Existence and multiplicity of solutions for critical elliptic equations with multi-polar potentials in symmetric domains

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In this paper, we consider the elliptic equations with critical Sobolev exponents and multi-polar potentials in bounded symmetric domains and prove the existence and multiplicity of symmetric positive solutions by using the Ekeland variational principle and the Lusternik–Schnirelmann category theory.

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1. Introduction

The critical elliptic problem has been studied extensively since the initial work [1] by Brézis and Nirenberg. The critical elliptic problem with one Hardy-type potential has also attracted much attention in recent years. We refer the interested readers to a partial list [2–11] and the references therein. Here, we are concerned with the critical elliptic problem with multi-polar (Hardy-type) potentials. In [12], Cao and Han considered the problem

$$
\begin{cases}
-\Delta u - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^2} u = K(x)|u|^{2^*-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

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and proved the existence of positive and sign-changing solutions in bounded smooth domains, where $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent. In [13], problem (1.1) with $K(x) \equiv 1$ was investigated by Felli and Terracini, and the existence of positive solutions with the smallest energy was deduced both in $\mathbb{R}^N$ and in bounded smooth domains. In [14], Felli and Terracini also studied problem (1.1) with symmetric multi-polar potentials in $\mathbb{R}^N$ when $K(x) \equiv 1$ and showed the existence of symmetric positive solutions. Other related results on critical elliptic problems with multiple Hardy-type potentials can be seen in [15–17] and the references therein.

However, as far as we know, there are few results about the multiplicity of solutions for the critical elliptic problem with multi-polar potentials. In this paper, motivated by [14,18], we consider the critical elliptic problem with symmetric multi-polar potentials in bounded symmetric domains and prove the existence and multiplicity of symmetric positive solutions.

More accurately, we are interested in the problem

$$
(\mathcal{P}_{\lambda_0,k}) \left\{ \begin{array}{ll}
-\Delta u - \frac{\mu_0 u}{|x|^2} - \sum_{i=1}^{m} \sum_{l=1}^{k} \mu_l \frac{u}{|x-d_l|^2} = \lambda_0 u + K(x)|u|^{2^*-2} u & \text{in } \Omega, \\
nu = 0 & \text{on } \partial \Omega,
\end{array} \right.
$$

where $\Omega \subset \mathbb{R}^N (N \geq 4)$ is a $\mathbb{Z}_k \times SO(\mathbb{N} - 2)$-invariant bounded smooth domain, $k \geq 3, \mu_0, \lambda_0 \in \mathbb{R}, \mu_l \in \mathbb{R}, l = 1, 2, \ldots, m$, and $K(x)$ is a $\mathbb{Z}_k \times SO(\mathbb{N} - 2)$-invariant positive bounded function on $\overline{\Omega}$. Here the domain $\Omega$ is said to be $\mathbb{Z}_k \times SO(\mathbb{N} - 2)$-invariant if $(e^{2\pi \sqrt{-1} T_i} y, T_2) \in \Omega, \forall x = (y, z) \in \Omega \subset \mathbb{R}^2 \times \mathbb{R}^{N-2}$ and $K(x)$ is said to be $\mathbb{Z}_k \times SO(\mathbb{N} - 2)$-invariant on $\overline{\Omega}$ if $K(y, z) = K(e^{2\pi \sqrt{-1} T_i} y, T_2), \forall x = (y, z) \in \overline{\Omega} \subset \mathbb{R}^2 \times \mathbb{R}^{N-2}$, where $T$ is any rotation of $\mathbb{R}^{N-2}$. Note that if $\Omega$ is $\mathbb{Z}_k \times SO(\mathbb{N} - 2)$-invariant, we can write $\Omega = \Omega^{(2)} \times B^{(N-2)}(0, R) \subset \mathbb{R}^2 \times \mathbb{R}^{N-2}$, where $\Omega^{(2)}$ is $\mathbb{Z}_k$-invariant in $\mathbb{R}^2$ (that is, $e^{2\pi \sqrt{-1} T_i} y \in \Omega^{(2)}, \forall y \in \Omega^{(2)}$) and $B^{(N-2)}(0, R)$ is a ball in $\mathbb{R}^{N-2}$ centered at the origin with radius $R$.

The group $\mathbb{Z}_k \times SO(\mathbb{N} - 2)$ acts on $H^1_0(\Omega)$ as $u(y, z) \rightarrow u(e^{2\pi \sqrt{-1} T_i} y, T_2)$. Given $m$ regular polygons with $k$ sides, centered at the origin and lying on the plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N$, we assume that

$(A_1). \, d^i_l \in \Omega, l = 1, 2, \ldots, m$,

$(A_2). \, \Omega \supset B^{(2)}(0, R) \times B^{(N-2)}(0, R)$ with $R : \max\{r_l, l = 1, 2, \ldots, m\}$, where $r_l = |d^i_l| = |d^j_l| = \cdots = |d^k_l|$.

It is easy to see that the above assumptions can be easily satisfied, such as, a domain

$\Omega = B^{(2)}(0, R) \times B^{(N-2)}(0, R)$

$= \{ (x_1, x_2, x_3, \ldots, x_N) | |x^2_1 + x^2_2 < R^2, x^2_3 + \cdots + x^2_N < R^2 \}$

with $d^i_l = \{d^i_l(0, 0) \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N, d^i_l(0, 0) = e^{2\pi i (l-1) \sqrt{-1} / k} r_l, 0 < r_1 < r_2 < \cdots < r_m < R, i = 1, 2, \ldots, k, l = 1, 2, \ldots, m$.

We will prove the existence and multiplicity of $\mathbb{Z}_k \times SO(\mathbb{N} - 2)$-invariant positive solutions for the problem $(\mathcal{P}_{\lambda_0,k})$.

Before that, we also consider the limiting case of the problem $(\mathcal{P}_{\lambda_0,k})$, that is,

$$
(\mathcal{P}^{\infty}_{\lambda_0,k}) \left\{ \begin{array}{ll}
-\Delta u - \frac{\mu_0 u}{|x|^2} - \sum_{i=1}^{m} k \mu_l \frac{u}{|x-d_l|^2} = \lambda_0 u + K(x)|u|^{2^*-2} u & \text{in } \Omega_\theta(R, \overline{\Omega}), \\
nu = 0 & \text{on } \partial \Omega_\theta(R, \overline{\Omega}),
\end{array} \right.
$$

where $\Omega_\theta := \{ (x, 0) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : |x| = r_l \}$, the distribution $\delta_{S_{r_l}} \in \mathcal{D}'(\mathbb{R}^N)$ supported in $S_{r_l}$ and defined by, as in [14],

$$
\mathcal{D}'(\mathbb{R}^N) \langle \delta_{S_{r_l}}, \psi \rangle = \frac{1}{2\pi r_l} \int_{S_{r_l}} \psi(x) d\sigma(x), \quad \forall \psi \in \mathcal{D}(\mathbb{R}^N)
$$

with $d\sigma$ the line element on $S_{r_l}$, $\mathcal{D}(\mathbb{R}^N)$ the space of smooth functions with compact support in $\mathbb{R}^N$. $\Omega_\theta(R, \overline{\Omega}) := B^{(2)}(0, R) \times B^{(N-2)}(0, R)$ with $R > \max\{r_l, l = 1, 2, \ldots, m\}, N \geq 4$ and $K(x)$ a
$\mathcal{O}(2) \times \mathcal{O}(N - 2)$-invariant positive bounded function on $\Omega(R, \hat{R})$. For simplicity of notation, we write $\Omega(R)$ instead of $\Omega(R, \hat{R})$ in the sequel. Here $K(x)$ is said to be $\mathcal{O}(2) \times \mathcal{O}(N - 2)$-invariant on $\Omega(R)$ if $K(y, z) = K(|y|, |z|)$, $\forall x = (y, z) \in \Omega(R) \subset \mathbb{R}^2 \times \mathbb{R}^{N - 2}$. We will prove the existence of $\mathcal{O}(2) \times \mathcal{O}(N - 2)$-invariant positive solutions for the problem $(\mathcal{P}_{\lambda_0, K})$.

The paper is organized as follows. In Section 2, we give some preliminary results. Section 3 is devoted to the existence of one $\mathcal{O}(2) \times \mathcal{O}(N - 2)$-invariant positive solution for the problem $(\mathcal{P}_{\lambda_0, K})$, provided that $K(x)$ satisfies some growth condition at zero; for details see Theorem 5.1. In Section 4, we show the existence of one $Z_k \times \mathcal{O}(N - 2)$-invariant positive solution for the problem $(\mathcal{P}_{\lambda_0, K})$ in Theorem 4.1. Some growth conditions on $K(x)$ are needed, of course. In Section 5, the multiplicity of $Z_k \times \mathcal{O}(N - 2)$-invariant positive solutions for the problem $(\mathcal{P}_{\lambda_0, K})$ is obtained by the Ekeland variational principle and the Lusternik–Schnirelmann category theory, respectively. Here, besides the growth conditions on $K(x)$, the parameters $\lambda_0, \mu_0, \mu_1, l = 1, 2, \ldots, m$, are requested to be close to zero; for details see Theorems 5.1 and 5.9. At last, a nonexistence result is proved in Appendix.

2. Notations and preliminary results

Throughout this paper, positive constants will be denoted by $C$.

