[0 0 0;1 0 0]
X=[p;p;p];
X=[X X X;X X X]
Y=zeros(4,9);
Y(1,4)=1; Y(2,1)=1; Y(2,7)=1;
Y(3,4)=1; Y(4,1)=1; Y(4,7)=1;
Y=[Y;Y;Y]
figure(1)
```
Results are shown in Fig. 9.6.

Figure 9.6: Rectangular periodic $x[m,n]$ and block periodic $y[m,n]$. 
9.15 **Image scaling system** Consider the image scaling system

\[ y[m, n] = \alpha + \beta \frac{x[m, n] - M}{N - M} \]

where the output is \( y[m, n] \) and the input is \( x[m, n] \) and \( \alpha, \beta, M \) and \( N \) are parameters to be determined for a certain image. Let the input \( x[m, n] \) be the image *clown* in a gray scale.

(a) Suppose that we wish to change the pixel values of the output \( y[m, n] \) so that they range from 6 to 256. Determine the values of \( M \) and \( N \) from the input image, and the parameters \( \alpha \) and \( \beta \) so that the desired scaling is possible. Use the functions `colormap` and `imagesc` to plot the output in a gray scale with a range \([0, 255]\).

(b) Let \( \alpha = 9 \) and \( \beta = 246 \), and the input signal \( x[m, n] \) be the test image *clown*. Is this scaling system linear? Explain why or why not. Plot the resulting image with a range of \([0, 255]\) for the pixel values.

(c) Compute and plot the histogram of the two images and verify the range of values of the pixels of the two images.

**Solution**

(a) The values needed to obtain the desired image are \( M = 1 \) and \( N = 81 \) corresponding to the minimum and maximum pixel values of *clown* and \( \alpha = 6 \) and \( \beta = 200 \).

(b) The equation relating the input and the output for the given values of \( \alpha \) and \( \beta \) is

\[ y[m, n] = 10 + 246 \frac{(x[m, n] - 1)}{80} \]

This system is non-linear because of the term 10 which is a bias that does not change with changes in the input.