Denote

$$(H_1^{\text{inc}}(\Omega(R))) := \{u(y, z) \in H_0^1(\Omega(R)) : u(y, z) = u(|y|, |z|)\},$$

where $(y, z) \in \Omega(R) \subset \mathbb{R}^2 \times \mathbb{R}^{N - 2}$, and

$$(H_1^1(\Omega)) := \{u(y, z) \in H_0^1(\Omega) : u(e^{2\pi \sqrt{-1}k}y, z) = u(y, |z|)\},$$

where $(y, z) \in \Omega \subset \mathbb{R}^2 \times \mathbb{R}^{N - 2}$.

It is known that the nonzero critical points of the energy functional

$$J_{\text{circ}}(u) := \frac{1}{2} \int_{\Omega(R)} \left( |\nabla u|^2 - \mu_0 \frac{u^2}{|x|^2} \right) dx - \frac{1}{2\pi} \int_{\Omega(R)} \frac{u^2(y)}{|x-y|^2} d\sigma(x) dy - \frac{k_0}{2} \int_{\Omega(R)} u^2 dx - \frac{1}{2} \int_{\Omega(R)} K(x)|u|^2^* dx$$

defined on $(H_1^{\text{inc}}(\Omega(R)))$, and the energy functional

$$J_{\lambda}(u) := \frac{1}{2} \int_{\Omega} \left( |\nabla u|^2 - \mu_0 \frac{u^2}{|x|^2} \right) dx - \sum_{i=1}^{m} \frac{k_i \mu_i}{2} \int_{\Omega} \frac{u^2(y)}{|x-a_i|^2} dy - \frac{\lambda_0}{2} \int_{\Omega} u^2 dx - \frac{1}{2} \int_{\Omega} K(x)|u|^2^* dx$$

defined on $(H_1^1(\Omega))$ are equivalent to the nontrivial weak solutions for the problem $(\mathcal{P}_{\lambda_0, K})$ and $(\mathcal{P}_{\lambda_0, K})$, respectively.

We show a Hardy-type inequality first, which is an improved version of Theorem 1.1 in [14].

Proposition 2.1. Let $\Omega' \subset \Omega \subset \mathbb{R}^{N}(N \geq 3)$, be bounded or not, and $R > r > 0$. Then, for any $u \in H_1^1(\Omega')$, the map $y \mapsto |u(y)| \int_{\Omega'} \frac{d\sigma(x)}{|x-y|^2} \in L^1(\Omega')$ and

$$\overline{\mu} \int_{\Omega'} |u(y)|^2 \left( \frac{1}{2\pi r} \int_{S_r} \frac{d\sigma(x)}{|x-y|^2} \right) dy \leq \int_{\Omega'} |\nabla u(y)|^2 dy,$$

where the constant $\overline{\mu} := \left( \frac{N-2}{N} \right)^2$ is optimal and not attained.
Proof. If $\Omega'$ is equal to $\Omega_b(R)$, we can prove as Theorem 1.1 in [14] that,
\[
\left(\frac{N-2}{2}\right)^2 = \inf_{u \in C^1_0(\Omega_b(R)) \setminus \{0\}} \frac{\int_{\partial \Omega_b(R)} |\nabla u|^2 \, dy}{\int_{\Omega_b(R)} |u|^2 \left(\frac{1}{2\pi} \int_{S_i} \frac{\sigma(x)}{|x-y|^2} \, dy\right) \, dx},
\]
and the constant $\left(\frac{N-2}{2}\right)^2$ is optimal and not attained. Then for any domain $\Omega'$, $\Omega_b(R) \subset \Omega' \subset \mathbb{R}^N$, the results follow. □

Remark 2.2. (1) If $\Omega' = \mathbb{R}^N$, then Proposition 2.1 is reduced to Theorem 1.1 in [14].
(2) For any domain $\Omega'$ (bounded or not) with $(z,0) \in \Omega' \subset \mathbb{R}^2 \times \mathbb{R}^{N-2}$, if $(x,0) \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$ : $|x-z| \leq r \in \Omega'$, then it is also easy to prove that
\[
\overline{\mu} \int_{\Omega'} |u(y)|^2 \left(\frac{1}{2\pi} \int_{S_i(z)} \frac{\sigma(x)}{|x-y|^2} \, dy\right) \, dy \leq \int_{\Omega'} |\nabla u(y)|^2 \, dy,
\]
where $S_i(z) \doteq (x,0) \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$ : $|x-z| = r$, the constant $\overline{\mu} = \left(\frac{N-2}{2}\right)^2$ is optimal and not attained.

Denote by $D^{1,2}(\mathbb{R}^N)$ the closure space of $C^\infty_0(\mathbb{R}^N)$ with respect to the norm
\[
\|u\|_{D^{1,2}(\mathbb{R}^N)} \doteq \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)^{\frac{1}{2}}.
\]
The limiting problem
\[
\begin{cases}
-\Delta u - \frac{\mu u}{|x|^2} = |u|^{2^* - 2} u & \text{in } \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N \setminus \{0\}, u \in D^{1,2}(\mathbb{R}^N),
\end{cases}
\]
(2.2)
where $\mu < \overline{\mu}$, admits a family of solutions
\[
U^\mu = C_\mu(N) \left(\frac{\epsilon}{\epsilon^2|x|^{(\sqrt{\overline{\mu} - \mu})/\sqrt{N}} + |x|^{(\sqrt{\overline{\mu} - \mu})/\sqrt{N}}}ight)^{\frac{N-2}{2}},
\]
with $\epsilon > 0$ and $C_\mu(N) = \left(\frac{2N\sqrt{\overline{\mu} - \mu}}{N-2}\right)^{\frac{N-2}{2}}$; see [19–22]. Moreover, for $0 \leq \mu < \overline{\mu}$, all solutions of (2.2) take the above form and these solutions minimize
\[
S(\mu) : = \min_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu u^2 \overline{\mu} \right) \, dx}{\left(\int_{\mathbb{R}^N} |u|^2 \, dx\right)^{2/2^*}}.
\]
and
\[
\int_{\mathbb{R}^N} \left(|\nabla U^\mu|^2 - \mu \frac{|U^\mu|^2}{|x|^2}\right) \, dx = \int_{\mathbb{R}^N} |U^\mu|^2 \, dx = S(\mu)^\frac{N}{2}.
\]
Note that $S(0) := S$ is the best Sobolev constant.

The following lemma is from [14].

Lemma 2.3. Let $N \geq 4$, $\mu < \overline{\mu}$. If $u \in D^{1,2}(\mathbb{R}^N)$ is a solution for problem (2.2), then there exist positive constants $\kappa_0(u)$ and $\kappa_\infty(u)$ such that
\[
u(x) = |x|^{-\frac{N-2}{2}(1-v)} [\kappa_0(u) + O(|x|^v)], \quad \text{as } x \to 0,
\]
\[
u(x) = |x|^{-\frac{N-2}{2}(1+v)} [\kappa_\infty(u) + O(|x|^{-v})], \quad \text{as } |x| \to \infty.
\]
for some $\alpha \in (0, 1)$, where $v_\mu = (1 - \frac{4u_\mu}{(N-2)^2})^{1/2}$. And hence there exists a positive constant $\kappa(u)$ such that
\[
\frac{1}{\kappa(u)} U^1_\mu \leq u(x) \leq \kappa(u) U^1_\mu. \tag{2.5}
\]

Denote, for any $u \in D^{1, 2}(\mathbb{R}^N)$,
\[
u_\epsilon(x) := \epsilon^{-\frac{N-2}{2}} u \left( \frac{x}{\epsilon} \right), \quad \text{for any } \epsilon > 0.
\]

For any solution $u^\mu \in D^{1, 2}(\mathbb{R}^N)$ for problem (2.2), denote $V(x) = \varphi(x)|u^\mu(x)|$ with $\varphi(x)$ satisfying
\[
\varphi(x) \in C^\infty_0(B(0, r)), \quad 0 \leq \varphi(x) \leq 1, \quad \varphi(x) \equiv 1 \text{ if } x \in B \left( 0, \frac{r}{2} \right), \quad |\nabla \varphi(x)| \leq C, \tag{2.6}
\]
where $0 < r < 1$ small enough.

Now we give the following lemma.

**Lemma 2.4.** Let $N \geq 4$, $\mu < \overline{\mu}$. Then there hold
\[
\int_{B(0, r)} |V|^2 \, dx = \begin{cases} O(\epsilon^2) & \text{if } \mu < \overline{\mu} - 1, \\ O(\epsilon^2) \ln \epsilon & \text{if } \mu = \overline{\mu} - 1, \\ O(\epsilon^2 \sqrt{\mu}) & \text{if } \mu > \overline{\mu} - 1, \end{cases} \tag{2.7}
\]
\[
\int_{B(0, r)} |V|^2 \, dx = \int_{\mathbb{R}^N} |u^\mu|^2 \, dx - O(\epsilon^2 \sqrt{\mu - \overline{\mu}}), \tag{2.8}
\]
\[
\int_{B(0, r)} \left( |\nabla V|^2 - \mu \frac{V^2}{|x|^2} \right) \, dx = \int_{\mathbb{R}^N} \left( |\nabla u^\mu|^2 - \mu \frac{|u^\mu|^2}{|x|^2} \right) \, dx + \begin{cases} O(\epsilon^2 \ln \epsilon) & \text{if } \mu = \overline{\mu} - 1, \\ O(\epsilon^2 \sqrt{\mu - \overline{\mu}}) & \text{if } \mu \neq \overline{\mu} - 1, \end{cases} \tag{2.9}
\]

and for any $\xi \in \mathbb{R}^N \setminus \{0\}$,
\[
\int_{B(0, r)} \frac{|V(x)|^2}{|x + \xi|^2} \, dx = \begin{cases} \epsilon^2 \int_{\mathbb{R}^N} |u^\mu|^2 \, dx + o(\epsilon^2) & \text{if } \mu < \overline{\mu} - 1, \\ \kappa_\infty(\mu^\mu) \epsilon^2 \ln \epsilon + O(\epsilon^2) & \text{if } \mu = \overline{\mu} - 1, \\ C(N, \mu, \mu^\mu) \epsilon^2 \sqrt{\mu - \overline{\mu}} + o(\epsilon^2 \sqrt{\mu - \overline{\mu}}) & \text{if } \mu > \overline{\mu} - 1, \end{cases} \tag{2.10}
\]

**Proof.** The proofs of (2.7)–(2.9) are essentially similarly to Lemmas A.1 and A.2 in [8]. By using Lemma 2.3,
\[
\int_{B(0, r)} |V|^2 \, dx \leq \epsilon^{-(N-2)} \int_{|y| < \epsilon} \left| \frac{u^\mu}{\epsilon} \right|^2 \, dy = \epsilon^2 \int_{|y| < \frac{\epsilon}{2}} |u^\mu(y)|^2 \, dy
\]
\[
\leq \kappa^2(u) \epsilon^2 \int_{|y| < \frac{\epsilon}{2}} |U^1_\mu(y)|^2 \, dy
\]
\[
= C^2_\mu(N) \kappa^2(u) \epsilon^2 \int_{|y| < \frac{\epsilon}{2}} \frac{1}{(|y|^2 - \sqrt{\mu - \overline{\mu}})^{N-2} + |y|^2 + \sqrt{\mu - \overline{\mu}} + |y|^2 + \sqrt{\mu - \overline{\mu}})^{N-2} \, dy
\]
\[
= C^2_\mu(N) \kappa^2(u) \epsilon^2 \int_{0}^{\frac{\epsilon}{2}} \left( \frac{t^{N-1}}{(t^2 - \sqrt{\mu - \overline{\mu}})^{N-2} + t^2 + \sqrt{\mu - \overline{\mu}})^{N-2}} \right) \, dt.
\]
\[ C^2(N)u_0 \epsilon^2 \left( \int_0^1 \frac{t^{N-1}}{t^N+\sqrt{t^N+\sqrt{t^N}}} \, dt + \frac{\epsilon^{-1}}{t^N+\sqrt{t^N+\sqrt{t^N}}} \right) \leq C^2(N)u_0 \epsilon^2 \left( \int_0^1 \frac{t^{N-1}}{t^N+\sqrt{t^N+\sqrt{t^N}}} \, dt + \frac{\epsilon^{-1}}{t^N+\sqrt{t^N+\sqrt{t^N}}} \right) \leq C^2(N)u_0 \epsilon^2 \left( \int_0^1 \frac{t^{N-1}}{t^N+\sqrt{t^N+\sqrt{t^N}}} \, dt + \frac{\epsilon^{-1}}{t^N+\sqrt{t^N+\sqrt{t^N}}} \right) \]

where \( \omega_N \) is the surface measure of the unit sphere in \( \mathbb{R}^N \). So (2.7) is obtained.

Notice that

\[ \int_{|y|<\frac{r}{2}} |\nabla|^2 \, dx \leq \int_{|y|\leq \frac{r}{2}} |u^\mu(y)|^2 \, dy = \int_{\mathbb{R}^N} |u^\mu(y)|^2 \, dy - \int_{|y|\geq \frac{r}{2}} |u^\mu(y)|^2 \, dy, \]

and

\[ \int_{|y|\leq \frac{r}{2}} |u^\mu(y)|^2 \, dy \geq \frac{1}{k^2 \mu(u)} \int_{|y|\geq \frac{r}{2}} |U^\mu_0(y)|^2 \, dy \]

\[ \int_{|y|\leq \frac{r}{2}} |u^\mu(y)|^2 \, dy \geq \frac{C^2(N)}{k^2 \mu(u)} \int_{|y|\geq \frac{r}{2}} \frac{1}{(t^N+\sqrt{t^N+\sqrt{t^N}})^N} \, dy \]

\[ \int_{|y|\leq \frac{r}{2}} |u^\mu(y)|^2 \, dy \geq \frac{C^2(N)}{2k^2 \mu(u)} \int_{|y|\geq \frac{r}{2}} \frac{1}{(t^N+\sqrt{t^N+\sqrt{t^N}})^N} \, dy \]

Then (2.8) follows.

Note that \( u^\mu \in D^{1,2}(\mathbb{R}^N) \) is a solution for problem (2.2), i.e.

\[-\Delta u^\mu - \mu \frac{u^\mu}{|y|^2} = |u^\mu|^2 u^\mu \quad \text{in } \mathbb{R}^N.\]

To prove (2.9), multiplying the above equation by \( \varphi^2(\epsilon \cdot) u^\mu(\cdot) \) and integrating by parts, it holds

\[ \int_{|y|<\frac{r}{2}} \varphi^2(\epsilon y) |u^\mu(y)|^2 \, dy = \int_{|y|<\frac{r}{2}} \left( \varphi^2(\epsilon y) |\nabla u^\mu(\cdot)|^2 - \mu \frac{\varphi^2(\epsilon y) |u^\mu(\cdot)|^2}{|y|^2} \right) \, dy \]

Hence

\[ \int_{B(0,r)} \left( |\nabla|^2 - \mu \frac{V^2}{|y|^2} \right) \, dx = \int_{|y|<\frac{r}{2}} \left( \varphi^2(\epsilon y) |\nabla u^\mu(y)|^2 - \mu \frac{\varphi^2(\epsilon y) |u^\mu(y)|^2}{|y|^2} \right) \, dy \]
To continue we distinguish three cases:

Similarly to (2.12),

\[
\int_{|y| \geq \frac{r}{2}} |u^\mu(y)|^2 dy = O(\epsilon^2 \sqrt{\pi - \mu}),
\]

\[
\int_{\frac{r}{2} \leq |y| < \frac{1}{\epsilon}} \psi^2(\epsilon y)|u^\mu(y)|^2 dy = O(\epsilon^2 \sqrt{\pi - \mu}).
\]

Hence (2.9) holds since

\[
\int_{|y| < \frac{1}{\epsilon}} |\nabla(\psi(\epsilon y))|^2 |u^\mu(y)|^2 dy \leq C\epsilon^2 \int_{\frac{r}{2} \leq |y| < \frac{1}{\epsilon}} |u^\mu(y)|^2 dy
\]

\[
= \begin{cases} O(\epsilon^2 \ln \epsilon) & \text{if } \mu = \pi - 1, \\ O(\epsilon^2 \sqrt{\pi - \mu}) & \text{if } \mu \neq \pi - 1, \end{cases}
\]

where the last equality can be obtained similarly to (2.11).

Now we prove (2.10). As in [12],

\[
\int_{|x| < r \over 0, r} \frac{|V(x)|^2}{|x + \xi|^2} dx = \frac{\epsilon^{-(N-2)}}{|\xi|^2} \int_{|x| < r} \left| u^\mu \left( \frac{x}{\epsilon} \right) \right|^2 dx + \epsilon^{-(N-2)}
\]

\[\times \int_{|x| < r} \left( \frac{1}{|x + \xi|^2} - \frac{1}{|\xi|^2} \right) \left| u^\mu \left( \frac{x}{\epsilon} \right) \right|^2 dx + \epsilon^{-(N-2)} \int_{\frac{r}{2} < |x| < r} \frac{(\psi^2 - 1) \left| u^\mu \left( \frac{x}{\epsilon} \right) \right|^2}{|x + \xi|^2} dx
\]

\[
:= A_1(\epsilon) + A_2(\epsilon) + A_3(\epsilon).
\]

To continue we distinguish three cases: \( \mu < \pi - 1, \mu = \pi - 1 \text{ and } \mu > \pi - 1. \)

For \( \mu < \pi - 1, \) by (2.5) we have that \( \int_{|x| < r} |u^\mu(x)|^2 dx < \infty. \) Then as (3.14) in [12],

\[
A_1(\epsilon) = \frac{\epsilon^2}{|\xi|^2} \int_{|x| < r} |u^\mu|^2 dx + o(\epsilon^2).
\]

(2.13)

Noticing (2.5), by using (3.15) and (3.17) in [12], we have

\[
|A_2(\epsilon)| \leq \epsilon^{-(N-2)} k^2(u^\mu) \int_{|x| < r} \left( \frac{1}{|x + \xi|^2} - \frac{1}{|\xi|^2} \right) \left| U_\nu^\mu \left( \frac{x}{\epsilon} \right) \right|^2 dx = o(\epsilon^2),
\]

(2.14)

\[
|A_3(\epsilon)| \leq \epsilon^{-(N-2)} k^2(u^\mu) \int_{\frac{r}{2} < |x| < r} \frac{\left| U_\nu^\mu \left( \frac{x}{\epsilon} \right) \right|^2}{|x + \xi|^2} dx = O(\epsilon^2 \sqrt{\pi - \mu}).
\]