```matlab
% Pr 9_15
clear all; clf
load clown; x=X;
M1=min(x(:))
M2=max(x(:))
for m=1:200.
    for n=1:320,
        y(m,n)=9+246*(x(m,n)-M1)/(M2-M1);
    end
end
M3=min(y(:))
M4y=max(y(:))
figure(1)
colormap(gray)
subplot(221)
imagesc(x,[0 256])
subplot(222)
```

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hist(x,100)
subplot(223)
imagesc(y,[0 256])
subplot(224)
hist(y,100)

Results are shown in Fig. 9.7.

Figure 9.7: Image scaling of *clown* image.
Chapter 10

The Z–transform

10.1 Basic Problems

10.1 The sign function extracts the sign of a real valued signal, i.e.,

\[ s[n] = \text{sign}(x[n]) = \begin{cases} 
-1 & x[n] < 0 \\
0 & x[n] = 0 \\
1 & x[n] > 0 
\end{cases} \]

(a) Let \( s[n] = s_1[n] + s_2[n] \), where \( s_1[n] \) is causal and \( s_2[n] \) anti-causal; find their Z-transforms and indicate the corresponding ROCs.

(b) Determine the Z-transform \( S(z) \).

Answers: \( s[n] = u[n] - u[-n] \) does not have Z-transform.

Solution

(a) We have \( s[n] = u[n] - u[-n] \), the Z-transform of \( s_1[n] = u[n] \) is

\[ S_1(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} \]

provided that \( |z^{-1}| < 1 \), thus \( |z| > 1 \) is the region of convergence for \( S_1(z) \). The Z-transform of \( s_2[n] = -u[-n] \) is given by

\[ S_2(z) = -\sum_{n=-\infty}^{0} z^{-n} = -\sum_{m=0}^{\infty} z^m = \frac{-1}{1 - z} \]
where the last sum converges for $|z| < 1$.

(b) The condition for

$$S(z) = S_1(z) + S_2(z) = \frac{1}{1 - z^{-1}} - \frac{1}{1 - z^{-1}} = \frac{1 + z^{-1}}{1 - z^{-1}}$$

to converge is that $|z| > 1$ and that $|z| < 1$ simultaneously, which is not possible. Since there is no region of convergence for $S(z)$, the Z-transform of $s[n]$ does not exist.
10.2 An analog pulse \( x(t) = u(t) - u(t - 1) \) is sampled using a sampling period \( T_s = 0.1 \).

(a) Obtain the discrete-time signal \( x(nT_s) = x(t)|_{t = nT_s} \) and plot it as a function of \( nT_s \)

(b) If the sampled signal is represented as an analog signal as

\[
x_s(t) = \sum_{n=0}^{N-1} x(nT_s) \delta(t - nT_s)
\]

determine the value of \( N \) in the above equation. Compute the Laplace transform of the sampled signal. i.e., \( X_s(s) = \mathcal{L}[x_s(t)] \)

(c) Determine the Z-transform of \( x(nT_s) \) or \( X(z) \). Indicate how to transform \( X_s(s) \) into \( X(z) \).

\textbf{Answers:} \( X_s(s) = \frac{(1 - e^{-1.1s})}{(1 - e^{-0.1s})} \); let \( z = e^{0.1s} \).

\textbf{Solution}

(a) If \( T_s = 0.1 \) the discrete-time signal is

\[
x(0.1n) = [u(t) - u(t - 1)] |_{t = 0.1n} = \begin{cases} 1 & 0 \leq 0.1n \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

(b) Expressing \( x[n] \) as indicated, then \( N = 11 \).

The Laplace transform of the sampled signal is

\[
X_s(s) = \sum_{n=0}^{10} \mathcal{L}[\delta(t - nT_s)] = \sum_{n=0}^{10} e^{-0.1ns} = \frac{1 - e^{-1.1s}}{1 - e^{-0.1s}}
\]

(c) The Z-transform of the discrete-time signal is

\[
X(z) = \sum_{n=0}^{10} z^{-n} = \frac{1 - z^{-11}}{1 - z^{-1}}
\]

To transform \( X_s(s) \) into \( X(z) \) we let \( z = e^{0.1s} \).
10.3 When finding the inverse Z-transform of a function with \( z^{-1} \) terms in the numerator, \( z^{-1} \) can be thought of a delay operator to simplify the calculation. For
\[
X(z) = \frac{1 - z^{-10}}{1 - z^{-1}}
\]
(a) Use the Z-transform of \( u[n] \) and the properties of the Z-transform to find \( x[n] \).
(b) Use your result for \( x[n] \) above to express \( X(z) \) as a polynomial in negative powers of \( z \).
(c) Find the poles and zeros of \( X(z) \) and plot them on the Z-plane. Is there a pole or zero at \( z = 1 \)? Explain.

Answers: \( x[n] = u[n] - u[n - 10] \); there is a pole/zero cancellation.

Solution
(a) Writing \( X(z) \) as
\[
X(z) = \frac{1}{1 - z^{-1}} - \frac{z^{-10}}{1 - z^{-1}}
\]
since the inverse Z-transform of the first term is \( u[n] \), then the inverse of the second is \( -u[n - 10] \) given that \( z^{-10} \) indicates a delay of 10 samples. Thus,
\[
x[n] = u[n] - u[n - 10] = \begin{cases} 1 & 0 \leq n \leq 9 \\ 0 & \text{otherwise} \end{cases}
\]
(b) Although \( X(z) \) has been shown as a ratio of two polynomials, using the above representation of \( x[n] \) its Z-transform is
\[
X(z) = 1 + z^{-1} + \cdots + z^{-9}
\]
i.e., a 9th-order polynomial in \( z^{-1} \).
(c) We can rewrite \( X(z) \) as
\[
X(z) = \frac{z^{10} - 1}{z^9(z - 1)} = \frac{(z - 1) \prod_{k=1}^{9} (z - e^{j\pi k/5})}{z^9(z - 1)} = \frac{\prod_{k=1}^{9} (z - e^{j\pi k/5})}{z^9}
\]
which is obtained by finding that the zeros of \( X(z) \) are values \( z_k^{10} = 1 \) or \( z_k = e^{j2\pi k/10} \) for \( k = 0, \cdots, 9 \). For \( k = 0 \) the zero is \( z_0 = 1 \), which cancels the pole at 1.
10.4 Consider the following problems related to LTI systems.

(a) The impulse response of an FIR filter is \( h[n] = \alpha^n (u[n] - u[n-M]) \)

i. Is it true that the transfer function for the filter is
\[
H(z) = \frac{1 - \alpha^M z^{-M}}{1 - \alpha z^{-1}} \text{ for any value of } \alpha?
\]

ii. Let \( M = 3, 0 \leq \alpha < 1 \) write \( H(z) \) as a polynomial, and then show that it equals
\[
\frac{1 - \alpha^3 z^{-3}}{1 - \alpha z^{-1}}. \text{ Determine the region of convergence of } H(z).
\]

(b) Consider the two-sided impulse response \( h[n] = 0.5^n |n|, \ - (N-1) \leq n \leq N - 1 \).

i. Determine the causal impulse response \( h_1[n] \), so that \( h[n] = h_1[n] + h_1[-n] \)

ii. Let \( N = 4 \), find the transfer function \( H(z) \) using the transfer function \( H_1[z] \), and determine the region of convergence of \( H(z) \) using the ROC of \( H_1(z) \).

iii. Let \( N \to \infty \), find the transfer function \( H(z) \) and its ROC.

(c) Given the finite length impulse response \( h[n] = 0.5^n(u[n] - u[n-2]) \).

i. Find the Z-transforms of the even, \( h_e[n] \), and the odd, \( h_o[n] \), components of \( h[n] \).

ii. Determine the regions of convergence of the Z-transforms of \( h_e[n] \) and \( h_o[n] \). How would the region of convergence of \( H(z) \) be obtained from the ROCs of \( X_e(z) \) and \( X_o(z) \)? Explain. Find \( h_e[n] \) and \( h_o[n] \).

Answers: (a) yes; (b) \( H_1(z) = \sum_{n=0}^{3} 0.5^n z^{-n} - 0.5, \text{ ROC } |z| > 0 \).

Solution

(a) i. Yes. Cross-multiplying
\[
H(z)(1 - \alpha z^{-1}) = (1 + \alpha z^{-1} + \cdots + \alpha^{M-1} z^{-(M-1)})(1 - \alpha z^{-1})
\]
\[
= 1 + \alpha z^{-1} + \cdots + \alpha^{M-1} z^{-(M-1)} - \alpha z^{-1} - \cdots - \alpha^M z^{-M}
\]
\[
= 1 - \alpha^M z^{-M}
\]

ii. \( H(z) = 1 + \alpha z^{-1} + \alpha^2 z^{-2} \text{ and } (1 - \alpha z^{-1})H(z) = 1 - \alpha^3 z^{-3}. \) Thus
\[
H(z) = \frac{1 - \alpha^3 z^{-3}}{1 - \alpha z^{-1}} = \frac{z^3 - \alpha^3}{z^2(z - \alpha)}
\]
so the zeros of \( H(z) \) are \( z_k = \alpha e^{j2\pi k/3} \) for \( k = 0, 1, 2 \), and the poles \( z = 0 \) of order 2, and \( z = \alpha \). The pole \( z = \alpha \) is cancelled by the zero at \( k = 0 \) so ROC is the whole \( Z \)-plane, except \( z = 0 \).
(b) i. If we let

\[ h_1[n] = \begin{cases} 
0.5 & n = 0 \\
0.5^n & 1 \leq n \leq N - 1 \\
0 & \text{otherwise}
\end{cases} \]

then \( h[n] = h_1[n] + h_1[-n] \)

ii. For \( N = 4 \), \( Z \)-transform of \( h_1[n] \) is

\[ H_1(z) = \sum_{n=0}^{3} 0.5^n z^{-n} - 0.5 = \frac{1 - 0.5^4 z^{-4}}{1 - 0.5 z^{-1}} - 0.5 = \frac{0.5 + 0.25 z^{-1} - 0.5^4 z^{-4}}{1 - 0.5 z^{-1}} \]

noticing that the initial expression for \( H_1(z) \) is a polynomial in \( z^{-1} \) then the ROC is \( |z| > 0 \). The final expression being rational indicates that if \( z^{-1} = 2 \) or \( z = 1/2 \) is a zero and a pole and they cancel so that only poles at the origin remain. The \( Z \)-transform

\[ \mathcal{Z}[h_1[-n]] = H_1(1/z) = \sum_{n=0}^{3} 0.5^n z^n - 0.5 = \frac{0.5 + 0.25 z - 0.5^4 z^4}{1 - 0.5 z} \]

with ROC the whole \( Z \)-plane as it is a polynomial in \( z \). Verify that the pole at \( z = 2 \) of the final expression for \( \mathcal{Z}[h_1[-n]] \) is cancelled by a zero at the same place. Thus

\[ H(z) = \frac{0.5 + 0.25 z^{-1} - 0.5^4 z^{-4}}{1 - 0.5 z^{-1}} + \frac{0.5 + 0.25 z - 0.5^4 z^4}{1 - 0.5 z} \]

with ROC: \( |z| > 0 \).

iii. If \( N \to \infty \), we have

\[ H(z) = \frac{1}{1 - 0.5 z^{-1}} + \frac{1}{1 - 0.5 z} - 1 = \frac{-0.25}{(1 - 0.5 z^{-1})(1 - 0.5 z)} \]

\[ = \frac{0.5 z}{(1 - 2 z)(1 - 0.5 z)} \quad \text{ROC} \quad 0.5 \leq |z| \leq 2 \]

(c) i. \( H(z) = 1 + 0.5 z^{-1} \), \( |z| > 0 \), \( H(z^{-1}) \) is \( Z \)-transform of \( h[-n] \) so

\[ H_c(z) = 0.5[H(z) + H(z^{-1})] = 0.5[2 + 0.5(z^{-1} + z)] = \frac{z + 0.25 z^2 + 0.25}{z} \quad |z| > 0 \]

\[ H_o(z) = 0.5[H(z) - H(z^{-1})] = 0.5[0.5(z^{-1} - z)] = \frac{0.25 - 0.25 z^2}{z} \quad |z| > 0 \]

The ROC of \( H(z) = H_c(z) + H_o(z) \) is the intersection of the ROCs of \( H_c(z) \) \( H_o(z) \) or \( |z| > 0 \).

ii. The inverse \( Z \)-transforms give

\[ h_c[n] = 0.25 \delta[n + 1] + \delta[n] + 0.25 \delta[n - 1] \]

\[ h_o[n] = -0.25 \delta[n + 1] + 0.25 \delta[n - 1] \]
10.5 Consider a discrete-time LTI system represented by the difference equation with the given initial condition

\[ y[n] + 0.5y[n - 1] = 2(x[n] - x[n - 1]), \quad n \geq 0, \quad y[-1] = 2 \]

where \( x[n] \) is the input and \( y[n] \) the output.

Suppose the input of the system is \( x[n] = (1 + 0.5^n \cos(\pi n))u[n] \). Determine the steady-state response \( y_{ss}[n] \). If the initial condition is changed, would you get the same steady-state response? Explain.

**Answers:** \( H(z) = \frac{2(1 - z^{-1})}{1 + 0.5z^{-1}} \); \( y_{ss}[n] = 0 \), initial conditions do not change this response.

**Solution**

We have

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{2(1 - z^{-1})}{1 + 0.5z^{-1}} \]

\[ Y(z) = H(z)(Z[u[n]] + Z[0.5^n \cos(\pi n)u[n]]) \]

only the first of the above terms would give steady state due to its pole at \( z = 1 \), so

\[ H(z) \frac{1}{1 - z^{-1}} = \frac{2(1 - z^{-1})}{(1 - z^{-1})(1 + 0.5z^{-1})} = \frac{2}{1 + 0.5z^{-1}} \]

which gives a transient, so \( y_{ss}[n] = 0 \). Since system is stable, changing the initial condition will not change the steady state–response.
10.6 Determine the impulse response \( h[n] \) of the feedback system shown in Fig. 10.1. Determine if the system is BIBO stable.

**Answer:** System is not BIBO stable.

![Figure 10.1: Problem 6](image)

**Solution** Impulse response

\[
\begin{align*}
y[n] &= e[n-1] = x[n-1] - y[n-1] \\
H(z) &= \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 + z^{-1}} \\
h[n] &= (-1)^{n-1}u[n-1]
\end{align*}
\]

System is unstable, \( h[n] \) is not absolutely summable:

\[
\sum_{n=1}^{\infty} |h[n]| \to \infty
\]

Moreover, the pole \( z = 1 \) is on the unit disc and as such the system is not BIBO stable.
10.7 Consider the following problems for LTI discrete-time systems.

(a) The input and the output of a LTI discrete-time system are

Input: \( x[n] = \begin{cases} 
1 & n = 0, 1 \\
0 & \text{otherwise} 
\end{cases} \)

Output: \( y[n] = \begin{cases} 
1 & n = 0, 3 \\
2 & n = 1, 2 \\
0 & \text{otherwise} 
\end{cases} \)

Find the transfer function \( H(z) \).

(b) The transfer function of an LTI discrete-time system is

\[ H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.25z^{-2}} \quad |z| > 0.5 \]

i. Is this system causal? Explain.

ii. Determine the impulse response of the system.

(c) The transfer function of an LTI system is

\[ H(z) = \frac{z + 1}{z(z - 1)} \quad |z| > 1 \]

What are the values of \( h[0] \), \( h[1] \) and \( h[1000] \) of the impulse response.

**Answer:** (a) \( H(z) = 1 + z^{-1} + z^{-2} \); (b) Causal; (c) \( h[1000] = 2 \)

**Solution**

(a) We have

\[ Y(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3} \]
\[ X(z) = 1 + z^{-1} \]
\[ H(z) = \frac{Y(z)}{X(z)} = 1 + z^{-1} + z^{-2} \]

after division.

(b) i. Yes, because of the ROC.

ii. There is a pole-zero cancellation

\[ H(z) = \frac{1 - 0.5z^{-1}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} \]

so that \( h[n] = (-0.5)^nu[n] \).

(c) By long division in negative powers of \( z \)

\[ H(z) = z^{-1} + 2z^{-2} + 2z^{-3} + \cdots + 2z^{-10000} + \cdots \]
so that $h[0] = 0$, $h[1] = 1$ and $h[10000] = 2$.

Another way is

$$H(z) = \frac{z^{-1}}{1 - z^{-1}} + \frac{z^{-2}}{1 - z^{-1}}$$

so that $h[n] = u[n - 1] + u[n - 2]$ giving the same values as above.
10.8 The impulse response of a LTI discrete-time system is \( h[n] = u[n] - u[n - 3] \). If the input to this system is

\[
x[n] = \begin{cases} 
0 & n < 0 \\
1 & n = 0, 1, 2 \\
1 & n \geq 3 
\end{cases}
\]

(a) Find the output \( y[n] \) of this system by calculating graphically the convolution sum.

(b) Express \( x[n] \) in terms of basic functions and determine \( X(z) \).

(c) Use the Z-transform to find the output \( y[n] \).

**Answers:** \( x[n] = \delta[n - 1] + 2\delta[n - 2] + u[n - 3] \); \( y[n] = x[n] + x[n - 1] + x[n - 2] \)

**Solution**

(a) The output is

\[
y[0] = 0 \quad y[1] = 1 \\
y[4] = 4 \quad y[n] = 3 \quad n \geq 5
\]

(b) \( x[n] \) can be written as

\[
x[n] = n(u[n] - u[n - 3]) + u[n - 3]) = \delta[n - 1] + 2\delta[n - 2] + u[n - 3]
\]

\[
X(z) = z^{-1} + 2z^{-2} + \frac{z^{-3}}{1 - z^{-1}}
\]

\[
H(z) = 1 + z^{-1} + z^{-2}
\]

thus

\[
Y(z) = X(z)H(z) = X(z) + z^{-1}X(z) + z^{-2}X(z)
\]

so that

\[
y[n] = x[n] + x[n - 1] + x[n - 2] = \delta[n - 1] + 3\delta[n - 2] + 4\delta[n - 3] + 4\delta[n - 4] + 3u[n - 5]
\]
The transfer function of an RLC circuit is $H(s) = Y(s)/X(s) = 2s/(s^2 + 2s + 1)$.

(a) Obtain the ordinary differential equation with input $x(t)$ and output $y(t)$. Approximating
the derivatives by differences (let $T = 1$) obtain the difference equation that approximates
the differential equation.

(b) Let the input be a constant source, so that $x[n] = u[n]$, and let the initial conditions for
the difference equation be zero. Solve the difference equation using the Z-transform.

**Answers:**
\[ y[n] - y[n-1] + 0.25y[n-2] = 0.5(x[n] - x[n-1]) \]

**Solution**

(a) Voltage division gives the transfer function

\[ H(s) = \frac{Y(s)}{X(s)} = \frac{2s}{s^2 + 2s + 1} \]

thus the differential equation relating $x(t)$ and $y(t)$ is

\[ \frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = 2\frac{dx(t)}{dt} \]

With $T = 1$,

\[ \frac{d\eta(t)}{dt} \approx \eta[n] - \eta[n-1] \]
\[ \frac{d^2\eta(t)}{dt^2} = \frac{d(dy(t)/dt)}{dt} \approx (\eta[n] - \eta[n-1]) - (\eta[n-1] - \eta[n-2]) \]
\[ = \eta[n] - 2\eta[n-1] + \eta[n-2] \]

so that the differential equation is approximated by

\[ (y[n] - 2y[n-1] + y[n-2]) + 2(y[n] - y[n-1]) + y[n] = 2(x[n] - x[n-1]) \]
\[ y[n] - y[n-1] + 0.25y[n-2] = 0.5(x[n] - x[n-1]) \]

(b) If $x[n] = u[n]$, then $x[n] - x[n-1] = \delta[n]$ and

\[ y[n] - y[n-1] + 0.25y[n-2] = 0.5\delta[n] \]

Using Z-transform

\[ Y(z)[1 - z^{-1} + 0.5z^{-2}] = 0.5 \]
\[ Y(z) = \frac{0.5}{(1 - 0.5z^{-1})^2} \quad \Rightarrow \quad y[n] = (n + 1)0.5^{n+1}u[n] \]
10.10 Suppose we cascade a “differentiator” and a “smoother” systems characterized by the following input/output equations

**Differentiator** \[ w[n] = x[n] - x[n-1] \]

**Smoother** \[ y[n] = \frac{y[n-1]}{3} + \frac{w[n]}{3} \]

where the output of the differentiator and input to the smoother is \( w[n] \), while \( x[n] \) is the input of the differentiator (and of the overall system) and \( y[n] \) is the output of the smoother (and of the overall system).

(a) If \( x[n] = u[n] \) and the initial conditions for the smoother are zero, find the output of the overall system \( y[n] \).

(b) If \( x[n] = (-1)^n, -\infty < n < \infty \), find the steady-state response \( y_{ss}[n] \) of the overall system \( y_{ss}[n] \).

**Answers:** The overall transfer function is \( H(z) = (1/3)(1-z^{-1})/(1-z^{-1}/3) \); \( y_{ss}[n] = 0.5(-1)^n \).

**Solution**

Let \( H_s(z) \) and \( H(z) \) be the transfer functions of the smoother and of the overall system.

(a) If \( x[n] = u[n] \) then \( w[n] = \delta[n] \) and \( y[n] = h_s[n] \). Thus, from

\[
H_s(z) = \frac{Y(z)}{W(z)} = \frac{1/3}{1-z^{-1}/3} \Rightarrow y[n] = h_s[n] = (1/3)^{n+1}u[n]
\]

(b) For the overall system

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \frac{W(z)}{X(z)} = \frac{(1/3)(1-z^{-1})}{1-z^{-1}/3}
\]

\( x[n] = (-1)^n = \cos(\pi n) \)

\( y_{ss}[n] = |H(e^{j\pi})| \cos(\pi n + \angle H(e^{j\pi})) \)

\( |H(e^{j\pi})| = 1/2, \quad \angle H(e^{j\pi}) = 0 \)

\( y_{ss}[n] = 0.5(-1)^n \)
10.11 Suppose we are given a finite-length sequence \( h[n] \) (it could be part of an infinite-length impulse response from a discrete system that has been windowed) and would like to obtain a rational approximation for it. This means that if \( H(z) = Z[h[n]] \), a rational approximation of it would be \( H(z) = B(z)/A(z) \), from which we get

\[
H(z)A(z) = B(z)
\]

Letting

\[
B(z) = \sum_{k=0}^{M-1} b_k z^{-k}, \quad A(z) = 1 + \sum_{k=1}^{N-1} a_k z^{-k}
\]

for some choice of \( M \) and \( N \), equations from \( H(z)A(z) = B(z) \) should allow us to find the \( M+N-1 \) coefficients \( \{a_k, b_k\} \).

(a) Find a matrix equation that would allow us to find the coefficients of \( B(z) \) and \( A(z) \).

(b) Let \( h[n] = 0.5^n(u[n] - u[n - 101]) \) be the sequence we wish to obtain a rational approximation and let \( B(z) = b_0 \) while \( A(z) = a_0 + a_1 z^{-1} \), find the equations to solve for the coefficients \( \{b_0, a_0, a_1\} \). Use MATLAB to solve the equations.

**Answers:**

\[
h[m] + \sum_{k=1}^{N-1} a_k h[m-k] = b_m \quad \text{where} \quad h[m] = 0 \quad \text{for} \quad m < 0 \quad \text{and} \quad b_m = 0 \quad \text{for} \quad m < 0 \quad \text{and} \quad m > M - 1.
\]

**Solution**

(a) The equation \( H(z)A(z) = B(z) \) using convolution and that \( a_0 = 1 \)

\[
h[m] + \sum_{k=1}^{N-1} a_k h[m-k] = b_m
\]

where \( h[m] = 0 \) for \( m < 0 \) and \( b_m = 0 \) for \( m < 0 \) and \( m > M - 1 \) or

\[
h[m] + \sum_{k=1}^{N-1} a_k h[m-k] = \begin{cases} b_m & m = 0, 1, \cdots, M - 1 \\ 0 & m = M, \cdots, M + N - 2 \end{cases}
\]

Solving first for the denominator coefficients (the bottom equations above), we get a matrix equation

\[
\begin{bmatrix}
  h[M] & h[M-1] & \cdots & h[M-N+1] \\
  h[M+1] & h[M] & \cdots & h[M-N+2] \\
  \vdots & \vdots & \ddots & \vdots \\
  h[M+N-2] & h[M+N-3] & \cdots & h[M-1]
\end{bmatrix}
\begin{bmatrix}
  1 \\
  a_1 \\
  \vdots \\
  a_{N-1}
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{bmatrix}
\]

To solve these equations we move the first column of the matrix on the left to replace the zero vector on the right, using the fact that \( a_0 = 1 \) is known.
Once the denominator coefficients are found, the following set of equations are solved for the numerator coefficients:

\[
\begin{bmatrix}
    h[0] & 0 & \cdots & 0 \\
h[1] & h[0] & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
h[M-1] & h[M-2] & \cdots & h[M-N]
\end{bmatrix}
\begin{bmatrix}
    1 \\
a_1 \\
\vdots \\
a_{N-1}
\end{bmatrix}
= 
\begin{bmatrix}
    b_0 \\
b_1 \\
\vdots \\
b_{M-1}
\end{bmatrix}
\]

(b) We let \(a_0 = 1\) and then the equation for \(a_1\) is \((M = 1, N = 2)\)

\[h[0]a_1 = -h[1]\]

so that \(a_1 = -h[1]/h[0] = -0.5/1 = -0.5\). Then the \(b_0\) is obtained \(h[0] = b_0\), so \(b_0 = 1\). The so-called “Pade approximant” is then

\[\hat{H}(z) = \frac{b_0}{1 + a_1 z^{-1}} = \frac{1}{1 - 0.5 z^{-1}}\]

with impulse response \(\hat{h}[n] = 0.5^n u[n]\) which approximates very well the given impulse response.
10.12 There are signals that do not have Z-transform. Consider
\[ x[m, n] = 2^{m+n}u_{12}[m, n]. \]

(a) Carefully draw the support of \( x[m, n] \).

(b) Attempt to determine the Z-transform of \( x[m, n] \), and indicate why it does not exist.

Solution

(a) The support of \( x[m, n] \) coincides with that of \( u_{12}[m, n] \) which is the first and second-quadrant in the \((m, n)\) plane.

(b) We have
\[
X(z_1, z_2) = \sum_{m=-\infty}^{\infty} 0.5^m z_1^{-m} \left[ \sum_{n=0}^{\infty} 0.5^n z_2^{-n} \right] = \frac{1}{1 - 0.5z_1^{-1}} \left[ \sum_{m=0}^{\infty} 0.5^m z_1^{-m} + \sum_{m=-\infty}^{-1} 0.5^m z_1^{-m} \right]
\]
\[
= \frac{1}{1 - 0.5z_1^{-1}} \left[ \frac{1}{1 - 0.5z_1^{-1}} + \left( \frac{1}{1 - 2z_1^{-1}} - 1 \right) \right]
\]
with the combined ROC:
\[
\text{ROC: } |z_2| > 0.5, \ (|z_1| > 0.5, \ \text{and} \ |z_1| < 0.5)
\]
and because the last two are contradictory the Z-transform does not exist.
The region of convergence of \( H(z_1, z_2) = 1/(1 - (z_1^{-1} + z_2^{-1})) \) is \(|z_1^{-1}| + |z_2^{-1}| < 1\). Determine if the unit bidisc is included in this ROC and sketch a possible ROC in a \(|z_1| \times |z_2|\) plane.

**Solution**

If \(|z_1| = |z_2| = 1\), the bidisc, their sum is 2 which is bigger than 1, so the bidisc is not in the ROC. The ROC can be expressed as

\[
|z_2| > \frac{|z_1|}{|z_1| - 1}
\]

and we have that

\[
\begin{array}{c|c|c}
|z_1| & |z_2| & \\
0 & > & 0 \\
1 & = & \infty \\
2 & > & 2 \\
3 & > & 1.5 \\
\infty & = & 1
\end{array}
\]

The ROC is thus a curve such that \(|z_2| \to \infty\) as \(|z_1| = 1\), it is above \(z_2 = 2\) when \(z_1 = 2\), is above \(z_2 = 1.5\) when \(z_1 = 3\) and as \(z_1 \to \infty\) it tends to \(z_2 = 1\).
10.14 For what values of $\alpha$ is the filter with the following transfer function not BIBO stable?

$$H(z_1, z_2) = \frac{1}{1 + 0.5z_1^{-1} + \alpha z_2^{-1}}$$

Solution

The condition for stability

$$A(z_1, 1) = 1 + 0.5z_1^{-1} + \alpha$$

has as root $z_1 = -0.5/(1 + \alpha)$ and if $|z_1| = |0.5/(1 + \alpha)| \geq 1$, i.e., on or outside the unit circle, the filter is unstable. So for any value of $\alpha$ that makes $|1 + \alpha| \leq 0.5$, or any $\alpha$ such that $-1.5 \leq \alpha \leq -0.5$. 
### 10.2 Problems using MATLAB

#### 10.15 Fibonacci sequence generation — Consider the Fibonacci sequence generated by the difference equation $f[n] = f[n-1] + f[n-2]$, $n \geq 0$ with initial conditions $f[-1] = 1$, $f[-2] = -1$.

(a) Find the Z-transform of $f[n]$, or $F(z)$. Find the poles $\phi_1$ and $\phi_2$ and the zeros of $F(z)$.

How are the poles connected? How are they related to the "golden-ratio"?

(b) The Fibonacci difference equation has zero input, but its response is a sequence of ever-increasing integers. Obtain a partial fraction expansion of $F(z)$ and find $f[n]$ in terms of the poles $\phi_1$ and $\phi_2$, and show that the result is always integer. Use MATLAB to implement the inverse in term of the poles.

**Answers:** $f[n] = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots \}$ for $n \geq 0$

**Solution**

(a) Using the difference equation and the given initial conditions we get the values of the Fibonacci series:

$$f[0] = f[-1] + f[-2] = 0,$$
$$f[1] = f[0] + f[-1] = 1,$$

or the sequence $\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots \}$ for $n \geq 0$. The Z-transform of $f[n]$ is then

$$F(z) = f[0] + f[1]z^{-1} + f[2]z^{-2} + f[3]z^{-3} + \cdots$$
$$= 0 + 1z^{-1} + 1z^{-2} + 2z^{-3} + \cdots$$

which is not in a closed-form. To get that, we use the Z-transform shift property in the given difference equation:

$$F(z) = (z^{-1}F(z) + f[-1]) + (z^{-2}F(z) + f[-1]z^{-1} + f[-2])$$
$$= [z^{-1} + z^{-2}]F(z) + z^{-1}$$

after replacing the initial conditions. We then have that

$$F(z) = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}$$

We have obtained the Z-transform of the Fibonacci sequence even though we know that the sequence blows up, and that the region of convergence for this sequence is not obvious from the development.
The zero of $F(z)$ is at zero, and the poles are

$$\phi_{1,2} = \frac{1 \pm \sqrt{1 + 4}}{2} = 1.6183, -0.6183$$

The value $\phi_1 = 1.6183$ is the golden ratio which in ancient Greece was considered the most pleasing ratio for many designs. The letter $\phi$ has been used in honor of the Greek sculptor Phidias who is said to have used it in his work.

(b) Notice that $\phi_1 + \phi_2 = 1$. The partial fraction expansion is

$$F(z) = \frac{A}{1 - \phi_1 z^{-1}} + \frac{B}{1 - \phi_2 z^{-1}}$$

The coefficients are found

$$A = F(z)(1 - \phi_1 z^{-1})|_{z^{-1} = 1/\phi_1} = \frac{1}{\phi_1 - \phi_2} = \frac{1}{\sqrt{5}}$$

$$B = F(z)(1 - \phi_2 z^{-1})|_{z^{-1} = 1/\phi_2} = \frac{1}{\phi_2 - \phi_1} = -\frac{1}{\sqrt{5}}$$

so for ROC: $|z| > \phi_1 = 1.6183$, the inverse is

$$f[n] = \frac{1}{\sqrt{5}}(\phi_1^n - \phi_2^n)u[n]$$

This formula was first presented by Leonhard Euler in 1765 as a formula to obtain the Fibonacci numbers. Let us verify that this equation gives the Fibonacci numbers, $f[0] = 0$, $f[1] = \frac{1}{\sqrt{5}}(\phi_1 - \phi_2) = 1$, and so on. For the region of convergence $|z| > \phi_1 = 1.6183$, the system represented by the Fibonacci difference equation is not BIBO stable as one of its poles is outside the unit circle.

Using the poles $\phi_i$ we compute the Fibonacci sequence. The obtained values do not coincide exactly with the sequence, only if we find integers that smaller using floor we obtain it.