(2.15)
for \( r > 0 \) small enough, where (2.15) holds for \( \mu < \bar{\mu} \). Then combining (2.13) with (2.14), (2.15), it gives
\[
\int_{B(0, r)} |V(x)|^2 \, dx = \frac{e^2}{|\xi|^2} \int_{\Omega_0} |u^\mu|^2 \, dx + o(e^2) \quad \text{if} \quad \mu < \bar{\mu} - 1.
\]

For \( \mu = \bar{\mu} - 1 \), by using (35) in [14],
\[
A_1(\epsilon) = \frac{\epsilon^2}{|\xi|^2} \int_{|\eta| < \frac{r}{2}} |u^\mu(\eta)|^2 \, d\eta = \frac{\epsilon^2}{|\xi|^2} (\kappa^2_{\infty}(u^\mu) \ln |\epsilon|) + O(\epsilon^2). \tag{2.16}
\]
By using (2.5), as (3.22) in [12],
\[
|A_2(\epsilon)| = O(\epsilon^2). \tag{2.17}
\]
Then (2.16), (2.17) and (2.15) imply that
\[
\int_{B(0, r)} |V(x)|^2 \, dx = \kappa^2_{\infty}(u^\mu) \frac{\epsilon^2}{|\xi|^2} |\ln |\epsilon| + O(\epsilon^2) \quad \text{if} \quad \mu = \bar{\mu} - 1.
\]

For \( \mu > \bar{\mu} - 1 \), by using (2.5),
\[
\int_{B(0, r)} |V(x)|^2 \, dx \leq \epsilon^{-(N-2)} \int_{|x| < r} \frac{|u^\mu(x)|^2}{|x + \xi|^2} \, dx
\]
\[
= \frac{4\kappa^2(u^\mu) \epsilon^2}{|\xi|^2} \int_{|\eta| < \frac{r}{2}} |U^1_\mu(\eta)|^2 \, d\eta \quad \text{if} \quad \mu < \bar{\mu} - 1,
\]
\[
= \frac{4\kappa^2(N)\kappa^2(u^\mu) \epsilon^2}{|\xi|^2} \int_0^1 \left( \int_{t\sqrt{\bar{\mu} - \mu} / \sqrt{\bar{\mu}} + t\sqrt{\bar{\mu} + \sqrt{\bar{\mu} - \mu}} / \sqrt{\bar{\mu}})^{N-2} dt \right)
\]
\[
= \frac{4\kappa^2(N)\kappa^2(u^\mu) \epsilon^2}{|\xi|^2} \left( \int_0^1 \frac{t^{N-1}}{(t\sqrt{\bar{\mu} - \mu} / \sqrt{\bar{\mu}} + t\sqrt{\bar{\mu} + \sqrt{\bar{\mu} - \mu}} / \sqrt{\bar{\mu}})^{N-2} dt \right)
\]
\[
\leq \frac{4\kappa^2(N)\kappa^2(u^\mu) \epsilon^2}{|\xi|^2} \left( \int_0^1 \frac{t^{N-1}}{t^{N-2}\sqrt{\bar{\mu} - \mu}} dt + \int_1^1 \frac{t^{N-1}}{t^{N-2}2\sqrt{\bar{\mu} - \mu}} dt \right)
\]
\[
= \frac{4\kappa^2(N)\kappa^2(u^\mu) \epsilon^2}{|\xi|^2} \left( \frac{1}{2} + \frac{1}{2} \right) + o(e^{2\sqrt{\bar{\mu} - \mu}})
\]
\[
= \frac{C(N, \mu, u^\mu)}{|\xi|^2} e^{2\sqrt{\bar{\mu} - \mu}} + o(e^{2\sqrt{\bar{\mu} - \mu}}). \tag{2.18}
\]
On the other hand,
\[
\int_{B(0, r)} |V(x)|^2 \, dx \geq \epsilon^{-(N-2)} \int_{|x| > \frac{r}{2}} \frac{|u^\mu(x)|^2}{|x + \xi|^2} \, dx
\]
\[
\geq \frac{4\epsilon^2}{|\xi|^2} \kappa^2(u^\mu) \int_{|\eta| > \frac{r}{2}} |U^1_\mu(\eta)|^2 \, d\eta \quad \text{if} \quad \mu < \bar{\mu} - 1,
\]
Then (2.18) and (2.19) imply that
\[
\int_{B(0,r)} |V(x)|^2 \, dx \geq C(N, \mu, u^{\mu}) \frac{1}{|x|^2} e^{|x|^2 + o(e^{|x|^2})} \quad \text{if } \mu > \overline{\mu} - 1. \quad \square
\]

3. Existence of one positive solution for \((\mathcal{P}_{\infty,K}^\infty)\)

In this section, we show the existence of one positive solution for the problem \((\mathcal{P}_{\infty,K}^\infty)\).

Let \(N \geq 4\) and \(K(x)\) be \(SO(2) \times SO(N-2)\)-invariant on \(\overline{\Omega}_\mu(R)\) in this section. Denote \(\mathcal{D}^{1,2}_0(\mathbb{R}^N) := \{u(y, z) \in \mathcal{D}^{1,2}(\mathbb{R}^N) : u(y, z) = u(|y|, |z|)\}\), where \((y, z) \in \mathbb{R}^2 \times \mathbb{R}^{N-2}\). For \(\mu_0 < \overline{\mu}\), define
\[
S_{\text{circ}}(\mu_0) := \inf_{u \in \mathcal{D}^{1,2}_0(\mathbb{R}^N) \backslash \{0\}} \frac{\int_{\Omega} |\nabla u|^2 \, dx - \mu_0 \int_{\Omega} \frac{u^2}{|x|^2} \, dx}{\left(\int_{\Omega} |u|^2 \, dx\right)^{2/2^*}},
\]
which is achieved in \(\mathcal{D}^{1,2}_0(\mathbb{R}^N)\) (see Lemma 6.1 in [14]). It is also easy to know that \(S_{\text{circ}}(\mu_0)\) is independent of \(\Omega_\mu(R) \subset \mathbb{R}^N\) in the sense that, for any \(\Omega_\mu(R) \subset \mathbb{R}^N\),
\[
S_{\text{circ}}(\mu_0) = \inf_{u \in (H^1_0)^{\text{circ}}(\Omega_\mu(R)) \backslash \{0\}} \frac{\int_{\Omega_\mu(R)} |\nabla u|^2 \, dx - \mu_0 \int_{\Omega_\mu(R)} \frac{u^2}{|x|^2} \, dx}{\left(\int_{\Omega_\mu(R)} |u|^2 \, dx\right)^{2/2^*}}.
\]

An assumption on \(K(x)\) is as follows.
\((\mathcal{K}_1)\). There exists \(a_1 > 0\) such that \(K(x) = K(0) + O(|x|^{a_1})\) as \(x \to 0\) and
\[
\begin{align*}
\alpha_1 &= 2 \quad \text{if } \mu_0 < \overline{\mu} - 1, \\
\alpha_1 &\geq 2 \quad \text{if } \mu_0 = \overline{\mu} - 1, \\
\alpha_1 &> 2 \sqrt{\overline{\mu} - \mu_0} \quad \text{if } \overline{\mu} > \mu_0 > \overline{\mu} - 1.
\end{align*}
\]

Set \(t^+ = \max\{t, 0\}\). We state the main result in this section.

**Theorem 3.1.** Let \(N \geq 4, \mu_0^1 + \sum_{i=1}^m \mu_i^1 < \overline{\mu}\) and \((\mathcal{K}_1)\) hold. For given \(\lambda_0 \in \mathbb{R}\), there exists \(\overline{\lambda} \geq 0\), such that if
\[
\sum_{i=1}^m k_i \mu_i^1 > \overline{\lambda},
\]
then the problem \((\mathcal{P}_{\infty,K}^\infty)\) admits a \(SO(2) \times SO(N-2)\)-invariant positive solution.

A lemma is crucial.

**Lemma 3.2.** Let \(N \geq 4, \mu_0^1 + \sum_{i=1}^m k_i \mu_i^1 < \overline{\mu}\). Assume that \([u_n] \subset (H^1_0)^{\text{circ}}(\Omega_\mu(R))\) is a Palais–Smale \((PS\ in\ short)\) sequence at level \(c\) for \(J_{\text{circ}}\) restricted to \((H^1_0)^{\text{circ}}(\Omega_\mu(R))\), that is
\[
J_{\text{circ}}(u_n) \to c, \quad J'_{\text{circ}}(u_n) \to 0 \quad \text{in the dual space } ((H^1_0)^{\text{circ}}(\Omega_\mu(R)))^*.
\]
If
\[
c < \frac{1}{N} \frac{\int_{\Omega} V(x)^2 u_n^2 \, dx}{\int_{\Omega} |u_n|^2 \, dx},
\]
then \([u_n]\) has a converging subsequence in \((H^1_0)^{\text{circ}}(\Omega_\mu(R))\).
Proof. The proof is omitted since it is standard and similarly to Theorem 4.2 in [23]. □

Denote

\[ \mathcal{M}_{\text{circ}} := \left\{ u \in (H^1_0)_{\text{circ}}(\Omega_R) \mid \int_{\Omega_R} \left( |\nabla u|^2 - \mu_0 \frac{|u|^2}{|x|^2} \right) \, dx - \sum_{i=1}^{m} k_{\mu_i} \int_{\Omega_R} \left( \frac{1}{2\pi r_l} \int_{S_l} |u|^2 |x-y|^2 \, d\sigma(x) \right) \, dy \right\}^{\frac{n-2}{2}} \]