```matlab
% Pr. 10.15
clear all; clf
phi1=1.6183; phi2=-0.6183;
n=0:100;
f=(phi1.^n-phi2.^n)/sqrt(5);
figure(1)
stem(n(1:10),f(1:10));grid;title('Fibonacci sequence');xlabel('n')
floor(f(1:10))
```

0 1 1 2 3 5 8 13 21 34
Figure 10.2: Problem 15: Approximate computation of Fibonacci sequence.
10.16 Generation of discrete-time sinusoid — Given that the Z-transform of a discrete-time cosine $A \cos(\omega_0 n) u[n]$ is

$$\frac{A(1 - \cos(\omega_0)z^{-1})}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$$

(a) Use the given Z-transform to find a difference equation whose output $y[n]$ is a discrete-time cosine $A \cos(\omega_0 n)$ and input $x[n] = \delta[n]$. What should you use as initial conditions?

(b) Verify your algorithm by generating a signal $y[n] = 2 \cos(\pi n/2)u[n]$ by implementing your algorithm in MATLAB. Plot the input and the output signals $x[n]$ and $y[n]$.

(c) Indicate how to change your previous algorithm to generate a sine function $y[n] = 2 \sin(\pi n/2)u[n]$. Use MATLAB to find $y[n]$, and to plot it.

**Answers:**

$$y[n] - 2\cos(\omega_0)y[n-1] + y[n-2] = Ax[n] - A\cos(\omega_0)x[n-1] \quad n \geq 0.$$

**Solution**

(a) Thinking of the given Z-transform as a transfer function with $x[n]$ as the input and $y[n]$ as the output we have that

$$AX(z)(1 - \cos(\omega_0)z^{-1}) = Y(z)(1 - 2\cos(\omega_0)z^{-1} + z^{-2})$$

which gives the following difference equation

$$y[n] - 2\cos(\omega_0)y[n-1] + y[n-2] = Ax[n] - A\cos(\omega_0)x[n-1] \quad n \geq 0$$

with zero initial conditions and input $x[n] = \delta[n]$ that will generate the cosine function.

(b) We have that $\omega_0 = \pi/2$ so that $\cos(\omega_0) = 0$ giving the difference equation

$$y[n] + y[n-2] = 2\delta[n] \quad n \geq 0$$

with zero initial conditions. Recursively this equation gives

$$y[0] = 2$$

$$y[1] = 0$$


$$y[3] = 0$$

$$\vdots$$

or $y[n] = 2 \cos(\pi n/2)u[n]$. 

---

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(c) The signal to be generated $y_1[n] = 2\sin(\pi n/2)$ gives for $n \geq 0$

$$
\begin{align*}
y_1[0] &= 0 \\
y_1[1] &= 2 \\
y_1[2] &= 0 \\
y_1[3] &= -2 \\
&\vdots
\end{align*}
$$

i.e., $y_1[n] = y[n-1]$, so that the above difference equation shifted by 1 gives

$$
\begin{align*}
y[n-1] + y[n-3] &= 2\delta[n-1] & n-1 \geq 0 \\
y_1[m] + y_1[m-2] &= 2\delta[m] & m \geq 0
\end{align*}
$$

which gives

$$
\begin{align*}
y_1[0] &= -y[-2] + 2 = 0 \\
y_1[1] &= -y[-1] + 0 = 2 \\
y_1[2] &= y_1[0] = 0 \\
&\vdots
\end{align*}
$$

so that the initial conditions are $y[-2] = 2$ and $y[-1] = -2$.

Another way to obtain the initial conditions to generate $2\sin(\pi n/2)$ with the same difference equation as before is in the z-domain

$$
Y(z) = \frac{2X(z) - y[-1]z^{-1} - y[-2]}{1 + z^{-2}}
$$

If we let $X(z) = 1$ (i.e. $x[n] = \delta[n]$) and initial conditions $y[-2] = 2$ and $y[-1] = -2$ we get

$$
Y(z) = \frac{2z^{-1}}{1 + z^{-2}}
$$

which is the Z-transform of the desired signal.

The following script computes and plots the two sinusoids. To generate the sine, we compute the first two values using the initial conditions and the input, and then use the difference equation to obtain the rest using a for loop.

```matlab
% Pr. 10.16
clear all; clf
% generation of 2cos(pi n/2)
b=2;
```

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a=[1 0 1];
x=[1 zeros(1,100)];
y=filter(b,a,x);
n=0:length(y)-1;
figure(1)
subplot(211)
stem(n,y); axis([0 100 -2.2 2.2]); grid; ylabel('y[n]')
hold on; plot(n,y,:r'); hold off
% generation of 2sin(pi n/2)
y1(1)=0;y1(2)=2;
for m=3:101,
    y1(m)=-y1(m-2);
end
subplot(212)
stem(n,y1); axis([0 100 -2.2 2.2]); grid; ylabel('y_1[n]');xlabel('n')
hold on; plot(n,y1,:r'); hold off

Figure 10.3: Problem 16: Generation of 2 cos(\pi n/2) (top) and 2 sin(\pi n/2) (bottom).
10.17 **Inverse Z-transform** — We are interested in the unit-step solution of a system represented by the following difference equation

\[y[n] = y[n-1] - 0.5y[n-2] + x[n] + x[n-1]\]

(a) Find an expression for \(Y(z)\).

(b) Do a partial expansion of \(Y(z)\).

(c) Find the inverse Z-transform \(y[n]\). Use MATLAB to verify your answer.

**Answers** \(y[n] = [4 + 3.16(0.707)^n \cos(\pi n/4 - 161.5^\circ)]u[n]\).

**Solution**

(a) To find the unit–step response, the input is \(x[n] = u[n]\) and the initial conditions are zero. We then have

\[Y(z)[1 - z^{-1} + 0.5z^{-2}] = (1 + z^{-1}) \frac{1}{1-z^{-1}}\]

\[Y(z) = \frac{1 + z^{-1}}{(1-z^{-1})(1-0.5(1+j)z^{-1})(1-0.5(1-j)z^{-1})}\]

(b) The partial fraction expansion is

\[Y(z) = \frac{A}{1-z^{-1}} + \frac{B}{1-0.5(1+j)z^{-1}} + \frac{B^*}{1-0.5(1-j)z^{-1}}\]

\[A = Y(z)(1-z^{-1})|_{z^{-1}=1} = 4\]

\[B = Y(z)(1-0.5(1+j)z^{-1})|_{z^{-1}=0.5(1+j)} = 1.58e^{-j161.5^\circ}\]

(c) The inverse Z-transform is then

\[y[n] = \left[A + B \left(\frac{\sqrt{2}}{2}e^{j\pi/4}\right)^n + B^* \left(\frac{\sqrt{2}}{2}e^{-j\pi/4}\right)^n\right]u[n]\]

\[= [A + |B| \left(\frac{\sqrt{2}}{2}\right)^n (e^{j(\pi n/4 + \angle B)} + e^{-j(\pi n/4 - \angle B)})]u[n]\]

\[= [A + 2|B| \left(\frac{\sqrt{2}}{2}\right)^n \cos(\pi n/4 + \angle B)]u[n]\]

\[= [4 + 3.16(0.707)^n \cos(\pi n/4 - 161.5^\circ)]u[n]\]

The following script is used to verify the above result using MATLAB

```matlab
%% Pr. 10.17
clear all; clf
N=[1 1 0 0];
D=conv([1 -1],[1 -1 0.5])
[r,p,k]=residuez(N,D)
abs(r(2))
```

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angle(r(2))*180/pi
n=0:49;
y=r(1)*(p(1).ˆn)+r(2)*p(2).ˆn)+r(3)*(p(3).ˆn);
figure(1)
subplot(311)
zplane(N,D)
subplot(312)
stem(n,y); ylabel('y[n]')
% verification
y1=4+3.16*(0.707).ˆn.*cos(pi.*n/4-161.5*pi/180);
subplot(313)
stem(n,y1); ylabel('y_1[n]'); xlabel('n')

Figure 10.4: Problem 17: Verification of unit-step response using residues and analytic solution. Poles and zeros of $Y(z)$; unit–step response found using the partial fraction expansion with MATLAB (middle) and analytically (bottom).
10.18 Prony's Rational Approximation — The Padé approximant provides an exact matching of $M + N - 1$ values of $h[n]$, where $M$ and $N$ are the orders of the numerator and denominator of the rational approximation. But there is no method for choosing the numerator and denominator orders, $M$ and $N$. Also, there is no guarantee on how well the rest of the signal is matched. Prony’s rational approximation considers how well the rest of the signal is approximated. Let $h[n] = 0.9^n u[n]$, be the impulse response we wish to find a rational approximation. Take the first 100 values of this signal as the impulse response.

(a) Assume the order of the numerator and the denominators are equal $M = N = 1$, use the MATLAB function `prony` to obtain the rational approximation, and then use `filter` to verify that the impulse response of the rational approximation is close to the given 100 values. Plot the error between $h[n]$ and the impulse response of the rational approximation for the first 100 samples. Plot the poles and zeros of the rational approximation and compare them to the poles and zeros of $H(z) = \mathcal{Z}(h[n])$.

(b) Suppose that $h[n] = (h_1 * h_2)[n]$, i.e., the convolution of $h_1[n] = 0.9^n u[n]$ and $h_2[n] = 0.8^n u[n]$. Use again Prony to find the rational approximation when the first 100 values of $h[n]$ are available. Use `conv` from MATLAB to compute $h[n]$. Compare the impulse response of the rational approximation to $h[n]$. Plot the poles and zeros of $H(z) = \mathcal{Z}(h[n])$ and of the rational approximation.

(c) Consider the $h[n]$ given above, and perform the Prony approximation using orders $M = N = 3$, explain your results. Plot poles and zeros.

Solution

(a) The following script obtains the Prony rational approximation of the first 100 value of $h[n] = 0.9^n u[n]$.

```matlab
% Pr. 10.18
clear all; clf
n=[0:100];
h=0.9.^n;
[b,a]=prony(h,1,1);  % rational approximation of order (1,1) of h
delta=[1 zeros(1,100)];
hd=filter(b,a,delta);  % approximate sequence
figure(1)
subplot(211)
plot(n,h,'o',n,hd,'+')
xlabel('n')
ylabel('h[n],hd[n]')
grid
p=roots(a)  % poles
z=roots(b)  % zeros
```

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subplot(223)
stem(n,hd-h); ylabel('error')
subplot(224)
zplane(b,a)

![Diagram]

Figure 10.5: Problem 18: Prony approximation, of order \((1, 1)\), of finite segment of \(h[n] = 0.9^n u[n]\).

The poles and zeros corresponding to the actual and the approximate are very close.

(b) The convolution of \(h_1[n] = 0.9^n u[n]\) and \(h_2[n] = 0.8^n u[n]\) is done using the MATLAB function `conv`. The Prony approximation of \(h_1[n] * h_2[n]\) gives an approximation to \(H_1(z)H_2(z)\) for the correct orders. Using `prony` of order \((2, 2)\) (in general this has to be guessed, but here we have the advantage of knowing the sequences, a big advantage!).

clear all; clf
n=[0:99];
h1=(0.9).^n;
h2=(0.8).^n;
h=conv(h1,h2);
[b,a]=prony(h,2,2) % rational approximation of order (2,2) of h
delta=[1 zeros(1,100)];
hd=filter(b,a,delta); % approximate sequence
subplot(211)
plot(n,h(1:100),’o’,n,hd(1:100),’+’)
xlabel(’n’) ylabel(’h(n),hd[n]’) grid
The following are the results obtained. Notice that the poles/zeros are those of $H_1(z)$ and $H_2(z)$ or the product $H(z) = H_1(z)H_2(z)$.

\[
\begin{align*}
  b &= 1.0000 \quad -0.0000 \quad -0.0000 \\
  a &= 1.0000 \quad -1.7000 \quad 0.7200 \\
  p &= 0.9000 \quad 0.8000 \\
  z &= 1.0e-004 \quad * \\
      &= \begin{pmatrix} 0.2674 \\ -0.2674 \end{pmatrix} \\
\end{align*}
\]

(c) When the chosen order is higher than the actual one, it is possible to have a pole-zero cancellations. For the $h[n] = [h_1 * h_2][n]$ when using \textit{prony} instead of a second-order approximation we choose a third-order, the results has a pole/zero cancellation.

\[
\begin{align*}
  [bl, al] &= \text{prony}(h, 3, 3) \\
  p &= \text{roots}(al) \\
  z &= \text{roots}(bl) \\
  \text{subplot}(211) \\
  \text{zplane}(z, p) \\
  \text{delta} &= [1 \text{ zeros}(1, 120)]; \\
  \text{hd} &= \text{filter}(bl, al, delta); \\
\end{align*}
\]
\begin{verbatim}
M=length(h);
n=[0:ceil(M/2)-1];
subplot(212)
plot(n,h(1:ceil(M/2)),'o',n,hd(1:ceil(M/2)),'+')
\end{verbatim}

The results in this case are

\begin{verbatim}
b1 =
    1.0000  0.0000 -0.0000 -0.0000
a1 =
    1.0000 -1.7000  0.7200  0.0000
p =
    0.9000
    0.8000
    -0.0000
z =
    1.0e-03 *
    0.9266
    -0.4633 + 0.8017i
    -0.4633 - 0.8017i
\end{verbatim}

showing the same poles as before, which means a third one has been cancelled.
Chapter 11

Fourier Analysis of Discrete–time Signals and Systems

11.1 Basic Problems

11.1 The frequency response of an ideal low-pass filter is

\[ H(e^{j\omega}) = \begin{cases} 1 & -\pi/2 < \omega < \pi/2 \\ 0 & \text{otherwise in } (-\pi, \pi] \end{cases} \]

(a) Find the impulse response \( h[n] \) of this filter. Is this filter causal?

(b) Suppose that the input of this filter \( x[n] \) has a DTFT \( X(e^{j\omega}) = H(e^{j\omega}) \), what is the output of the filter \( y[n] \).

(c) According to the above result, does that mean that \( (h * h)[n] = h[n] \)? Explain.

Answers: \( h[n] \) is non-causal; yes, \( h[n] = (h * h)[n] \).

Solution

(a) Impulse response

\[ h[n] = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j\omega n} d\omega = \begin{cases} 0.5 \sin(\pi n/2)/(\pi n) & n \neq 0 \\ 1 & n = 0 \end{cases} \]

\( h[n] \) is non-causal as \( h[n] \neq 0 \) for \( n < 0 \).

(b) \( Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = H(e^{j\omega}) \) so \( y[n] = h[n] \)

(c) Yes, \( H(e^{j\omega}) = H(e^{j\omega})H(e^{j\omega}) \) so \( h[n] = (h * h)[n] \).
11.2 Find the DTFT $X(e^{j\omega})$ of $x[n] = \delta[n] - \delta[n-2]$.

(a) Sketch and label carefully the magnitude spectrum $|X(e^{j\omega})|$ for $0 \leq \omega < 2\pi$.
(b) Sketch and label carefully the magnitude spectrum $|X(e^{j\omega})|$ for $-\pi \leq \omega < \pi$.
(c) Sketch and label carefully the phase spectrum $\angle X(e^{j\omega})$ for $-\pi \leq \omega < \pi$.

Answer: $X(e^{j\omega}) = 2je^{-j\omega} \sin(\omega)$.

Solution

(a) (b) DTFT

$$X(e^{j\omega}) = 1 - e^{-j2\omega} = 2je^{-j\omega} \sin(\omega)$$

Yes, $X(e^{j\omega})$ is periodic of period $2\pi$ since $\omega + 2k\pi = \omega$. $|X(e^{j\omega})| = 2|\sin(\omega)|$ is periodic of period $\pi$, but also periodic of period $2\pi$.

(b) Phase

$$\angle X(e^{j\omega}) = \begin{cases} 
-\omega + \pi/2 & \text{if } \sin(\omega) > 0 \\
-\omega + 3\pi/2 = -\omega - \pi/2 & \text{if } \sin(\omega) \leq 0
\end{cases}$$

and because of the odd symmetry of the phase it is zero at $\omega = 0$. See Fig. 11.1.
11.3 The transfer function of an FIR filter is $H(z) = z^{-2}(0.5z + 1.2 + 0.5z^{-1})$.

(a) Find the frequency response $H(e^{j\omega})$ of this filter. Is the phase response of this filter linear?

(b) Find the impulse response $h[n]$ of this filter. Is $h[n]$ symmetric with respect to some $n$? How does this relate to the phase?

**Answers:** Phase is linear; $h[n] = 0.5\delta[n - 1] + 1.2\delta[n - 2] + 0.5\delta[n - 3]$.

**Solution**

(a) Frequency response

$$H(e^{j\omega}) = e^{-j2\omega}(1.2 + \cos(\omega))$$

$$\angle H(e^{j\omega}) = -2\omega \text{ linear, as } 1.2 + \cos(\omega) > 0 \ \omega \in (-\pi, \pi]$$

(b) Impulse response

$$H(z) = \frac{0.5}{h[1]}z^{-1} + \frac{1.2}{h[2]}z^{-2} + \frac{0.5}{h[3]}z^{-3}$$

$h[n]$ is symmetric with respect to $n = 2$, so the phase is linear.
11.4 A periodic discrete-time signal \( x[n] \) with a fundamental period \( N = 3 \) is passed through a filter with impulse response \( h[n] = (1/3)(u[n] - u[n - 3]) \). Let \( y[n] \) be the filter output. We begin the filtering at \( n = 0 \), i.e., we are interested in \( y[n] \), \( n \geq 0 \) and could assume \( y[n] = 0 \), \( n < 0 \). For a period, \( x[n] \) is \( x[0] = 1 \), \( x[1] = -2 \), \( x[2] = 1 \).

(a) Determine the values of \( y[0] \), \( y[1] \) and \( y[2] \). Because \( x[n] \) is periodic you can assume its values for \( n < 0 \) are given.

(b) Use the convolution sum to calculate \( y[n] \), \( n \geq 0 \). What is the steady state output? When is the steady-state of \( y[n] \) attained?

(c) Compute the output of the filter using the DTFT.

**Answers:** \( y[0] = y[1] = y[2] = 0 \); \( H(e^{j0}) = 1 \), \( H(e^{j2\pi/3}) = 0 \).

**Solution**

(a) The impulse response is \( h[n] = (1/3)[\delta[n] + \delta[n-1] + \delta[n-2]] \) so that \( y[n] = (1/3)(x[n] + x[n-1] + x[n-2]) \) for \( n \geq 0 \), so

\[
y[0] = (1/3)(x[0] + x[-1] + x[-2]) = 0
\]
\[
y[1] = (1/3)(x[1] + x[0] + x[-1]) = 0
\]
\[
\]

(b) Convolution sum:

Impulse response \( h[n] = (1/3)[\delta[n] + \delta[n-1] + \delta[n-2]] \)

\[
y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \frac{1}{3} \sum_{k=-\infty}^{n} x[k](\delta[n-k] + \delta[n-k-1] + \delta[n-k-2])
\]
\[
= \frac{1}{3} (x[n] + x[n-1] + x[n-2])
\]
which is the average of three consecutive values of \( x[n] \) or 0. The steady state is \( y[n] = 0 \) which is attained at \( n = 0 \).

(c) The Fourier series coefficients of \( x[n] \) are

\[
X[k] = \frac{1}{3} (1 - 2e^{-1} + z^{-2})|_{z=e^{j2\pi k/3}}
\]
\[
= \frac{e^{-j2\pi k/3}}{3} (e^{j2\pi k/3} - 2 + e^{-j2\pi k/3}) = \frac{-2e^{-j2\pi k/3}}{3} [1 - \cos(2\pi k/3)]
\]
giving \( X[0] = 0 \) and \( X[1] = X[-1] = -(2/3)e^{-j2\pi/3}(1 - \cos(2\pi/3)) = e^{-j2\pi/3} \), and \( \omega_0 = 2\pi/3 \). So that

\[
X(e^{j\omega}) = \mathcal{F} \left[ \sum_{k=-1}^{1} X[k]e^{j\omega k} \right] = \sum_{k=-1}^{1} 2\pi X[k] \delta(\omega - k\omega_0)
\]
\[
= 2\pi X[-1] \delta(\omega + \omega_0) + 2\pi X[1] \delta(\omega - \omega_0)
\]
The filter frequency response is

\[
H(e^{j\omega}) = \frac{e^{-j\omega}}{3} (e^{j\omega} + 1 + e^{-j\omega}) = \frac{e^{-j\omega}}{3} (1 + 2\cos(\omega))
\]
giving $H(e^{j0}) = 1, H(e^{j2\pi/3}) = 0$. Thus the output is

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = 2\pi X[-1]H(e^{j2\pi/3}) + 2\pi X[1]H(e^{-j2\pi/3}) = 0$$
From the definition of the DFT
\[ X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \quad k = 0, 1, \ldots, N - 1 \]

one can obtain a matrix equation \( X = Fx \).

(a) Suppose that \( N = 2 \) and \( x[n] = 1 \) for \( n = 0, 1 \), express the DFT as a matrix equation.

(b) Check if the matrix \( F \) in your representation is invertible, and if so obtain the its inverse so that you have a representation of the IDFT.

(c) Is it true that \( F^*F = \alpha I \), i.e., the product of the transpose of the complex conjugate \( F^* \) and \( F \) is the identity multiplied by a constant \( \alpha \)? What is \( \alpha \)? Would the inverse of \( F \) be \((1/\alpha)F^*\)? Is this related to the orthonormality of the Fourier exponential basis? Explain.

Answers:
\[
F = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad F^TF = -2I
\]

Solution

(a) Matrix equation
\[
\begin{bmatrix} X[0] \\ X[1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

which gives \( X[0] = 2 \) and \( X[1] = 0 \). Since \( \text{det}(F) = -2 \neq 0 \) the matrix is invertible with inverse \( 0.5F^T = 0.5F \) and

\[
F^{-1} \begin{bmatrix} X[0] \\ X[1] \end{bmatrix} = \begin{bmatrix} x[0] \\ x[1] \end{bmatrix}
\]

(b) We have
\[
\frac{F^T}{\text{det}(F)} F = I
\]

so \( \alpha = 2 \).
The input of a discrete-time system is \( x[n] = u[n] - u[n - 4] \) and the impulse of the system is \( h[n] = \delta[n] + \delta[n - 1] + \delta[n - 2] \).

(a) Calculate the DFTs of \( h[n], x[n] \) of length \( N = 7 \). Call them \( H(k) \) and \( X(k) \). Calculate \( X(k)H(k) = \hat{Y}(k) \).

(b) If \( y(n) = (x * h)[n] \), would you expect \( \hat{y}[n] = y[n] \)? Explain. If you had chosen the lengths of the DFTs to be \( N = 4 \) would you have had the circular and the linear convolutions coincide? Explain.

Answers: (a) \( X[k] = \sum_{n=0}^{3} e^{-j \frac{2\pi nk}{7}}, H[k] = \sum_{n=0}^{2} e^{-j \frac{2\pi nk}{7}} \); (b) yes, they coincide.

Solution

(a) The DFTs of length 7 are

\[
H[k] = \sum_{n=0}^{2} e^{-j \frac{2\pi nk}{7}}
\]

\[
X[k] = \sum_{n=0}^{3} e^{-j \frac{2\pi nk}{7}}
\]

\[
Y[k] = X[k]H[k] = 1 + 2e^{-j \frac{2\pi k}{7}} + 3e^{-j \frac{4\pi k}{7}} + 3e^{-j \frac{6\pi k}{7}} + 2e^{-j \frac{8\pi k}{7}} + e^{-j \frac{10\pi k}{7}}
\]

(b) Yes, the circular convolution coincides with the linear convolution if the DFTs are of length 7, which is larger than the length of the linear convolution (4 + 3 - 1 = 6). If \( N < 6 \) the two convolutions will not coincide.
11.2 Problems using MATLAB

11.7 Frequency shift of FIR filters — Consider a moving average FIR filter with an impulse response \( h[n] = \frac{1}{3}(\delta[n] + \delta[n-1] + \delta[n-2]) \). Let \( H(z) \) be the Z-transform of \( h[n] \).

(a) Find the frequency response \( H(e^{j \omega}) \) of the FIR filter.

(b) Let the impulse response of a new filter be given by \( h_1[n] = (-1)^n h[n] \), use again the eigenfunction property to find the frequency response \( H_1(e^{j \omega}) \) of the new FIR filter.

(c) Use the MATLAB functions \( \text{freqz} \) and \( \text{abs} \) to compute the magnitude response of the two filters. Plot them and determine the poles and the zeros of the two filters. What type of filters are these?

Answers: \( H_1(e^{j \omega}) = \frac{1}{3}e^{-j \omega}(2 \cos(\omega) - 1) \), high-pass.

Solution

(a) The Z-transform of \( h[n] \) is

\[
H(z) = \frac{1}{3}(1 + z^{-1} + z^{-2})
\]

with ROC the whole Z-plane, except the origin, so we can let \( z = e^{j \omega} \) to get

\[
H(e^{j \omega}) = \frac{1}{3}e^{-j \omega}(1 + 2 \cos(\omega))
\]

(b)

\[
H_1(e^{j \omega}) = \sum_n h[n]e^{-j(\omega-\pi)n} = H(e^{j(\omega-\pi)}) = \frac{1}{3}e^{-j(\omega-\pi)}(1 + 2 \cos(\omega-\pi))
\]

\[
= \frac{1}{3}e^{-j \omega}(2 \cos(\omega) - 1)
\]

(c) The frequency response of the two filters is computed and plotted using the following script

```matlab
% Pr. 11_7
clear all; clf
b1=(1/3)*[1 1 1];
b2=(1/3)*[1 -1 1];
[H1,w]=freqz(b1,1);
[H2,w]=freqz(b2,1);
figure(1)
subplot(221)
pplot(w/pi,abs(H1));ylabel('|H_1|')
subplot(222)
pplot(w/pi,angle(H1));ylabel('<H_1')
subplot(223)
pplot(w/pi,abs(H2));ylabel('|H_2|');xlabel('\omega/\pi')
subplot(224)
pplot(w/pi,angle(H2));ylabel('<H_2');xlabel('\omega/\pi')
```
Figure 11.2: Problem 7: Magnitude and phase of low-pass $H_1(z)$ (top) and high-pass $H_2(z)$ filters
11.8 Computations from DTFT definition — For simple signals it is possible to obtain some information on their DTFTs without computing it. Let

\[ x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]. \]

(a) Find \( X(e^{j0}) \) and \( X(e^{j\pi}) \) without computing the DTFT \( X(e^{j\omega}) \).

(b) Find

\[ \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \]

(c) Find the phase of \( X(e^{j\omega}) \). Is it linear?

Use MATLAB to verify your results.

Answers: \( X(e^{j0}) = 9 \), \( X(e^{j\pi}) = 1 \), phase is linear.

Solution

(a) Using

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} \]

we have

\[ X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] = 1 + 2 + 3 + 2 + 1 = 9 \]

\[ X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} (-1)^n x[n] = 1 - 2 + 3 - 2 + 1 = 1 \]

(b) By Parseval’s result

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \sum_{n} |x[n]|^2 \]

we get that

\[ \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 2\pi(1 + 4 + 9 + 4 + 1) = 38\pi \]

(c) The Z–transform of \( x[n] \) is

\[ X(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4} = z^{-2}[z^2 + 2z^1 + 3 + 2z^{-1} + z^{-2}] \]

and the DTFT is

\[ X(e^{j\omega}) = e^{-j2\omega} \left[ \frac{3 + 2\cos(\omega) + 4\cos(2\omega)}{\text{real-valued}} \right] \]

so that the phase is \(-2\omega\) when above term is positive and \(-2\omega \pm \pi\) when negative. Some of the above results are verified using the following script.

```matlab
% Pr. 11_8
clear all;clf
x=[1 2 3 2 1];
N=256; X=fft(x,N);
```
Figure 11.3: Problem 8: Magnitude and phase responses for $x[n]$. 

```matlab
w=[0:N-1]*2*pi/N;
figure(1)
subplot(211)
plot(w,abs(X));axis([min(w) max(w) -0.1 10]);grid
ylabel('|X|')
subplot(212)
plot(w,unwrap(angle(X)));axis([min(w) max(w) -15 1]);grid
ylabel('<X');xlabel('\omega')
```
11.9 Linear equations and Fourier series — The Fourier series of a signal \( x[n] \) and its coefficients \( X_k \) are both periodic of some value \( N \) and as such can be written

\[
(i) \quad x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi nk/N} \quad 0 \leq n \leq N - 1
\]

\[
(ii) \quad X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \quad 0 \leq k \leq N - 1
\]

(a) To find the \( x[n], 0 \leq n \leq N - 1 \) given \( X_k, 0 \leq k \leq N - 1 \), write a set of \( N \) linear equations. Indicate how you would find the \( x[n] \) from the matrix equation. There is duality in the Fourier series and its coefficients, so see it consider the reverse problem: how would you solve for the \( X_k \) given the \( x[n] \)?

(b) Let \( x[n] = n \) for \( n = 0, 1, 2 \) and 0 for \( n = 3 \), be a period of a periodic signal \( x[n] \) of fundamental period \( N = 4 \), use the above method to solve for the Fourier series coefficients \( X_k, 0 \leq k \leq 3 \). Use MATLAB to find the inverse of the complex exponential matrix.

(c) Suppose that when computing the \( X_k \) for the \( x[n] \) signal given above, you separate the sum into 2 sums, one for the even values of \( n \), i.e., \( n = 0, 2 \) and the other for the odd values of \( n \), i.e., \( n = 1, 3 \). Write an equivalent matrix expression for the \( X_k \).

Solution

(a) Equation (ii) can be written in matrix form as

\[
X = Sx
\]

where \( X = [X_0 \ X_1 \cdots X_{N-1}]^T \) is an \( N \) vector containing the FS coefficients, \( x = [x[0]/N \ x[1]/N \cdots x[N-1]/N]^T \) are the signal values divided by \( N \), and the \( N \times N \) matrix \( S \) contains the complex exponentials. Then

\[
x = S^{-1}X
\]

Calling \( \hat{S} = S^{-1} \) we would then have

\[
x = \hat{S}X
\]

and then

\[
X = \hat{S}^{-1}x
\]

where \( \hat{SS} = I \), where \( I \) is the \( N \times N \) unit matrix, i.e., \( \hat{S} \) and \( S \) are orthonormal matrices.

(b) The following script computes the Fourier series coefficients from the first period of the periodic signal, and computes the first period using the Fourier series. Notice the way the matrix \( S \) is generated.