Define

\[ \pi_{\text{circ}} : (H^1_0)_{\text{circ}}(\Omega_R) \setminus \{0\} \to \mathcal{M}_{\text{circ}}. \]

\[ \pi_{\text{circ}}(u) = \left( \int_{\Omega_R} \left( |\nabla u|^2 - \mu_0 \frac{|u|^2}{|x|^2} - \lambda_0 |u|^2 \right) \, dx - \sum_{i=1}^{m} k_{\mu_i} \int_{\Omega_R} \left( \frac{1}{2\pi r_l} \int_{S_l} |u|^2 |x-y|^2 \, d\sigma(x) \right) \, dy \right)^{\frac{n-2}{2}} \]

for all \( u \in (H^1_0)_{\text{circ}}(\Omega_R) \setminus \{0\} \). Then

\[ f_{\text{circ}}(\pi_{\text{circ}}(u)) = \frac{1}{N} \left( \int_{\Omega_R} \left( |\nabla u|^2 - \mu_0 \frac{|u|^2}{|x|^2} - \lambda_0 |u|^2 \right) \, dx - \sum_{i=1}^{m} k_{\mu_i} \int_{\Omega_R} \left( \frac{1}{2\pi r_l} \int_{S_l} |u|^2 |x-y|^2 \, d\sigma(x) \right) \, dy \right)^{\frac{2}{2}} \]

for all \( u \in (H^1_0)_{\text{circ}}(\Omega_R) \setminus \{0\} \). Denote

\[ m_{\text{circ}} := \inf_{\mathcal{M}_{\text{circ}}} f_{\text{circ}}. \]

Now we estimate \( m_{\text{circ}} \).

Proposition 3.3. Let \( N \geq 4, \mu_0^+ + \sum_{i=1}^{m} k_{\mu_i}^+ < \overline{\mu} \) and (\( \mathcal{K}_1 \)) hold. For given \( \lambda_0 \in \mathbb{R} \), there exists \( \overline{l} \geq 0 \), such that if

\[ \sum_{i=1}^{m} \frac{\mu_i}{r_l^2} > \overline{l}, \]

then

\[ m_{\text{circ}} < \frac{1}{N} S_{\text{circ}}^N(\mu_0)^{\frac{N-2}{N}}. \]

Proof. Assume \( S_{\text{circ}}(\mu_0) \) is attained by some \( u^{\mu_0} \in \mathcal{D}^{1,2}_{\text{circ}}(\mathbb{R}^N) \). For simplicity we assume that \( f_{\text{circ}}(u^{\mu_0}) = 1 \). Therefore the function \( \nu^{\mu_0} = S_{\text{circ}}(\mu_0)^{1/(2^*-2)} |u^{\mu_0}| \) is a nonnegative solution for (2.2). Take \( U_{\text{circ}}(x) = \psi(x) |u^{\mu_0}_{\circ}(x)| \) with \( \psi(x) \) satisfying (2.6). Then it follows from (2.10) that, for some
positive constants $\kappa_1, \kappa_2$.

$$\int_{\Omega}(\frac{1}{2\pi r} \int_{S_1} |U_{\text{circ}}(y)|^2 \frac{1}{|x-y|^2} \, d\sigma(x)) \, dy = \begin{cases} \frac{\epsilon^2}{\pi} \int_{\partial \Omega} |u^{\mu_0}|^2 \, dx + o(\epsilon^2) & \text{if } \mu_0 < \bar{\mu} - 1, \\ \frac{\epsilon^2}{\pi} \int_{\partial \Omega} |\ln |x|\, d\sigma + O(\epsilon^2) & \text{if } \mu_0 = \bar{\mu} - 1, \\ \frac{\epsilon^2}{\pi} \int_{\partial \Omega} |\ln |x|\, d\sigma + o(\epsilon^2) & \text{if } \mu_0 > \bar{\mu} - 1. \end{cases}$$

On the other hand, $(K_1)$ and $(2.8)$ imply that

$$\int_{\Omega} K(x) |U_{\text{circ}}|^2 \, dx = K(0) \int_{\Omega} |U_{\text{circ}}|^2 \, dx + \int_{\Omega} (K(x) - K(0)) |U_{\text{circ}}|^2 \, dx = K(0) + O(\epsilon^{1+\sqrt{\mu_0}}) + O(\epsilon^{\alpha_1}).$$

Combining the above two equalities and Lemma 2.4, for $\mu_0 < \bar{\mu} - 1$, we have

$$J_{\text{circ}}(\pi_{\text{circ}}(U_{\text{circ}})) = \frac{1}{N} \left( \int_{\Omega} \left( |\nabla U_{\text{circ}}|^2 - \mu_0 \frac{|\mu_0|^2}{|x|^2} - \lambda_0 U_{\text{circ}}^2 \right) \, dx - \sum_{l=1}^{m} k_l \int_{\Omega} \left( \frac{1}{2\pi r} \int_{S_1} |U_{\text{circ}}|^2 \frac{1}{|x-y|^2} \, d\sigma(x) \right) \, dy \right)^\frac{1}{2}$$

$$= \frac{1}{N} \left( \int_{\Omega} \left( |\nabla u_{\mu_0}|^2 - \mu_0 \frac{|\mu_0|^2}{|x|^2} \right) \, dx + O(\epsilon^{1+\sqrt{\mu_0}}) + O(\epsilon^2) - \epsilon^2 k \int_{\Omega} |u_{\mu_0}|^2 \, dx \left( \sum_{l=1}^{m} \frac{\mu_l^2}{1} + o(1) \right) \right)^\frac{1}{2}.$$

Hence there exists $\bar{L} > 0$, such that if

$$\sum_{l=1}^{m} \frac{\mu_l^2}{1} > \bar{L},$$

then

$$J_{\text{circ}}(\pi_{\text{circ}}(U_{\text{circ}})) < \frac{1}{N} \frac{S_{\text{circ}}(\mu_0)}{K(0)} \frac{1}{\sqrt{2}}.$$

Hence $m_{\text{circ}} < \frac{1}{N} \frac{S_{\text{circ}}(\mu_0)}{K(0)} \frac{1}{\sqrt{2}}$.

When $\mu_0 = \bar{\mu} - 1$ and $\mu_0 > \bar{\mu} - 1$, the proofs are similar. \( \square \)

**Proof of Theorem 3.1.** Let $\{u_{\mu_0}\} \subset (H^1_0)_{\text{circ}}(\Omega_0(\bar{R}))$ be a minimizing sequence for $J_{\text{circ}}$ on $\mathcal{M}_{\text{circ}}$. By the Ekeland variational principle [24], we can assume $\{u_{\mu_0}\}$ is a PS sequence. Proposition 3.3 gives

$$m_{\text{circ}} < \frac{1}{N} \frac{S_{\text{circ}}(\mu_0)}{K(0)} \frac{1}{\sqrt{2}}.$$

Hence Lemma 3.2 implies that there exists $u \in \mathcal{M}_{\text{circ}}$, then $|u| \in \mathcal{M}_{\text{circ}}$, such that

$$J_{\text{circ}}(u) = J_{\text{circ}}(|u|) = m_{\text{circ}}.$$

Therefore we end the proof by the maximum principle. \( \square \)

4. Existence of one positive solution for $(\mathcal{P}_{\mu_0, K})$

In this section, we show the existence of one positive solution for the problem $(\mathcal{P}_{\mu_0, K})$.

Denote $K_0 := \max_{x \in \Omega} K(x)$. Let $N \geq 4$ in this section and assume that $K(x), \Omega$ are $Z_k \times S^O(N-2)$-invariant in the sequel. Denote $D^{1,2}_1(\mathbb{R}^N) := \{u(y, z) \in D^{1,2}(\mathbb{R}^N) : u(e^{2\pi \sqrt{-1}l}y, z) = u(y, |z|)\}$, where $(y, z) \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$. 
As in [14], for any $\mu < \overline{\mu}$, define
\[
S_k(\mu) := \min_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_\Omega \left( |\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) \, dx}{\left( \int_\Omega |u|^2 \, dx \right)^{1/2}}.
\]  
(4.1)

It is easy to see that $S_k(\mu)$ is independent of the $\mathbb{Z}_k \times \mathbb{S}\mathbb{O}(N - 2)$-invariant domain $\Omega \subset \mathbb{R}^N$ in the sense that, for any $\mathbb{Z}_k \times \mathbb{S}\mathbb{O}(N - 2)$-invariant domain $\Omega$,
\[
S_k(\mu) = \inf_{u \in H^1_0(\widetilde{\Omega}) \setminus \{0\}} \frac{\int_{\widetilde{\Omega}} \left( |\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) \, dx}{\left( \int_{\widetilde{\Omega}} |u|^2 \, dx \right)^{1/2}}.
\]  
(4.2)

Two assumptions are needed.

(\mathcal{K}_2). There exist $x_0 \in \Omega \setminus \{0, a_i^j, \ i = 1, \ldots, k, \ l = 1, \ldots, m\}$ and $\alpha_2 > 2$ such that
\[K(x) = K_M + O(|x - x_0|^\alpha_2)\] as $x \to x_0$.