```
% Pr. 11_9
clear all; clf
x1=[0 1 2 0];
N=4;n=0:N-1;k=n;
S=exp(-j*2*pi.*(n'*k)/N)
Xk=(1/N)*S*x1'
x=inv(S)*N*Xk
```

\[
S = \begin{bmatrix}
1.0000 & 1.0000 & 1.0000 & 1.0000 \\
1.0000 & 0.0000 - 1.0000i & -1.0000 & 0.0000i \\
1.0000 & -0.0000 + 1.0000i & -0.0000 & 1.0000i \\
\end{bmatrix}
\]
(c) For \( N = 4 \) we can write the Fourier series coefficients multiplied by 4 as

\[
4X_k = x[0] + x[1]e^{-j2\pi k/4} + x[2]e^{-j2\pi 2k/4} + x[3]e^{-j2\pi 3k/4}
\]

where we have simplified some of the exponentials. Using that \( e^{-j2\pi k/2} = (-1)^k \) and \( e^{-j2\pi k/4} = (-j)^k \) we finally obtain

\[
X_k = 0.25 \{ x[0] + x[2](-1)^k \} + (-j)^k \{ x[1] + x[3](-1)^k \} \quad 0 \leq k \leq 3
\]

We thus have using the above results

\[
X = \frac{1}{4} S \begin{bmatrix} x[0] & x[1] & x[2] & x[3] \end{bmatrix}^T = \frac{1}{4} SP \begin{bmatrix} x[0] & x[2] & x[1] & x[3] \end{bmatrix}^T
\]

where \( \frac{1}{4}SP = S_1 \) and \( P \) is a permutation matrix

\[
S_1 = \frac{1}{4} S = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} P
\]

so that

\[
\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = S_1 \begin{bmatrix} x[0] \\ x[2] \\ x[1] \\ x[3] \end{bmatrix}
\]
11.10 DFT and IIR filters — A definite advantage of the FFT is that it reduces considerably the computation in the convolution sum. Thus if \( x[n], 0 \leq n \leq N - 1 \), is the input of an FIR filter with impulse response \( h[n], 0 \leq n \leq M - 1 \), their convolution sum \( y[n] = (x * h)[n] \) will be of length \( M + N - 1 \). Now if \( X[k] \) and \( H[k] \) are the DFTs (computed by the FFT) of \( x[n] \) and \( h[n] \), then \( Y[k] = X[k]H[k] \) is the DFT of the convolution sum, of length \( M + N - 1 \). To multiply the DFTs \( X[k] \) and \( H[k] \) they both should be of the same length as \( Y[k], \) i.e., \( M + N - 1 \). Consider what happen when the filter is IIR which has possibly an impulse response of very large length. Let

\[
y[n] = -1.755y[n-1] + 0.81y[n-2] = x[n] + 0.5x[n]
\]

be the difference equation representing an IIR filter with input \( x[n] \) and output \( y[n] \). Assume the initial conditions are zero, and the input is \( x[n] = u[n] - u[n-50] \). Use MATLAB to obtain the filter output.

(a) Compute using \textit{filter} the first 40 values of the impulse response \( h[n], \) and call them \( \tilde{h}[n], \) an approximate of \( h[n] \). Compute the filter output \( \tilde{y}[n] \) using the FFT as indicated above. In this case, we are approximating the IIR filter by and FIR filter of length 40. Plot the input and the output. Use FFTs of length 128.

(b) Suppose now that we do not want to approximate \( h[n], \) so consider the following procedure. Find the transfer function of the IIR filter, say \( H(z) = B(z)/A(z), \) and if \( X(z) \) is the \( Z \)-transform of the input then

\[
Y(z) = \frac{B(z)X(z)}{A(z)}
\]

Compute as before the FFT for \( x[n], \) of length 128, call it \( X[k], \) and compute the 128-length of the coefficients of \( B(z) \) and \( A(z) \) to obtain DFTs \( B[k] \) and \( A[k] \). Multiplying \( X[k] \) by \( B[k] \) and dividing by \( A[k], \) all of length 128, results in a sequence of length 128 that should correspond to \( Y[k], \) the DFT of \( y[n], \) Compute the inverse FFT to get \( y[n] \) and plot it.

(c) Use \textit{filter} to solve the difference equation and obtain \( y[n] \) for \( x[n] = u[n] - u[n-50] \). Considering this the exact solution, calculate the error with respect to the other responses in (a) and (b). Comment on your results.

\textbf{Answers:} \( h[n] \) decreases by \( n = 40 \).

\textbf{Solution}

(a) The impulse response of the IIR filter appears to taper down around \( n = 30 \) so the approximation of the impulse response seems good.

\begin{verbatim}
% Pr. 11_10
clear all; clf
x=ones(1,50);
b=[1 0.5 0]; a=[1 -1.755 0.81];
delta=[1 zeros(1,30)];
h=filter(b,a,delta);
N=128;
X=fft(x,N);H=fft(h,N);
Y=X.*H; y=real(ifft(Y));
figure(1)
subplot(311)
stem([x zeros(1,10)])
subplot(312)
\end{verbatim}
(b)(c) In this case we do not truncate the impulse response, and the results $y_1[n]$ are better than before $y[n]$, (the control is the solution obtained using filter, $y_c[n]$). See right figure in Fig. 11.4. The error is clearly seen when compared to the first result, but there is no visible error when compared to the second.

```matlab
% part 2
X=fft(x,N);A=fft(a,N);B=fft(b,N);
Y1=X.*B./A;
y1=ifft(Y1);
yc=filter(b,a,x);
figure(2)
subplot(211)
plot(y)
hold on
plot(yc,'r');grid
hold off
subplot(212)
plot(y1)
hold on
plot(yc,'r');grid
hold off
```

Figure 11.4: Problem 10: Input $x[n]$, impulse response $\hat{h}[n]$ and output $y[n]$ (left); right: exact response compared to solution when $h[n]$ is truncated (top) and when it is not (bottom)
11.11 Fourier transform of circular windows — Circular windows are of interest in the design of 2D-filters. Suppose that \( r = \sqrt{x^2 + y^2} \) and let the circular window be defined as

\[
  w_c(x, y) = \begin{cases} 
  1 & r < 1 \\
  0.5 & r = 1 \\
  0 & r > 1 
\end{cases}
\]

Its 2D-Fourier transform is given by

\[
  W(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_c(x, y) e^{-2\pi j (\omega_1 x + \omega_2 y)} dx dy
\]

Since the symmetry of the window is circular, express the above equation in terms of polar coordinates, i.e., let \( x + jy = re^{j\theta} \) and \( \omega_1 + j\omega_2 = qe^{j\phi} \).

(a) Using polar coordinates show that the 2D-Fourier transform becomes

\[
  W(q) = \int_{0}^{2\pi} \int_{0}^{\infty} w_c(r) \exp(-2\pi q (\cos(\theta - \phi))) r dr d\theta
\]

(b) Letting \( \rho = \theta - \phi \), show that the above equation for \( W(q) \) can be written

\[
  W(q) = \int_{0}^{\infty} r w_c(r) J_0(2\pi qr) dr
\]

where

\[
  J_0(2\pi qr) = \int_{0}^{2\pi} e^{-2\pi qr \cos(\rho)} d\rho
\]

is the zero-order Bessel function of the first kind.

(c) Replacing the circular window we finally obtain

\[
  W(q) = \int_{0}^{1} r J_0(2\pi qr) dr = \frac{1}{(2\pi q)^2} \int_{0}^{2\pi q} r' J_0(r') dr' = \frac{J_1(2\pi q)}{2\pi q}
\]

where \( J_1(.) \) is the first order Bessel function of the first kind.

(d) Use MATLAB to generate a circular discrete window \( w_c[m, n] \) and compute its 2D-FFT to get \( W_c(k, \ell) \). Determine what type of symmetry \( W_c(k, \ell) \) has. What does it look like?

Solution

```matlab
%Pr. 11_11
clear all; clf
N=125; [m,n] = freqspace(N,'meshgrid');
w_c = ones(N);
r = sqrt(m.^2 + n.^2); w_c(r>0.05)=0;
figure(1)
subplot(311)
colormap('gray')
```
contour(wc); title('Circular window')
Wc=fft2(wc)
subplot(312)
mesh(fftshift(abs(Wc))); title('Magnitude response')
subplot(313)
contour(fftshift(abs(Wc)),5); title('Contour of magnitude response')

Results are shown in Fig. 11.

Figure 11.5: Problem 11: Magnitude response of circular windows. Contour plot of circular window (top), magnitude response of window (middle) and its contour.
11.12 Separable and non-separable cosines — Consider the separable and non-separable cosine signals

\[ x_1[m,n] = \cos(0.3\pi m) \cos(0.2\pi n) \]
\[ x_2[m,n] = \cos(0.25\pi n + 0.5\pi m) \]

computed in a support \((0 \leq m \leq 123, 0 \leq n \leq 123)\).

(a) Use the MATLAB function \texttt{fft2} to compute the FFTs of these two functions, and plot their magnitude functions \(|X_i(\omega_1, \omega_2)|, \ i = 1, 2\) using the function \texttt{mesh} (let the frequencies \(\omega_1, \omega_2\) in the plots be in the domain \([-\pi, \pi] \times [-\pi, \pi]\)) and comment on their difference (use the function \texttt{contour} to help you).

(b) Given a signal

\[ x[m,n] = \begin{cases} 
1 & 0 \leq m, n \leq 99 \\
0 & 100 \leq m, n \leq 123 
\end{cases} \]

Generate modulated signals

\[ y_1[m,n] = x[m,n]x_1[m,n] \]
\[ y_2[m,n] = x[m,n]x_2[m,n] \]

and using \texttt{fft2} determine the magnitude functions \(|Y_i(\omega_1, \omega_2)|, i = 1, 2\) and comment on their differences.

Solution

```matlab
%Pr. 11_12
% 2D-FFT of separable, non-separable and modulated cosines
clear all; clf
% (a) separable and non-separable cosines
for m=1:124,
    for n=1:124,
        x1(m,n)=cos(0.2*pi*(n-1))*cos(0.3*pi*(m-1)); % separable
        x2(m,n)=cos(0.25*pi*(n-1)+0.5*pi*(m-1)); % non-separable
    end
end
X1=fft2(x1); X2=fft2(x2);
X1a=abs(fftshift(X1)); X2a=abs(fftshift(X2));
w1=-pi:2*pi/124:pi-2*pi/124;w1=w1/pi; w2=w1;
figure(1)
subplot(221)
mesh(w1,w2,X1a);
axis([min(w1) max(w1) min(w2) max(w2) 0 max(max(X1a))])
xlabel('\omega_1/\pi'), ylabel('\omega_2/\pi')
title('Magnitude spectrum of x1[m,n]')
subplot(222)
contour(w1,w2,X1a);grid
xlabel('\omega_1/\pi'), ylabel('\omega_2/\pi')
title('Contour plot of the magnitude spectrum')
subplot(223)
mesh(w1,w2,X2a); view(25,20);
```
\[ M = \max(\max(X2a)); \]
\[
\text{axis}([\min(w1) \hspace{1em} \max(w1) \hspace{1em} \min(w2) \hspace{1em} \max(w2) \hspace{1em} 0 \hspace{1em} M]) \\
xlabel(\text{\textbackslash omega \_1 /\pi}), \ ylabel(\text{\textbackslash omega \_2 /\pi}) \\
\text{title(\textquoteleft Magnitude spectrum of } x2[m,n]\textquoteleft) \]
\[
\text{subplot}(224) \\
\text{contour}(w1,w2,X2a); \text{grid} \\
xlabel(\text{\textbackslash omega \_1 /\pi}), \ ylabel(\text{\textbackslash omega \_2 /\pi}) \\
\text{title(\textquoteleft Contour plot of the magnitude spectrum\textquoteleft)}
\]

\% (b) modulated cosines
\[
x = \text{zeros}(124,124); \\
\text{for } m=1:100, \\
\text{\hspace{1em} for } n=1:100, \\
\text{\hspace{2em} } x(m,n)=1; \\
\text{end}
\]
\[
y1 = x*x1; \ y2 = x*x2; \\
Y1 = \text{fft2}(y1); \ Y1a = \text{abs}(\text{fftshift}(Y1)); \\
Y2 = \text{fft2}(y2); \ Y2a = \text{abs}(\text{fftshift}(Y2));
\]
\[
\text{figure}(2) \\
\text{subplot}(221) \\
\text{mesh}(w1,w2,Y1a); \\
\text{axis}([\min(w1) \hspace{1em} \max(w1) \hspace{1em} \min(w2) \hspace{1em} \max(w2) \hspace{1em} 0 \hspace{1em} \max(\max(Y1a))]) \\
xlabel(\text{\textbackslash omega \_1 /\pi}), \ ylabel(\text{\textbackslash omega \_2 /\pi}) \\
\text{title(\textquoteleft Magnitude spectrum of } y1[m,n]\textquoteleft) \\
\text{subplot}(222) \\
\text{contour}(w1,w2,Y1a); \text{grid} \\
xlabel(\text{\textbackslash omega \_1 /\pi}), \ ylabel(\text{\textbackslash omega \_2 /\pi}) \\
\text{title(\textquoteleft Contour plot of the } |Y_1(k,l)|\textquoteleft) \\
\text{subplot}(223) \\
\text{mesh}(w1,w2,Y2a); \\
\text{axis}([\min(w1) \hspace{1em} \max(w1) \hspace{1em} \min(w2) \hspace{1em} \max(w2) \hspace{1em} 0 \hspace{1em} \max(\max(Y2a))]) \\
xlabel(\text{\textbackslash omega \_1 /\pi}), \ ylabel(\text{\textbackslash omega \_2 /\pi}) \\
\text{title(\textquoteleft Magnitude spectrum of } y2[m,n]\textquoteleft) \\
\text{subplot}(224) \\
\text{contour}(w1,w2,Y2a); \text{grid} \\
xlabel(\text{\textbackslash omega \_1 /\pi}), \ ylabel(\text{\textbackslash omega \_2 /\pi}) \\
\text{title(\textquoteleft Contour plot of the } |Y_2(k,l)|\textquoteleft)
\]

Results are shown in Fig. 12.
Figure 11.6: Problem 12: Results of parts (a) separable and non-separable and (b) modulated.
11.13 **DCT vs DFT** — To see the advantage of the DCT over the DFT consider a one-dimensional edge having the values

\[ x = [8 \ 16 \ 24 \ 32 \ 40 \ 48 \ 4 \ 8] \]

Compute the FFT and DCT of length 8 and choose the first 4 values of the FFT and DCT sequences and set the other 4 to zero. Compute the IFFT and IDCT of the modified FFT and DCT sequences to obtain sequences that approximates the edge sequence. Plot the original sequence, and the approximate sequences obtained from the IFFT and IDCT and plot the approximation error. Find the mean square error corresponding to the two approximating sequences and determine which gives the smallest error. Why could you say that the FFT-based procedure has an ‘unfair’ advantage over the DCT-based procedure.

**Solution**

```matlab
% Pro 11.13
% comparison of DCT and FFT

x=[8 16 24 32 40 48 4 8];
X=fft(x);
Cx=dct(x);
X1=zeros(1,8); X1(1:4)=X(1:4);
x1=floor(real(ifft(X1)));
Cx1=zeros(1,8); Cx1(1:4)=Cx(1:4);
xx=floor(idct(Cx1));

figure(1)
subplot(211)
stem(x,'filled'); hold on; stem(x1,'red','filled'); hold on; plot(x);grid;hold on;plot(x1,'r'); hold off; title('DFT approximation')
subplot(212)
stem(x,'filled'); hold on; stem(xx,'red','filled');hold on; plot(x);grid; hold on plot(xx,'r'); hold off; title('DCT approximation')

figure(2)
subplot(211)
stem(x-x1,'filled');hold on; plot(x-x1);hold off;title('DFT error')
subplot(212)
stem(x-xx,'filled');hold on; plot(x-xx); hold off; title('DCT error')
edft=sum((x-x1).*(x-x1))/8
edct=sum((x-xx).*(x-xx))/8
```

The FFT-based procedure is using complex values and the DCT-based procedure real values, so the first is using twice as many values as the second one.
Figure 11.7: Problem 13: DCT vs DFT approximation and errors
Chapter 12

Introduction to the Design of Discrete Filters

12.1 Basic Problems

12.1 The frequency response of a filter is \( H(e^{j\omega}) = 1.5 + \cos(2\omega), \quad -\pi \leq \omega \leq \pi \).

(a) Is \( |H(e^{j\omega})| = H(e^{j\omega}) \)? Is phase zero?
(b) Find the impulse response \( h[n] \) of this filter. What type of filter is it? FIR? IIR? Is it causal? Explain.
(c) Let \( H_1(e^{j\omega}) = e^{-jN\omega}H(e^{j\omega}) \) determine the value of the positive integer \( N \) so that \( H_1(e^{j\omega}) \) has linear phase. Use the impulse response \( h[n] \) found above to find the impulse response \( h_1[n] \).

Answers: (a) Yes to both; (b) \( h[n] = 0.5\delta[n + 2] + 1.5 + 0.5\delta[n - 2] \).

Solution

(a) Yes, \( H(e^{j\omega}) > 0 \) and real with zero phase.
(b) Impulse response

\[
H(e^{j\omega}) = \frac{1}{h[0]} + \frac{0.5}{h[-2]} e^{j2\omega} + \frac{0.5}{h[2]} e^{-j2\omega}
\]

It can also be found using the IDTFT

\[
h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1.5 + \cos(2\omega)) e^{jn\omega} d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} 1.5e^{jn\omega} d\omega + \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(2\omega)e^{jn\omega} d\omega
\]
which is different from zero for \( n = 0 \) and \( n = \pm 2 \), and \( h[0] = 1.5 \) and \( h[-2] = h[2] = 0.5 \).

Indeed, then

\[
H(z) = 0.5z^2 + 1.5 + 0.5z^{-2}
\]

\[
H(e^{j\omega}) = 1.5 + \cos(2\omega)
\]

The filter is a non-causal FIR.

(c) Since \( H(e^{j\omega}) \) has zero phase, then \( H_1(e^{j\omega}) = e^{-jN\omega}H(e^{j\omega}) \) has linear phase for any \( N \geq 1 \), and \( h_1[n] = h[n - N] \).
12.2 An FIR filter has a transfer function \( H(z) = z^{-2}(z - e^{j\pi/2})(z - e^{-j\pi/2}) \).

(a) Find and plot the poles and zeros of this filter.
(b) Expressing the frequency response at some frequency \( \omega_0 \) as
\[
H(e^{j\omega_0}) = \frac{\vec{Z}_1(\omega_0)\vec{Z}_2(\omega_0)}{\vec{P}_1(\omega_0)\vec{P}_2(\omega_0)}
\]
carefully draw the vectors on the pole-zero plot.
(c) Consider the case when \( \omega_0 = 0 \), find \( H(e^{j0}) \) analytically and using the vectors.
(d) Repeat the above calculations for \( \omega_0 = \pi \) and \( \omega_0 = \pm \pi/2 \) (verify your result from the given Z-transform). Can you classify this filter?

Answers: \( H(e^{j0}) = 2; H(e^{j\pi}) = 2; H(e^{\pm j\pi/2}) = 0. \)

Solution
(a) Poles \( z = 0 \) double; zeros \( z_{1,2} = e^{\pm j\pi/2} = \pm j \).
(b) Vectors \( \vec{Z}_1(\omega_0) \) and \( \vec{Z}_2(\omega_0) \) go from the zeros to the frequency where we are finding the frequency response. Likewise, vectors \( \vec{P}_1(\omega_0) \) and \( \vec{P}_2(\omega_0) \) go from the poles to the frequency for which we are finding the frequency response.
(c) Analytically, \( |H(e^{j0})| = (1 - j)(1 + j)/1 = 2 \). Graphically
\[
|H(e^{j0})| = \frac{|\vec{Z}_1(0)||\vec{Z}_2(0)|}{(|\vec{P}_1(0)|^2} = \frac{\sqrt{2}\sqrt{2}}{1} = 2
\]
and the phase would be
\[
\angle\vec{Z}_1(0) + \angle\vec{Z}_2(0) - 2\angle\vec{P}_1(0) = (2\pi - \phi) + \phi + 0 = 2\pi
\]
so \( H(e^{j0}) = 2. \) Notice that when \( \omega = 0 \) then \( H(e^{j0}) = H(z)|_{z=1} \).

![Figure 12.1: Problem 2: Poles and zeros and vectors to \( \omega = 0 \) to find \( H(e^{j0}) \).](image)

(d) Magnitude response: analytically and graphically we have
\[
H(e^{j\pi/2}) = 0 \quad H(e^{j\pi}) = 2e^{j0} = 2
\]
The filter is a notch filter with notch frequency \( \pi/2 \).
12.3 The transfer function of an FIR filter is \( H(z) = z^{-2}(z - 2)(z - 0.5) \).

(a) Find the impulse response \( h[n] \) of this filter and plot it. Comment on any symmetries it might have.

(b) Find the phase \( \angle H(e^{j\omega}) \) of the filter and carefully plot it. Is this phase linear for all frequencies \( -\pi < \omega \leq \pi \)?

**Answers:** \( h[n] = \delta[n] - 2.5\delta[n - 1] + \delta[n - 2] \), phase is linear.

**Solution**

(a) Transfer function can be written as

\[
H(z) = \frac{1}{h[0]} + \left( -\frac{2.5}{h[1]} \right) z^{-1} + \frac{1}{h[2]} z^{-2}
\]

thus the impulse response is

\[
h[n] = \delta[n] - 2.5\delta[n - 1] + \delta[n - 2]
\]

(b) Frequency response

\[
H(e^{j\omega}) = e^{-j\omega}(e^{j\omega} - 2.5 + e^{-j\omega})
\]

\[
= e^{-j\omega}(2\cos(\omega) - 2.5)
\]

\[
= e^{j(-\omega \pm \pi)} \left( -2\cos(\omega) + 2.5 \right) \quad >0
\]

so the phase is

\[
\angle H(e^{j\omega}) = -\omega \pm \pi
\]

or linear. This corresponds to the impulse response being symmetric about \( n = 1 \) and having no zeros on the unit circle.
12.4 Consider the following problems related to the specification of IIR filters

(a) The magnitude specifications for a low-pass filter are

\[ 1 - \delta \leq |H(e^{j\omega})| \leq 1 \quad 0 \leq \omega \leq 0.5\pi \]

\[ 0 < |H(e^{j\omega})| \leq \delta \quad 0.75\pi \leq \omega \leq \pi \]

i. Find the value of \( \delta \) that gives the following equivalent loss specifications for this filter

\[ \alpha(e^{j0}) = 0 \text{ dB}, \quad \alpha_{max} = 0.92 \text{ dB}, \quad \alpha_{min} = 20 \text{ dB} \]

ii. What are the values of \( \omega_p \) and \( \omega_{st} \)?

(b) The following are specifications for a low-pass discrete IIR filter that will be used in processing analog signals

\[ 10 \leq \alpha(e^{j\omega}) \leq 10.1 \text{ dBs} \quad 0 \leq f \leq 2 \text{ kHz} \]

\[ \alpha(e^{j\omega}) \geq 60 \text{ dBs} \quad 4 \text{ kHz} \leq f \leq 5 \text{ kHz} \]

i. Find the values of \( \alpha_{max} \), \( \alpha_{min} \) and the dc loss in dBs.

ii. What is the sampling frequency \( f_s \) in kHz in this filter specifications?

iii. Obtain the equivalent specifications in discrete frequencies for 0 dB dc loss for this filter.

**Answers:** (a) \( \delta = 0.1 \); \( \omega_p = 0.5\pi \); (b) \( \alpha_{max} = 0.1 \), \( \alpha_{min} = 50 \text{ dB} \) and 10 dB dc loss.

**Solution**

(a) i. From \( \alpha_{min} = 20 = -20\log_{10}(\delta) \) then \( \delta = 10^{-1} = 0.1 \) and it can be verified that \( \alpha_{max} = 0.92 = -20\log_{10}(1 - \delta) = -20\log_{10}0.9 \).

ii. \( \omega_p = 0.5\pi \) and \( \omega_{st} = 0.75\pi \)

(b) i. The dc loss \( \alpha(e^{j0}) = 10 \text{ dB} \), and subtracting it we get that

\[ \alpha_{max} = 0.1 \text{ dB} \]

\[ \alpha_{min} = 50 \text{ dB}. \]

ii. The maximum frequency \( 5 \text{ kHz} \leq f_s/2 \), so the sampling frequency is \( f_s \geq 10 \text{ kHz} \).

iii. Using \( \omega = 2\pi f/f_s \) and defining \( \hat{\alpha}(e^{j\omega}) = \alpha(e^{j\omega}) - 10 \) we get

\[ 0 \leq \hat{\alpha}(e^{j\omega}) \leq 0.1 \text{ dBs} \quad 0 \leq \omega \leq 0.4\pi \]

\[ \hat{\alpha}(e^{j\omega}) \geq 50 \text{ dBs} \quad 0.8\pi \leq \omega \leq \pi \]

with dc loss of 0 dB.
12.5 A low-pass IIR discrete filter has a transfer function

\[ H(z) = \sum_{n=0}^{\infty} 0.5^n z^{-n} \]

(a) Find the poles and zeros of this filter.

(b) Suppose that you multiply the impulse response of the low-pass filter by \((-1)^n\) so that you obtain a new transfer function

\[ H_1(z) = \sum_{n=0}^{\infty} (-0.5)^n z^{-n} \]

find the poles and zeros for \(H_1(z)\). What type of filter is it?

(c) Is it true, in general, that if every \(z\) in a low-pass filter transfer function is changed into \(-z\) you obtain a high-pass filter? Explain.

**Answers:** \(H(z) = z/(z - 0.5); H_1(z)\) is high-pass filter.

**Solution:** (a), (b) In rational form

\[ H(z) = \sum_{n=0}^{\infty} 0.5^n z^{-n} = \frac{1}{1 - 0.5z^{-1}} = \frac{z}{z - 0.5} \]

with a zero at \(z = 0\) and a pole at \(z = 0.5\). The filter is a low-pass filter.

\[ H_1(z) = \sum_{n=0}^{\infty} (-0.5)^n z^{-n} = \frac{1}{1 + 0.5z^{-1}} = \frac{z}{z + 0.5} \]

with zero at \(z = 0\) and pole at \(z = -0.5\). The filter is a high-pass filter, as the frequency response of \(h_1[n] = e^{j\pi n}h[n]\) so that \(H_1(e^{j\omega}) = H(e^{j(\omega - \pi)})\) which is the low-pass filter shifted to \(\pi\).

(c) Suppose

\[ H(Z) = \sum_n h[n]Z^{-n} \]

is the transfer function of a LPF, if we let \(Z = -z = (-1)z\) we obtain

\[ \sum_n h[n][(-1)z]^{-n} = \sum_n (h[n](-1)^n)z^{-n} \]

so that the resulting filter has \(h[n](-1)^n\) as impulse response, therefore it is high-pass.
12.6 Consider the following transfer function:

\[ H(z) = \frac{2(z - 1)(z^2 + \sqrt{2}z + 1)}{(z + 0.5)(z^2 - 0.9z + 0.81)} \]

(a) Develop a cascade realization of \( H(z) \) using a first-order and a second-order sections. Use minimal direct form to realize each of the sections.

(b) Develop a parallel realization of \( H(z) \) by considering first and second-order sections, each realized using minimal direct form.

**Answers:** Parallel \( H(z) = -4.94 + 2.16/(1 + 0.5z^{-1}) + (4.78 - 1.6z^{-1})/(1 - 0.9z^{-1} + 0.81z^{-2}) \)

**Solution:** (a) Let

\[ H(z) = \frac{2(1 - z^{-1})}{1 + 0.5z^{-1}} \frac{1 + \sqrt{2}z^{-1} + z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}} \]

(b) To obtain the parallel realization we do a partial fraction expansion of \( H(z) \). Notice that \( H(z) \) is not proper rational and so we have to obtain a constant term before performing the partial fraction expansion

\[ H(z) = -4.94 + \frac{2.16}{1 + 0.5z^{-1}} + \frac{4.78 - 1.6z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}} \]
Figure 12.3: Problem 6: Parallel realization of $H(z)$. 

$H(z) = -4.94 + 2.16 z^{-1} - 0.5 + 4.78 z^{-1} + 0.9 z^{-1} - 1.6 z^{-1} - 0.81$
12.2 Problems using MATLAB

12.7 FIR filters: causality and phase — A three-point moving–average filter is of the form:

\[ y[n] = \beta (\alpha x[n-1] + x[n] + \alpha x[n+1]) \]

where \(\alpha\) and \(\beta\) are constants, and \(x[n]\) is the input and \(y[n]\) is the output of the filter.

(a) Determine the transfer function \(H(z) = \frac{Y(z)}{X(z)}\) of the filter and from it find the frequency response \(H(e^{j\omega})\) of the filter in terms of \(\alpha\) and \(\beta\).

(b) Let \(\alpha = 0.5\), find then \(\beta\) so that the d.c. gain of the filter is unity, and the filter has a zero phase. For the given and obtained values of \(\alpha\) and \(\beta\), sketch \(H(e^{j\omega})\) and find the poles and zeros of \(H(z)\) and plot them in the Z-plane.

(c) Suppose we let \(v[n] = y[n-1]\) be the output of a second filter. Is this filter causal? Find its transfer function \(G(z) = \frac{V(z)}{X(z)}\). Use MATLAB to compute the unwrapped phases of \(G(z)\) and to plot the poles and zeros of \(G(z)\) and \(H(z)\) and explain the relation between \(G(z)\) and \(H(z)\).

Answers: \(H(e^{j\omega}) = \beta(1 + 2\alpha \cos(\omega)); \beta = 0.5\)

Solution: (a) The transfer function is

\[ H(z) = \frac{Y(z)}{X(z)} = \beta(\alpha z^{-1} + 1 + \alpha z) \]

letting \(z = e^{j\omega}\)

\[ H(e^{j\omega}) = \beta(\alpha e^{-j\omega} + 1 + \alpha e^{j\omega}) = \beta(1 + 2\alpha \cos(\omega)) \]

(b) The dc gain is

\[ H(e^{j0}) = \beta(1 + 2\alpha) = 1 \]

so that

\[ \beta = \frac{1}{1 + 2\alpha} \]

For the phase to be zero \(H(e^{j\omega}) \geq 0\) and so we could let \(\beta > 0\), and \(1 + 2\alpha \cos(\omega) \geq 0\). The last equation requires that for all \(\omega\) the amplitude of the \(\cos(\omega)\) be \(2\alpha \leq 1\) or \(\alpha \leq 1/2\).

If we let \(\alpha = 0.5\) then \(\beta = 0.5\) and

\[ H(e^{j\omega}) = 0.5(1 + \cos(\omega)) \quad -\pi \leq \omega < \pi \]

having a unit dc and zero phase as \(H(e^{j\omega}) \geq 0\).

For \(\alpha = \beta = 0.5\), the transfer function is

\[ H(z) = 0.5(0.5z^{-1} + 1 + 0.5z) = 0.25z^{-1}(1 + 2z + z^2) = 0.25\frac{(1 + z)^2}{z} \]

so pole \(z = 0\) and double zero \(z = -1\).

(c) We have that \(v[n] = y[n-1] = \beta(\alpha x[n-2] + x[n-1] + \alpha x[n])\) so the new filter is causal. The previous filter was non-causal. The transfer function

\[ G(z) = \frac{V(z)}{X(z)} = z^{-1}H(z) = \frac{0.25(z + 1)^2}{z^2} \]
with double poles at $z = 0$ and double zero at $z = -1$. Since $H(z)$ is zero phase, $G(z)$ has linear phase $\angle G(e^{j\omega}) = -\omega$.

The following script computes and plots the frequency responses of the two filters.