Denote $x_0 := (x_0^{(1)}, x_0^{(N-2)}) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} \cap \Omega$. Since $K(x)$ is $\mathbb{Z}_k \times \mathbb{S}\mathbb{O}(N - 2)$-invariant, then for any $x_0, j = (x_0^{(1)}, \sqrt{x_0^{(N-2)}})$, $x_1, \ldots, x_k$, there holds $K(x) = K_M + O(|x - x_0|^\alpha_2)$ as $x \to x_0$.

(\mathcal{K}_3). There exists $\alpha_3 > 0$ such that $K(x) = K(a_1^j) + O(|x - a_1^j|^\alpha_2)$ as $x \to a_1^j$, $i = 1, \ldots, k$, with
\[
\begin{align*}
\alpha_3 > 2 & \quad \text{if } \mu_L < \overline{\mu} - 1, \\
\alpha_3 > 2 & \quad \text{if } \mu_L = \overline{\mu} - 1, \\
\alpha_3 > 2 & \quad \text{if } \mu_L > \overline{\mu} - 1,
\end{align*}
\]

where $L(1 \leq L \leq m)$ is a positive natural number such that $\mu_L < \overline{\mu}$ and
\[
\frac{S^N(\mu_l)}{K(a_1^j)^{\frac{\alpha_3}{2}}} = \min \left\{ \frac{S^N(\mu_l)}{K(a_1^j)^{\frac{\alpha_3}{2}}}, \ l = 1, \ldots, m \right\}.
\]  
(4.3)

The main result in this section is as follows.

**Theorem 4.1.** Let $N \geq 4$, $\mu_0^+ + \sum_{i=1}^m k_1^+ < \overline{\mu}$.

(i). When
\[
\min \left\{ k \frac{S^N(\mu_1)}{K(a_1^j)^{\frac{\alpha_3}{2}}}, \frac{S^N(\mu_2)}{K(a_2^j)^{\frac{\alpha_3}{2}}}, \frac{S^N(\mu_3)}{K(a_3^j)^{\frac{\alpha_3}{2}}} \right\} = k \frac{S^N(\mu_1)}{K(a_1^j)^{\frac{\alpha_3}{2}}},
\]
assume (\mathcal{K}_2) holds, then for given $\lambda_0 \in \mathbb{R}$, there exists $\Lambda \geq 0$, such that if $\sum_{i=1}^m \sum_{j=1}^k \frac{\mu_1}{|a_1^j - x_0|^2} > \Lambda$, the problem (\mathcal{P}_{\lambda_0, \mu}) admits a $\mathbb{Z}_k \times \mathbb{S}\mathbb{O}(N - 2)$-invariant positive solution.

(ii). When
\[
\min \left\{ k \frac{S^N(\mu_1)}{K(a_1^j)^{\frac{\alpha_3}{2}}}, \frac{S^N(\mu_2)}{K(a_2^j)^{\frac{\alpha_3}{2}}}, \frac{S^N(\mu_3)}{K(a_3^j)^{\frac{\alpha_3}{2}}} \right\} = k \frac{S^N(\mu_1)}{K(a_1^j)^{\frac{\alpha_3}{2}}},
\]
assume (\mathcal{K}_3) holds, then for given $\lambda_0 \in \mathbb{R}$, there exists $\Lambda \geq 0$, such that if $\sum_{i=1, j \neq 1}^k \frac{\mu_1}{|a_1^j - x_0|^2} > \Lambda$, the problem (\mathcal{P}_{\lambda_0, \mu}) admits a $\mathbb{Z}_k \times \mathbb{S}\mathbb{O}(N - 2)$-invariant positive solution.

(iii). For $k$ large enough, assume (\mathcal{K}_3) holds, then for given $\lambda_0 \in \mathbb{R}$, there exists $\Lambda \geq 0$, such that if $\sum_{i=1}^m \frac{\mu_1}{|a_1^j|} > \Lambda$ with $\eta = |a_1^i| = |a_2^i| = \cdots = |a_k^i|$, the problem (\mathcal{P}_{\lambda_0, \mu}) admits a $\mathbb{Z}_k \times \mathbb{S}\mathbb{O}(N - 2)$-invariant positive solution.

The following lemma is important.
Lemma 4.2. Let $N \geq 4$, $\mu_+ := \sum_{i=1}^{m} k_i \mu_i^+ < \overline{\mu}$. Assume that $\{u_n\} \subset (H^k_0)^{\infty}_0(\Omega)$ is a PS sequence at level $c$ for $J_k$ restricted to $(H^k_0)^{\infty}_0(\Omega)$, that is, $J_k(u_n) \to c$, $J'_k(u_n) \to 0$ in the dual space $((H^k_0)^{\infty}_0(\Omega))^*$. Then

\[ J_k(u) \to c, \quad J'_k(u) \to 0 \quad \text{in the dual space } ((H^k_0)^{\infty}_0(\Omega))^*. \]

If

\[ c < \tilde{c}(\mu_0, \mu_L) := \frac{1}{N} \min \left\{ k \frac{S_2^N}{K_M} \cdot \frac{S_2^N(\mu_L)}{K(\alpha_L^1)^{\frac{N}{2}}}, \frac{S_2^N(\mu_0)}{K(0)^{\frac{N}{2}}} \right\}, \]

then $\{u_n\}$ has a converging subsequence in $(H^k_0)^{\infty}_0(\Omega)$.

**Proof.** We omit the proof here since it is standard and similarly to Theorem 4.1 in [23].

Denote

\[ \mathcal{N}_k(\mu_0, \mu_1, \lambda_0) := \left\{ u \in (H^k_0)^{\infty}_0(\Omega) : \int_\Omega \left( |\nabla u|^2 - \mu_0 \frac{u^2}{|x|^2} \right) dx - \sum_{i=1}^{m} \sum_{j=1}^{k} \mu_i \int_\Omega \frac{u^2}{|x - \alpha_j|^2} dx = \lambda_0 \int_\Omega u^2 dx \right\}. \]

Define

\[ \pi_k : (H^k_0)^{\infty}_0(\Omega) \setminus \{0\} \to \mathcal{N}_k(\mu_0, \mu_1, \lambda_0), \]

\[ \pi_k(u) = \left( \frac{\int_\Omega \left( |\nabla u|^2 - \mu_0 \frac{u^2}{|x|^2} - \lambda_0 u^2 \right) dx - \sum_{i=1}^{m} \sum_{j=1}^{k} \mu_i \int_\Omega \frac{u^2}{|x - \alpha_j|^2} dx}{\int_\Omega K(x)|u|^2 dx} \right)^{\frac{N-2}{4}} u. \]

Then

\[ J_k(\pi_k(u)) = \frac{1}{N} \left( \frac{\int_\Omega \left( |\nabla u|^2 - \mu_0 \frac{u^2}{|x|^2} - \lambda_0 u^2 \right) dx - \sum_{i=1}^{m} \sum_{j=1}^{k} \mu_i \int_\Omega \frac{u^2}{|x - \alpha_j|^2} dx}{\left( \int_\Omega K(x)|u|^2 dx \right)^{\frac{N}{2}}} \right)^{\frac{N}{4}}, \]

for all $u \in (H^k_0)^{\infty}_0(\Omega) \setminus \{0\}$. Denote

\[ m_k := \inf_{\mathcal{N}_k(\mu_0, \mu_1, \lambda_0)} J_k. \]

**Proposition 4.3.** Let $N \geq 4$, $\mu_+ := \sum_{i=1}^{m} k_i \mu_i^+ < \overline{\mu}$. 

(1) If the assumptions in (i) of Theorem 4.1 hold, then for given $\lambda_0 \in \mathbb{R}$, there exists $\tilde{L} \geq 0$, such that if $\sum_{i=1}^{m} \sum_{j=1}^{k} \mu_i \frac{|\alpha_i - \alpha_j|^2}{|\alpha_i - \alpha_j|^2} > \tilde{L}$, there holds

\[ m_k < k \frac{\overline{\mu}}{\overline{\mu} - \overline{\mu}_+}. \]

(II) If the assumptions in (ii) of Theorem 4.1 hold, then for given $\lambda_0 \in \mathbb{R}$, there exists $\tilde{L} \geq 0$, such that if $\sum_{i=1}^{m} \sum_{j=1}^{k} \mu_i \frac{|\alpha_i - \alpha_j|^2}{|\alpha_i - \alpha_j|^2} + \sum_{j=1, j \neq 1}^{k} \frac{\mu_j \mu_{j+1}}{|\alpha_i - \alpha_j|^2} > \tilde{L}$, there holds

\[ m_k < k \frac{\overline{\mu}}{\overline{\mu} - \overline{\mu}_+}. \]
Hence on the other hand, the condition where

\[ \sum_{i=1}^{m} \frac{\mu_i}{\mu_L} > L \]

there exists \( \tilde{L} \geq 0 \), such that if \( \sum_{i=1}^{m} \frac{\mu_i}{\mu_L} > \tilde{L} \), there holds

\[ m_k < \frac{1}{N} \frac{S^N}{K(0)^{\frac{N-2}{2}}} \]

**Proof.** (I) For \( x_0 \) given in \((\mathcal{K}_2)\), take \( U(x) = \sum_{j=1}^{k} \varphi(x) |U_0^j (x-x_0,j)| \) with \( \varphi(x) \) satisfying

\[ 0 \leq \varphi \leq 1, \quad \varphi = 1 \text{ if } x \in \bigcup_{j=1}^{k} B \left( x_0, \frac{r}{2} \right), \quad \varphi = 0 \text{ if } x \notin \bigcup_{j=1}^{k} B \left( x_0, r \right), \]

\[ |\nabla \varphi| \leq \frac{4}{r} \]

where \( r > 0 \) small enough, \( x_{0,j} = (e^{2\pi(j-1)/k}x_0^{(2)}, x_0^{(N-2)}) \), \( j = 1, 2, \ldots, k \).

Then (2.10) gives

\[ \int_{\Omega} \frac{|U(x)|^2}{|x-a_i|^2} dx = \sum_{j=1}^{k} \frac{e^2}{|a_i-x_0,j|^2} \int_{\mathbb{R}^N} |U_0^j|^2 dx + o(e^2). \]

On the other hand, the condition \((\mathcal{K}_2)\) and (2.8) imply that

\[ \int_{\Omega} K(x)|U|^2 dx = K_M k S^\frac{N}{2} + O(e^N) + O(e^{\alpha_2}). \]

By summing the above two equalities and Lemma 2.4, we have

\[ J_k(\pi_k(U)) = \frac{1}{N} \left( \int_{\Omega} \left( |\nabla U|^2 - \mu_0 \mu_L^2 |U|^2 - \lambda_0 U^2 \right) dx - \sum_{i=1}^{k} \mu_i \int_{\Omega} \frac{U^2}{|x-a_i|^2} dx \right) \left( \frac{S^N}{K(0)^{\frac{N-2}{2}}} \right)^\frac{1}{2} \]

\[ = \frac{1}{N} \left( k S^\frac{N}{2} + O(e^{-2}) - O(e^2) - e^2 k \int_{\Omega} |U_0|^2 dx \left( \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{\mu_i}{|a_i-x_0,j|^2} + o(1) \right) \left( K_M k S^\frac{N}{2} + O(e^N) + O(e^{\alpha_2}) \right)^\frac{1}{2} \right) \]

Therefore there exists \( \tilde{L} \geq 0 \), such that if \( \sum_{i=1}^{m} \mu_i (x_0, a_i) > \tilde{L} \), then

\[ J_k(\pi_k(U)) < \frac{1}{N} \frac{S^N}{K_M^{\frac{N-2}{2}}} \]

Hence \( m_k < \frac{1}{N} \frac{S^N}{K_M^{\frac{N-2}{2}}} \).

(II) It is easy to see that \( \mu_L > 0 \) in this case since \( S \leq S(\mu_L) \) if \( \mu_L = 0 \).

Take \( U_k(x) = \sum_{j=1}^{k} \varphi(x) |U_0^{\mu_k} (x-a_i^k)| \) with \( \varphi(x) \) satisfying

\[ 0 \leq \varphi \leq 1, \quad \varphi = 1 \text{ if } x \in \bigcup_{j=1}^{k} B \left( a_i, \frac{r}{2} \right), \quad \varphi = 0 \text{ if } x \notin \bigcup_{i=1}^{k} B (a_i, r), \]

\[ |\nabla \varphi| \leq \frac{4}{r} \]
with \( r > 0 \) small enough. Then as in Lemma 6.2 in [13], by (2.10), for some positive constants \( \kappa_1, \kappa_2, \kappa_3, \kappa_4 \), it follows

\[
\int_{\Omega} |U_k(x)|^2 \, dx = \begin{cases} 
\frac{k}{\lambda_1} \sum_{j=1}^{k} \frac{\epsilon^2}{|a_j^1 - a_j^2|^2} \int_{\Omega} |U_{\mu_L}^1|^2 \, dx + o(\epsilon^2) & \text{if } \mu_L < \mu - 1, \\
\kappa_1 \sum_{j=1}^{k} \frac{\epsilon^2}{|a_j^1 - a_j^2|^2} |\ln \epsilon| + O(\epsilon^2) & \text{if } \mu_L = \mu - 1, \text{ for } \lambda \neq \lambda_L, \\
\kappa_2 \sum_{j=1}^{k} \frac{\epsilon^2}{|a_j^1 - a_j^2|^2} + o(\epsilon^2) & \text{if } \mu_L > \mu - 1, 
\end{cases}
\]

By using Lemma 2.4, for \( \mu_L < \mu - 1 \),

\[
\int_{\Omega} \left( |\nabla U_k|^2 - \mu_0 \frac{U_k^2}{|x|^2} - \lambda_0 U_k \right) \, dx = \int_{\Omega} \sum_{j=1}^{k} \mu_L \int_{\Omega} \frac{U_k^2}{|x|} \, dx - \sum_{j=1}^{k} \sum_{j=1}^{k} \mu_L \int_{\Omega} \frac{U_k^2}{|x|} \, dx
\]

\[
= kS^N_\mu (\mu_L) + O(\epsilon^{2\sqrt{\mu - \mu_1}}) - O(\epsilon^2)
\]

\[
- \epsilon^2 k \int_{\Omega} |U_k|^2 \, dx \left( \sum_{l=1}^{m} \sum_{j=1}^{k} \frac{\mu_L}{|a_j^1 - a_j^2|^2} + \sum_{l=1}^{m} \sum_{j=1}^{k} \frac{\mu_L}{|a_j^1 - a_j^2|^2} \right) + o(1)
\]

\[
+ O(\epsilon^{N-2} - \epsilon^2)
\]

\[
= kS^N_\mu (\mu_L) - \epsilon^2 k \int_{\Omega} |U_k|^2 \, dx \left( \sum_{l=1}^{m} \sum_{j=1}^{k} \frac{\mu_L}{|a_j^1 - a_j^2|^2} + \sum_{l=1}^{m} \sum_{j=1}^{k} \frac{\mu_L}{|a_j^1 - a_j^2|^2} \right)
\]

\[
+ \sum_{l=1}^{m} \sum_{j=1}^{k} \frac{\mu_L}{|a_j^1 - a_j^2|^2} + o(1) \right) + O(\epsilon^2).
\]

On the other hand, the condition \((X_3)\) and (2.8) imply that

\[
\int_{\Omega} K(x)|U_k|^2 \, dx = K(a_1^1)S^N_\mu (\mu_L) + O(\epsilon^{2\sqrt{\mu - \mu_1}}) + O(\epsilon^2).
\]

Hence there exists \( \tilde{\epsilon} \geq 0 \), such that if \( \sum_{l=1, \neq L}^{m} \sum_{j=1}^{k} \frac{\mu_L}{|a_j^1 - a_j^2|^2} + \sum_{l=1, \neq L}^{m} \sum_{j=1}^{k} \frac{\mu_L}{|a_j^1 - a_j^2|^2} > \tilde{\epsilon}, \) then

\[
f_{k}(\pi_k(U_k)) = \frac{1}{N} \left( \int_{\Omega} \left( |\nabla U_k|^2 - \mu_0 \frac{U_k^2}{|x|^2} - \lambda_0 U_k \right) \, dx - \sum_{l=1}^{m} \sum_{j=1}^{k} \mu_L \int_{\Omega} \frac{U_k^2}{|x|} \, dx \right)^2
\]

\[
\left( \int_{\Omega} K(x)|U_k|^2 \, dx \right)^{\frac{2}{N}}.
\]
Theorem 4.1.\footnote{\textit{Proof of Theorem 4.1.}} For $k$ large enough, we have
\begin{align*}
\text{Proposition 4.3} & = \begin{cases}
\frac{S^N_k (\mu_L) - \varepsilon^2 k \int_{\Omega} \| \nabla u \|^2 \, dx}{N} + \frac{\sum_{i=1}^k \left( \int_{\Omega} | \nabla u |^2 \, dx - \lambda_0 \int_{\Omega} u^2 \, dx \right)}{k} \left( \frac{\int_{\Omega} \left| \nabla u \right|^2 \, dx}{\int_{\Omega} \left| u \right|^2 \, dx} \right)^{\frac{\mu}{N}} \\
\left( \frac{\sum_{i=1}^k \mu_i}{\mu} \right) + o(1) + O(\varepsilon^2) = S^N_k (\mu_L) + O(\varepsilon^2) + O(\varepsilon^4)
\end{cases}
\end{align*}
which gives $m_k < \frac{S^N_k (\mu_L) - \varepsilon^2 k \int_{\Omega} \| \nabla u \|^2 \, dx}{N}$. When $\mu_L = \overline{\mu} - 1$ and $\mu_L > \overline{\mu} - 1$, the proofs are similar.

(III) For $k$ large enough, we have
\begin{align*}
\min \left\{ \frac{S^N_k (\mu_L)}{K(a_k)^{n-2}}, \frac{S^N_k (\mu_L)}{K(a_k)^{n-2}} \right\} = \frac{S^N_k (\mu_0)}{K(0)^{n-2}}.
\end{align*}
Take $u_0 (x) = \psi(x) |u_0^\mu (x)|$ with $\psi(x)$ satisfying (2.6) and argue as in Proposition 3.3, for $\mu_0 < \overline{\mu} - 1$.

\begin{align*}
J_k (\pi_k (U_{\text{circ}})) = \frac{1}{N} \left( \int_{\Omega} \left\{ | \nabla U_{\text{circ}} |^2 - \mu_0 | u_{\text{circ}} |^2 - \lambda_0 U_{\text{circ}} \right\} \, dx \right)^{\frac{N}{2}} + \frac{\sum_{i=1}^k \mu_i}{\mu} \left( \frac{\int_{\Omega} \left| \nabla u \right|^2 \, dx}{\int_{\Omega} \left| u \right|^2 \, dx} \right)^{\frac{\mu}{N}} - \frac{\sum_{i=1}^k \mu_i}{\mu} \left( \frac{\int_{\Omega} \left| \nabla u \right|^2 \, dx}{\int_{\Omega} \left| u \right|^2 \, dx} \right)^{\frac{\mu}{N}} + o(1) = S^N_k (\mu_0) + O(\varepsilon^2) + O(\varepsilon^4)
\end{align*}
which gives $m_k < \frac{S^N_k (\mu_0)}{N} K(0)^{\frac{n-2}{2}}$.