```matlab
% Pr. 12.7
clear all; clf
N=128; w=[0:N/2-1]*2*pi/N;
H=0.5*(1+cos(w));
figure(1)
subplot(221)
plot(w, abs(H)); axis([0 pi 0 1]); ylabel('|H|')
subplot(222)
plot(w,angle(H));axis([0 pi -0.1 0.1]); ylabel('<H')
beta=0.5;alpha=0.5;
b=[0.25 0.5 0.1]
[G,w]=freqz(b,1);
subplot(223)
plot(w,abs(G)); axis([0 pi 0 1]); ylabel('|G|')
subplot(224)
plot(w,unwrap(angle(G))); axis([0 pi -pi 0]); ylabel('<G'); xlabel('

Figure 12.4: Problem 7: Frequency responses of $H(e^{j\omega})$ (top) and $G(e^{j\omega})$ (bottom).
```
12.8 Effect of phase on filtering — Consider two filters with transfer functions

\[(i) \quad H_1(z) = z^{-100}, \quad (ii) \quad H_2(z) = \left(\frac{1 - 2z^{-1}}{1 - 0.5z^{-1}}\right)^{10}\]

(a) The magnitude response of these two filters is unity, but that they have different phases. Find analytically the phase of \(H_1(e^{j\omega})\) and use MATLAB to find the unwrapped phase of \(H_2(e^{j\omega})\) and to plot it.

(b) Consider the MATLAB signal handel.mat (a short piece of the Messiah by composer George Handel), use the MATLAB function filter to filter it with the two given filters. Listen to the outputs, plot them and compare them. What is the difference (look at the first 200 samples of the outputs from the two filters)?

(c) Can you recover the original signal by advancing either of the outputs? Explain.

Answers: Use MATLAB function conv to find coefficients \(H_2(z)\).

Solution: (a) Clearly \(H_1(z)\) has unit magnitude. \(H_2(z)\) is all-pass filter: let \(H_2(z) = (F(z))^{10}\),

\[|F(e^{j\omega})|^2 = \left(\frac{0.5 - 2e^{-j\omega}}{1 - 0.5e^{-j\omega}}\right) \left(\frac{0.5 - 2e^{j\omega}}{1 - 0.5e^{j\omega}}\right) = \left(\frac{0.5 - 2e^{j\omega}(1 - 0.5e^{-j\omega})}{1 - 0.5e^{j\omega}(1 - 2e^{-j\omega})}\right) = 1\]

so \(|H_2(e^{j\omega})| = |F(e^{j\omega})|^{10} = 1\). Since \(H_2(z)\) is the product of \(F(z)\) ten times, its coefficients can be computed using the function conv to find the coefficients.

(b) The following script computes the coefficients of the filters and their frequency responses and then processes handel.

```matlab
% Pr. 12.8
clear all; clf
b1=[zeros(1,99) 1];
b=[0.5 -1];a=[1 -0.5];
b2=b;a2=a;
for k=2:10; % computing coefficients of H_2(z)
    b2=conv(b,b2);
    a2=conv(a,a2);
end
[H1,w]=freqz(b1,1);
[H2,w]=freqz(b2,a2);
figure(1)
subplot(221)
plot(w,abs(H1));ylabel('|H1|');axis([0 pi 0 1.2])
subplot(222)
plot(w,unwrap(angle(H1)));ylabel('<H1');axis([0 pi -320 0])
subplot(223)
plot(w,abs(H2));ylabel('|H2|');axis([0 pi 0 1.2])
subplot(224)
plot(w,unwrap(angle(H2)));ylabel('<H1');axis([0 pi -32 0])

load handel; x=y';
y1=filter(b1,1,x);
y2=filter(b2,a2,x);
figure(2)
```
The output of $H_1(z)$ is the input signal delayed 100 samples, the output of $H_2(z)$ is not due to its phase not being exactly linear, although the filter is all-pass.

(c) If the phase is linear advancing the signal 100 samples gives the original signal, while that is not possible for the non-linear phase.

Figure 12.5: Problem 8: Magnitude and phase of filters (left). Part of handel signal (top right) and outputs of linear–phase and non–linear phase filters.
12.9 Bilinear transformation and pole location — Find the poles of the discrete filter obtained by applying the bilinear transformation with \( K = 1 \) to an analog second–order Butterworth low-pass filter. Determine the half-power frequency \( \omega_{hp} \) of the resulting discrete filter. Use the MATLAB function \texttt{bilinear} to verify your results.

**Answers:** Double zero at \( z = -1 \) and poles at \( z_{1,2} = \pm j \sqrt{(2 - \sqrt{2})/(2 + \sqrt{2})} \).

**Solution:** The second order low-pass filter (normalized in frequency) is

\[
H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}
\]

Applying the bilinear transformation \( s = (1 - z^{-1})/(1 + z^{-1}) \) we obtain the following discrete filter

\[
H(z) = \frac{(1 + z^{-1})^2}{(1 - z^{-1})^2 + \sqrt{2}(1 - z^{-2}) + (1 + z^{-1})^2} = \frac{(z + 1)^2}{(2 + \sqrt{2})(z^2 + (2 - \sqrt{2})/(2 + \sqrt{2}))}
\]

so we have a double zero at \( z = -1 \) and poles at \( z_{1,2} = \pm j \sqrt{(2 - \sqrt{2})/(2 + \sqrt{2})} \).

We have that \( K_b \) transforms the normalized half-power analog frequency \( \Omega_{hp} = 1 \) into the normalized half-power frequency \( \omega_{hp} \) of the discrete filter, so that

\[
K_b = 1 = \frac{1}{\tan(\omega_{hp}/2)} \quad \Rightarrow \quad \omega_{hp} = \pi/2
\]

The following script verifies the analytic results

```matlab
% Pr. 12.9
clear all;clf
n=1;d=[1 sqrt(2) 1];
[b,a]=bilinear(n,d,0.5)
[H,w]=freqz(b,a)
figure(1)
subplot(211)
zplane(b,a)
subplot(212)
plot(w/pi,abs(H));grid;ylabel('|H(e^{j\omega})|'); xlabel('\omega/\pi')
```

The obtained coefficients are

\[
\begin{align*}
b &= 0.2929 & 0.5858 & 0.2929 \\
a &= 1.0000 & 0.0000 & 0.1716
\end{align*}
\]
Figure 12.6: Problem 9: Poles/zeros of $H(z)$ with $\omega_{hp} = \pi/2$. 
12.10 **Discrete Butterworth filter for analog processing** — Design a Butterworth low-pass discrete filter that satisfies the following specifications

\[ 0 \leq \alpha(e^{j\omega}) \leq 3 \text{ dB for } 0 \leq f \leq 25 \text{ Hz} \]
\[ \alpha(e^{j\omega}) \geq 38 \text{ dB for } 50 \leq f \leq Fs/2 \text{ Hz} \]

and the sampling frequency is \( F_s = 2000 \text{ Hz} \). Express the transfer function \( H(z) \) of the designed filter as a cascade of filters. Use first the design formulas and then use MATLAB to confirm your results. Show that the designed filter satisfies the specifications, plotting the loss function of the designed filter.

**Answers:** \( N = 7; H(z) = G \prod_{i=1}^{7} H_i(z); H_1(z) = 0.04(1 + z^{-1})/(1 - 0.92z^{-1}) \).

**Solution:** We have that \( T_s = 1/F_s = 0.5 \times 10^{-3} \text{ sec/sample}, so } K_b = \cot(0.5\omega_{hp}) = \cot(2\pi f_{hp}/(2F_s)) = 25.45. \) In this case, we use the loss function for the Butterworth to find the order

\[ \alpha(e^{j\omega_{st}}) = 10 \log [1 + (K_b \tan(2\pi f_{st}/(2F_s)))]^{2N} \geq 38 \]

The minimum order is

\[ N \geq \frac{\log(10^{3.8} - 1)}{2\log(0.078K_b)} \approx 6.298 \]

so that \( N = 7 \).

The frequency normalized analog filter in factorized form is

\[ H_7(s) = \frac{G}{(s + 1)(s^2 + 1.8s + 1)(s^2 + 1.25s + 1)(s^2 + 0.45s + 1)} = GH_1(s)H_2(s)H_3(s)H_4(s) \]

We have then

\[ H_1(z) = \frac{1}{s + 1}|_{s=K_b(1-z^{-1})/(1+z^{-1})} = \frac{(1/(K_b - 1)(1 + z^{-1}))}{(K_b + 1)/(K_b - 1) - z^{-1}} \]

For the others which are second-order we have

\[ H_i(z) = H_i(s)|_{s=K_b(1-z^{-1})/(1+z^{-1})} = \frac{1}{s^2 + (1/Q_i)s + 1|_{BT}} \]

\[ = \frac{(1 + z^{-1})^2}{(K_b^2 + K_b/Q_i + 1) - 2(K_b^2 - 1)z^{-1} + (K_b^2 - K_b^2/Q_i + 1)z^{-2}} \]

for \( i = 2, 3, 4 \) and \( Q_2 = 1/1.8, Q_3 = 1/1.25, \) and \( Q_4 = 1/0.45 \). Replacing \( K_b \) and \( Q_i \) we get

\[ H_1(z) = \frac{0.04(1 + z^{-1})}{1 - 0.92z^{-1}} \]
\[ H_2(z) = \frac{1.44 \times 10^{-3}(1 + z^{-1})^2}{1 - 1.86z^{-1} + 0.87z^{-2}} \]
\[ H_3(z) = \frac{1.47 \times 10^{-3}(1 + z^{-1})^2}{1 - 1.9z^{-1} + 0.91z^{-2}} \]
\[ H_4(z) = \frac{1.51 \times 10^{-3}(1 + z^{-1})^2}{1 - 1.96z^{-1} + 0.91z^{-2}} \]

To get \( H_7(e^{j\omega}) = 1 \) so that the dc loss is 0 db,

\[ GH_1(1)H_2(1)H_3(1)H_4(1) = 1 \]

which gives \( G = 1. \)
% Pr. 12.10
N = 7; fhp = 25; Fs = 2000;
whp = 2*pi*fhp/Fs;
[b, a] = butter(N, whp/pi);
% analytic results
% coefficients of H1
b1 = [0.04, 0.04]; a1 = [1, -0.92];
% coefficients of H2
b2 = 1.44e-3*[1, 2, 1]; a2 = [1, -1.86, 0.87];
% coefficients of H3
b3 = 1.47e-3*[1, 2, 1]; a3 = [1, -1.9, 0.91];
% coefficients of H4
b4 = 1.51e-3*[1, 2, 1]; a4 = [1, -1.96, 0.91];
% comparison of analytic and MATLAB results
analyticalB = conv(conv(conv(b1, b2), b3), b4)
matlabB = b
analyticalA = conv(conv(conv(a1, a2), a3), a4)
matlabA = a

The comparison of analytic and computational results gives

analyticalB = 1.0e-08 *
0.0128 0.0895 0.2685 0.4475 0.4475 0.2685 0.0895 0.0128
matlabB = 1.0e-08 *
0.0121 0.0848 0.2544 0.4240 0.4240 0.2544 0.0848 0.0121

analyticalA = 1.0000 -6.6400 18.8560 -29.6888 27.9928 -15.8063 4.9490 -0.6628
matlabA = 1.0000 -6.6471 18.9442 -30.0084 28.5329 -16.2848 5.1656 -0.7025
12.11 Butterworth filtering of analog signal — We wish to design a discrete Butterworth filter that can be used in filtering a continuous-time signal. The frequency components of interest in this signal are between 0 and 1 kHz, so we would like the filter to have a maximum passband attenuation of 3 dB within that band. The undesirable components of the input signal occur beyond 2 kHz, and we like to attenuate them by at least 10 dB. The maximum frequency present in the input signal is 5 kHz. Finally, we would like the d.c. gain of the filter to be 10. Choose the Nyquist sampling frequency to process the input signal. Use MATLAB to design the filter. Give the transfer function of the filter, plot its poles and zeros, and its magnitude and unwrapped phase response using an analog frequency scale in kHz.

**Answers:** Low-pass, with $f_p = 1$ kHz ($f_p = f_{hp}$), $\alpha_{max} = 3$ dB, $f_{st} = 2$ kHz, $\alpha_{min} = 10$ dB.

**Solution:** The desired filter is a low-pass, with $f_p = 1$ kHz, $\alpha_{max} = 3$ dB, so that $f_p = f_{hp}$; $f_{st} = 2$ kHz, $\alpha_{min} = 10$ dB, and d.c. gain of 10 after the design with d.c. of 0 dB. Since $f_{max} = 5$ kHz, then we can choose $F_s = 10$ kHz. The coefficients of the resulting filter are shown below.

```matlab
% Pr. 12.11
Fs=10000;fp=1000;fst=2000;
wp=2*pi*fp/Fs/pi; ws=2*pi*fst/Fs/pi;
alphamax=3; alphamin=10;
[N,wn]=buttord(wp,ws,alphamax,alphamin)
[b,a]=butter(N,wp); % to guarantee that wp is half-power frequency (wn\ne wp)
G=10*sum(b)/sum(a);
b=G*b;
[H,w]=freqz(b,a);
figure(1)
subplot(221)
zplane(b,a)
subplot(222)
plot(w/pi*Fs/2, abs(H));grid; axis([0 Fs/2 -0.1 11])
subplot(223)
plot(w/pi*Fs/2,unwrap(angle(H)));grid; axis([0 Fs/2 -4 0])

b = 0.6746 1.3491 0.6746
a = 1.0000 -1.1430 0.4128
```

![Figure 12.7: Problem 11: Poles/zeros and magnitude and phase responses.](image-url)
12.12 Butterworth vs Chebyshev filtering — If we wish to preserve low frequencies components of the input, a low-pass Butterworth filter could perform better than a Chebyshev filter. MATLAB provides a second Chebyshev filter function `cheby2` that has a flat response in the passband and a rippled one in the stopband. Let the signal to be filtered be the first 100 samples from MATLAB’s `train` signal. To this signal add some Gaussian noise to be generated by `randn`, multiply it by 0.1 and add it to the 100 samples of the train signal. Design three discrete filters, each of order 20, and half-frequency (for Butterworth butter) and the passband frequency (for the Chebyshev filters) of $\omega_n = 0.5$. For the design with `cheby1` let the maximum passband attenuation be 0.01 dB and for the design with `cheby2` let the minimum stopband attenuation be 60 dB. Obtain the three filters and use them to filter the noisy train signal.

Using MATLAB plot the following for each of the 3 filters:

- Using the `fft` function compute the DFT of the original signal, the noisy signal and the noise, and plot their magnitudes. Is the cutoff frequency of the filters adequate to get rid of the noise? Explain.
- Compute and plot the magnitude and the unwrapped phase, as well as the poles and zeros for each of the three filters. Comment on the differences in the magnitude responses.
- Use the function `filter` to obtain the output of each of the filters, and plot the original noiseless signal and the filtered signals. Compare them.

**Solution:** The following script designs the desired filters, computes and plots the output of the filtered signals and different spectra:

```matlab
% Pr. 12.12
clear all; clf
load train; y=y'; N=256;
noise=0.1*randn(1,N);
yc=y(1:N);
yn=y+noise; % noisy signal
[b,a]=butter(20,0.5); % butterworth
[b,a]=cheby2(20,60,0.5); % cheby2
[b,a]=cheby1(20,0.01,0.5) % cheby1
figure(1)
[H,w]=freqz(b,a);
b=b/H(1);
[H,w]=freqz(b,a); % unit dc gain
subplot(311)
plot(w/pi, abs(H)); grid
hold on
plot(w/pi,0.707*ones(1,length(w)),'r')
hold off
subplot(312)
plot(w/pi,unwrap(angle(H)))
subplot(313)
zplane(b,a)

k=0:N-1;wl=k*2*pi/N;wl=wl/pi-1;
Ns=fftshift(abs(fft(noise))); % noise spectrum
Yc=fftshift(abs(fft(yc))); % clean signal spectrum
Yn=fftshift(abs(fft(yn))); % noisy signal spectrum
```
Figure 12.8: Problem 12: Chebyshev 2 case: Filter frequency response and poles zeros (top left), spectra of different signals, signals (bottom)
12.13 Notch and all-pass filters — Notch filters is a family of filters that include the all-pass filter. For the filter
\[
H(z) = K \frac{(1 - \alpha_1 z^{-1})(1 + \alpha_2 z^{-1})}{(1 - 0.5 z^{-1})(1 + 0.5 z^{-1})}
\]
(a) Determine the values of \(\alpha_1, \alpha_2\) and \(K\) that would make \(H(z)\) an all-pass filter of unit magnitude. Use MATLAB to compute and plot the magnitude response of \(H(z)\) using the obtained values for \(\alpha\) and \(K\). Plot the poles and zeros of this filter.
(b) If we would like the filter \(H(z)\) to be a notch filter, of unit gain at \(\omega = \pi/2\) (rad), and notches at \(\omega = 0\) and \(\pi\), determine the values of \(\alpha\) and \(K\) to achieve this. Use MATLAB functions to verify that the filter is a notch filter, and to plot the poles and zeros.
(c) Use MATLAB to show that when \(\alpha_1 = \alpha_2 = \alpha\) and \(1 \leq \alpha \leq 2\), the given \(H(z)\) correspond to a family of notch filters with different attenuations. Determine \(K\) so that these filters are unity gain at \(\omega = \pi/2\).
(d) Suppose we use the transformation \(z^{-1} = jZ^{-1}\) to obtain a filter \(H(Z)\) using \(H(z)\) obtained in the previous item. Where are the notches of this new filter? What is the difference between the all-filters \(H(z)\) and \(H(Z)\)?

Answers: \(H(z) = K(1 - \alpha^2 z^{-2})/(1 - 0.25 z^{-2})\), \(1 \leq \alpha \leq 2\), notch filters with notches at \(\omega = 0, \pi\).

Solution: (a) For the filter to be all-pass the zeros must be the inverse of the poles 0.5 and -0.5 or 2 and -2. Thus
\[
H(z) = K \frac{(1 - 2z^{-1})(1 + 2z^{-1})}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} = K \frac{(1 - 4z^{-2})}{(1 - 0.25z^{-2})} \quad K > 0
\]
As an all-pass \(H(z)\) has a constant magnitude (at any frequency) and in this case we want it to be unity. For instance, for this filter to have unit gain at \(\omega = \pi/2\) we need that
\[
|H(e^{j\pi/2})| = K \left| \frac{1 - 4j^2}{1 - 0.25j^2} \right| = K \left| \frac{5}{1.25} \right| = 1 \quad \Rightarrow \quad K = \frac{1.25}{5} = 0.25
\]
(b) The zeros of \(H(z)\) are \(\alpha_1\) and \(-\alpha_2\) which are real. For \(H(z)\) to be a notch filter \(|\alpha_i| = 1\), and so we let \(\alpha_1 = \alpha_2 = 1\) for notches at \(\omega = 0, \pi\). Thus
\[
H(z) = K \frac{(1 - z^{-1})(1 + z^{-1})}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} = K \frac{(1 - z^{-2})}{(1 - 0.25z^{-2})}
\]
and for it to have unit gain at \(\omega = \pi/2\)
\[
|H(e^{j\pi/2})| = K \left| \frac{1 - j^2}{1 - 0.25j^2} \right| = K \left| \frac{2}{1.25} \right| = 1 \quad \Rightarrow \quad K = \frac{1.25}{2} = 0.25
\]
(c) In general, the filter
\[
H(z) = K \frac{(1 - \alpha^2 z^{-2})}{(1 - 0.25z^{-2})}
\]
for \(1 \leq \alpha \leq 2\) is a family of notch filters with notches at \(\omega = 0, \pi\), including an all-pass filter when \(\alpha = 2\). The gain \(K > 0\) is
\[
|H(e^{j\pi/2})| = K \left| \frac{1 - \alpha^2j^2}{1 - 0.25j^2} \right| = K \left| \frac{1 + \alpha^2}{1.25} \right| = 1 \quad \Rightarrow \quad K = \frac{1.25}{1 + \alpha^2}
\]
(d) If we let \( z^{-1} = jZ^{-1} \) the filter becomes

\[
H(Z) = K \frac{(1 - \alpha^2 j^2 Z^{-2})}{(1 - 0.25 j^2 Z^{-2})} = K \frac{(1 + \alpha^2 Z^{-2})}{(1 + 0.25 Z^{-2})}
\]

with poles at \( \pm 0.5e^{j\pi/2} \) and zeros at \( \pm \alpha e^{j\pi/2} \), i.e., shifted 90° with respect to the ones before. For values of \( 1 \leq \alpha \leq 2 \), \( H(Z) \) is a family of notch filters at \( \omega = \pi/2 \), including an all-pass filter when \( \alpha = 2 \). When running the following script make the numerator and the denominator correspond to \( H(z) \) or to \( H(Z) \) by deleting the comment symbol.

```matlab
% Pr. 12.13
clear all;clf
a=[1 0 -0.25]; % denominator of H(z)
a=[1 0 0.25]; % denominator of H(Z)
figure(1)
for alpha=2:-0.1:1;
    alpha
    K=1.25/(1+alpha^2)
    b=[K 0 -(alpha^2)*K]; % numerator of H(z)
    b=[K 0 (alpha^2)*K]; % numerator of H(Z)
    [H,w]=freqz(b,a);
    subplot(211)
    zplane(b,a); axis([-2 2 -2.5 2.5])
    hold on
    subplot(212)
    plot(w/pi,abs(H));axis([0 1 0 1.5])
    hold on
    pause(1)
end
hold off
```
Figure 12.9: Problem 13: Loci of poles and zeros and magnitude responses of $H(z)$ (top) and of $H(Z)$ (bottom).
12.14 Modulation property transformation for IIR filters — The modulation-based frequency transformation of the DTFT is applicable to IIR filters. It is obvious in the case of FIR filters, but requires a few more steps in the case of IIR filters. In fact, if we have that the transfer function of the prototype IIR low-pass filter is \( H(z) = B(z)/A(z) \), with impulse response \( h[n] \), let the transformed filter be \( \hat{H}(z) = Z(2h[n] \cos(\omega_0 n)) \) for some frequency \( \omega_0 \)

(a) Find the transfer function \( \hat{H}(z) \) in terms of \( H(z) \)

(b) Consider an IIR low-pass filter \( H(z) = 1/(1 - 0.5z^{-1}) \). If \( \omega_0 = \pi/2 \) determine \( \hat{H}(z) \).

(c) How would you obtain a high-pass filter from \( H(z) \) given in the previous item? Use MATLAB to plot the resulting filters here and in the past item.

Answers: (b) \( \hat{H}(z) = 2/(1 + 0.25z^{-2}) \).

Solution: (a) Modulating the impulse response \( h[n] \) of an IIR filter gives a new filter with transfer function

\[
\hat{H}(z) = \sum_{n=0}^{\infty} 2h[n] \cos(\omega_0 n) z^{-n}
\]

so that if \( H(z) = B(z)/A(z) \) then

\[
\hat{H}(z) = \frac{B(e^{j\omega_0} z)}{A(e^{j\omega_0} z)} + \frac{B(e^{-j\omega_0} z)}{A(e^{-j\omega_0} z)}.
\]

(b) For the given transfer function

\[
H(z) = \frac{1}{1 - az^{-1}}
\]

for \( 0 < a < 1 \). The transformed filter is

\[
\hat{H}(z) = H(e^{-j\omega_0} z) + H(e^{j\omega_0} z)
\]

\[
= \frac{1}{1 - ae^{-j\omega_0} z^{-1}} + \frac{1}{1 - a\cos(\omega_0)z^{-1}}
\]

\[
= \frac{2 - 2a \cos(\omega_0) z^{-1}}{1 - 2a \cos(\omega_0) z^{-1} + a^2 z^{-2}}
\]

If \( a = 0.5 \) and \( \omega_0 = \pi/2 \), i.e., we want to convert the prototype low-pass filter into a band-pass filter, we get

\[
\hat{H}(z) = \frac{2}{1 + 0.25z^{-2}} = \frac{2z^2}{z^2 + 0.25}
\]

with two zeros at \( z = 0 \) and two poles at \( z = \pm j0.5 \), so that the filter is a bandpass filter with center frequency \( \pi/2 \).

(c) To get a high-pass filter we let \( \omega_0 = \pi \). The following script computes and plots the different frequency responses.
% Pr. 12.14
clear all; clf
b=1 ; a=[1 -0.5];
[H,w]=freqz(b,a);
alpha=0.5; w0=pi/2
b1=[2 -2*alpha*cos(w0)]; a1=[1 -2*alpha*cos(w0) alpha^2];
w0=pi
b2=[2 -2*alpha*cos(w0)]; a2=[1 -2*alpha*cos(w0) alpha^2];
[H1,w]=freqz(b1,a1);
[H2,w]=freqz(b2,a2);
figure(1)
lot(311)
plot(w/pi,abs(H)); axis([0 1 0 2]); grid
subplot(312)
plot(w/pi,abs(H1)); axis([0 1 0 3]); grid
subplot(313)
plot(w/pi,abs(H2)); axis([0 1 0 4]); grid

Figure 12.10: Problem 14: Magnitude responses of low-pass prototype (top) and band-pass and high-pass filters (middle and bottom).
12.15 **Effect of size of filter in denoising** — The size of an FIR averaging filter has significant effect on denoising. Consider as the clean image the *peppers.png*, use the function `rgb2gray` and `double` to get a double precision gray-level image $x[m, n]$. Suppose it is distorted by additive Gaussian noise with zero mean and 0.1 variance (use the function `imnoise`). Consider $3 \times 3$ and $9 \times 9$ low-pass averaging filters (normalized so that the maximum value of the impulse response is unity) to filter the noisy image. Call $y_3[m, n]$ and $y_9[m, n]$ of the $3 \times 3$ and $9 \times 9$ filters. To determine which of the two filters is more efficient in denoising the image, determine the mean-square errors

$$
\epsilon_i = \sum_{m,n} (y_i[m, n] - x[m, n])^2 / M_i,
$$

for $i = 3, 9$, $M_3 = 3^2$ and $M_9 = 9^2$. Indicate which of the two filters, under this criterion, is more effective in denoising the image.

**Solution**

```matlab
% Pr. 12.15
% Effect of size of filter in denoising
% 3x3 vs 9x9 averaging filtering of
% signal with added Gaussian noise

clc; clear all; clf;
% clean and nosisy images
I=imread('peppers.png');
I=im2double(I);
I2=rgb2gray(I);
figure(1)
subplot(221)
imshow(I2);title('CLEAN')
I3=imnoise(I2,'gaussian',0, 0.1);
subplot(222)
imshow(I3);title('NOISY')

% filtering
h3=1/9*ones(3,3);
h9=1/81*ones(9,9);
y3=filter2(h3,I3);
y9=filter2(h9,I3);
subplot(223)
imshow(y3); title('3x3 FILTERING')
subplot(224)
imshow(y9); title('9x9 FILTERING')
% errors
M=size(I2); M1=M(1);
e3=sum(sum((y3-I2).^2)); e3=e3/M1^2
e9=sum(sum((y9-I2).^2)); e9=e9/M1^2

Figure 15 shows the results of filtering with a $3 \times 3$ and a $9 \times 9$ averaging FIR filters. The mean-square errors are $\epsilon_3 = 0.4890$ and $\epsilon_9 = 0.4337$ a slightly better result for the $9 \times 9$ averaging filter.

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Figure 12.11: Problem 15: Effect of size of filter in denoising.
12.16 Laplacian filtering Consider the mask of a Laplacian filter:

\[
H_L = \begin{bmatrix}
0 & 1 & 0 \\
1 & -4 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

Generate an image of size $100 \times 100$ that is a block of ones of size $50 \times 50$ surrounded by zeros, call it $x[m,n]$.

(a) Filter $x[m,n]$ and display it and the Laplacian filtered image $y[m,n]$. Use then Sobel’s horizontal and vertical filters to filter $x[m,n]$ and obtain a second image $z[m,n]$ by adding their outputs. Which of these two provide a better edge detection?

(b) Find the magnitude response of the Laplacian filter associated with $H_L$.

(c) Apply the Laplacian filter mask $H_L$ to the peppers image and show the original and filtered images.

Solution

```matlab
% Pr. 12.16
% Laplacian Filtering
% clc; clear all; close all;
x=zeros(100,100);
x(26:75,26:75)=ones(50,50);
figure(1)
colormap(’gray’)
subplot(221)
imagesc(x); title(’Original Image’)
% Laplacian
HL=[0 1 0; 1 -4 1; 0 1 0];
% Sobel
hx=[1 0 -1;2 0 -2;1 0 -1];
hy=hx’;
yx=imfilter(x,hx);
yy=imfilter(x,hy);
yS=sqrt(yx.^2 + yy.^2);
subplot(222)
imshow(yS);title(’Sobel Filtered Image’)
yHL=imfilter(x,HL);
subplot(223)
imshow(yHL);title(’Laplacian Filtered Image’)
HLf=fftshift(abs(fft2(HL,256,256)));
w=-1+1/128:1/128:1;
subplot(224)
mesh(w,w,HLf); title(’Frequency response of the Laplacian Filter’)
xlabel(’\omega_1/\pi’)
ylabel(’\omega_2/\pi’)
I=imread(’peppers.png’);
I2=im2double(I);
```

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I3=rgb2gray(I2);
figure(2)
colormap('gray')
subplot(211)
imshow(I3);title('Peppers Image')
yL=imfilter(I3,HL);
subplot(212)
yL=yL>0.05;
imshow(yL);
title('Laplacian Filtered Peppers Image')

Figure 12.12: Problem 16: Laplacian vs Sobel filtering (left), and edge detection using Laplacian (right).
12.17 Circular filter
Design a 2D FIR filter, using the function `fwind1`, that approximates the desired magnitude response

\[ H_d(e^{j\omega_1}, e^{j\omega_2}) = \begin{cases} 
1 & r < 0.5 \\
0 & \text{otherwise}
\end{cases} \]

where \( r = \sqrt{f_1^2 + f_2^2} < 0.5 \) for normalized frequencies \(-1 \leq f_1 \leq 1\) and \(-1 \leq f_2 \leq 1\) and \( f_i = \omega_i/\pi, i = 1, 2 \). Use a Hamming window. Determine the magnitude response of the filter, and plot the desired frequency magnitude response and that of the designed filter using the function `contour`. Normalized the frequencies to be \((-1, 1) \times (-1, 1)\).

Solution

```matlab
% Pr. 12.17
% Circular filter design using fwind1
clear all; clf
% filter specifications
N=125;
[f1,f2] = freqspace(N,’meshgrid’);
Hd = ones(N);
r = sqrt(f1.^2 + f2.^2);
% type of filter
Hd(r>0.5)=0; %low-pass
% Hamming window
h1 = fwind1(Hd,hamming(N));
H=freqz2(h1,124); w1=-1:2/124:1-2/124;w2=w1
figure(1)
colormap(’gray’)
subplot(221)
mesh(f1,f2,Hd); title(’Magnitude response of desired filter’)
subplot(222)
contour(f1,f2,Hd);grid
subplot(223)
mesh(f1,f2,Hd); title(’Magnitude response of designed filter’)
subplot(224)
contour(w1,w2,H); grid;
```

The results are shown in Fig. 17.
Figure 12.13: Problem 17: Magnitude responses of desired (top) and of designed filters (bottom). Circular plots are shown of each of these.