On the other hand, Theorem 7.3 in [14] indicates $\lim_{k \to +\infty} S_k (\mu_0) = S_{\text{circ}} (\mu_0)$. Then if $k$ large enough,
\begin{align*}
m_k < \frac{S^N_k (\mu_0)}{N} K(0)^{\frac{n-2}{2}}.
\end{align*}
When $\mu_0 = \overline{\mu} - 1$ and $\mu_0 > \overline{\mu} - 1$, the proofs are similar. \hfill \Box

\textbf{Proof of Theorem 4.1.} Let $\{u_k\} \subset (H_0^1 (\Omega))^3$ be a minimizing sequence for $J_k$ on $\mathcal{A}_k (\mu_0, \mu_1, \lambda_0)$. Then by using Proposition 4.3 and Lemma 4.2, the results can be obtained as in the proof of Theorem 3.1. \hfill \Box

5. Existence of multiple positive solutions for $(\mathcal{P}_k)$

This section is devoted to the multiplicity of positive solutions for the problem $(\mathcal{P}_k)$. The results here consist of two parts. We state them as follows respectively.

We always assume $N \geq 5$ and $3 \leq k < \left( \frac{K(0)^{\frac{n-2}{2}}}{\lambda_0} \right)$ in this section.
5.1. Part I

Denote $C(K) = \{ b \in \Omega \mid |K(b) = \max_{x \in \Omega} K(x)\}$. We state some assumptions first.

(K$_4$). $K(x) \in C(\Omega)$, $K_0 = \max_{x \in \Omega} K(x) > \max \{ K(0), K(a_i) \}$, $i = 1, \ldots, k$, $l = 1, \ldots, m$.

(K$_5$). The set $C(K)$ is finite and $b \in \Omega \cap R^2 \times \{0\}$ for every $b \in C(K)$, say $C(K) = \{ b_{l_s}, 1 \leq i \leq k, 1 \leq s \leq \frac{1}{k} \text{Card}(C(K)) \}$, where $b_{l_s} = (b_{l_s}^{(2)}, 0) = (e^{2\pi i (l-1)k^{-1}}b_{l_s}^{(2)}, 0) \in R^2 \times \{0\}$.

(K$_6$). There exists $\alpha_4 > 2$ such that if $b_{l_s} \in C(K)$, then $K(x) = K(b_{l_s}) + O(|x - b_{l_s}|^\alpha)$ as $x \to b_{l_s}$.

(H). The first eigenvalue of operator $-\Delta - \mu \frac{1}{|x|^2} - \sum^m_{i=1} \frac{1}{|x - a_i|^2}$ is positive, that is, there exists $\lambda_0^i > 0$ such that

$$
\int_\Omega |\nabla u|^2 - \mu_0^i \frac{u^2}{|x|^2} \, dx - \sum^m_{i=1} \int_\Omega \sum^k_{j=1} \frac{\mu_j}{|x - a_j|^2} \, dx \geq \lambda_0^i \int_\Omega \frac{u^2}{|x|^2} \, dx, \quad \forall u \in H^1_0(\Omega).
$$

Let $I$ be the positive natural number appeared in (4.3).

Theorem 5.1. Let $N \geq 5$, $\mu_0^i + \sum^m_{i=1} k \mu_j^i < \bar{\mu}$. If (K$_4$), (K$_5$), (K$_6$), (H) hold and for every $1 \leq s \leq \frac{1}{k} \text{Card}(C(K))$

$$
\sum^m_{i=1} \sum^k_{j=1} \frac{\mu_j}{|a_j|^2 - b_{l_s}|^2} > 0,
$$

then there exist $\epsilon_{\mu_0} > 0$, $\epsilon_{\mu_j^i} > 0$ such that for all $0 < \mu_0 < \epsilon_{\mu_0}$, $0 \leq \mu_j < \epsilon_{\mu_j}$, $|\mu_j| < \epsilon_{\mu_j}$, $l = 1, \ldots, m$, $l \neq i$, $0 \leq \lambda_0 < \epsilon_{\lambda_0}$, the problem (P$_{\mu_0, k}$) admits $\frac{1}{k} \text{Card}(C(K))$ positive solutions which are $Z_k \times SO(N - 2)$-invariant.

To prove the above theorem, we follow the arguments of [18].

By using Lemma 4.2, we have immediately the following lemma.

Lemma 5.2. If $\mu_0^i + \sum^m_{i=1} k \mu_j^i < \bar{\mu}$ and (K$_4$) hold, then there exist $\epsilon_{\mu_0}^0 > 0$, $\epsilon_{\mu_j}^0 > 0$ such that

$$
\frac{k}{k_{2,2}} - \min \{ \frac{(1 - \frac{\mu_0}{K(0)}) \frac{\mu_j}{K(a_j)} + \frac{\mu_j}{K(0)} - \frac{\mu_j}{K(a_j)}\} \}
$$

for all $0 < \mu_0 \leq \epsilon_{\mu_0}^0$, $0 \leq \mu_j \leq \epsilon_{\mu_j}^0$.

Choose $r_0 > 0$ small enough such that $B(b_{l_s}, r_0) \cap B(b_{l_t}, r_0) = \emptyset$ for all $i \neq j$ or $s \neq t$, $1 \leq i, j \leq k$, $1 \leq s, t \leq \frac{1}{k} \text{Card}(C(K))$. Let $\delta = \frac{r_0}{3}$ and denote, for any $1 \leq s \leq \frac{1}{k} \text{Card}(C(K))$,

$$
T_s(u) := \frac{\int_\Omega \psi(x)|\nabla u|^2 \, dx}{\int_\Omega |\nabla u|^2 \, dx}, \quad \text{where } \psi(x) = \min\{1, |x - b_{l_s}|, i = 1, \ldots, k\}.
$$

As in [18], if $u \not\equiv 0$ and $T_s(u) \leq \delta$, then

$$
r_0 \int_\Omega \frac{1}{k} \sum_{i=1}^k |\nabla u|^2 \, dx \leq \frac{r_0}{3} \int_\Omega |\nabla u|^2 \, dx.
$$

Therefore we obtain immediately the following lemma.

Lemma 5.3. If $u \in H^1_0(\Omega)$ such that $T_s(u) \leq \delta$, then

$$
\int_\Omega |\nabla u|^2 \, dx \geq \frac{3}{k} \int_\Omega \frac{1}{k} \sum_{i=1}^k |\nabla u|^2 \, dx.
$$
Corollary 5.4. If \( u \in H^1_0(\Omega) \), \( u \neq 0 \) such that \( T^t(u) \leq \delta \), \( T^t(u) \leq \delta \), then \( s = t \).

It is easy to prove that if (\( \mathcal{H} \)) holds and \( 0 \leq \lambda_0 < \lambda'_0 \), \( \mu'_0 + \sum_{i=1}^m k \mu_i I_1 < 1 \), then there exists \( c > 0 \) such that
\[
\|u\|_{H^1_0(\Omega)} \geq c, \quad \text{for any } u \in \mathcal{N}_c(\mu_0, \mu_1, \lambda_0).
\]

Definition 5.5. For any \( 1 \leq i \leq k, 1 \leq s \leq \frac{1}{k} \text{Card}(\mathcal{C}(K)) \), consider the set
\[
M^s(\mu_0, \mu_1, \lambda_0) := \{ u \in \mathcal{N}_c(\mu_0, \mu_1, \lambda_0) : T^s(u) < \delta \}
\]
and its boundary
\[
\Gamma^s(\mu_0, \mu_1, \lambda_0) := \{ u \in \mathcal{N}_c(\mu_0, \mu_1, \lambda_0) : T^s(u) = \delta \}.
\]
Define
\[
m^s := \inf \{ \lambda_k : u \in M^s(\mu_0, \mu_1, \lambda_0) \}, \quad \eta^s := \inf \{ \lambda_k : u \in \Gamma^s(\mu_0, \mu_1, \lambda_0) \}.
\]

Lemma 5.6. Let \( N \geq 5 \), \( \mu_0^+ + \sum_{i=1}^m k \mu_i I_1 < 1 \). If (\( \mathcal{H}_4 \)), (\( \mathcal{H}_5 \)), (\( \mathcal{H}_6 \)) hold and
\[
\sum_{i=1}^m \sum_{j=1}^k \frac{\mu_i}{|a_i - b_{ij}|^2} > 0, \quad 1 \leq s \leq \frac{1}{k} \text{Card}(\mathcal{C}(K)),
\]
then \( M^s(\mu_0, \mu_1, \lambda_0) \neq \emptyset \) and there exist \( \epsilon^1_{\mu_0} > 0, \epsilon^0_{\mu_0} > 0 \) such that
\[
m^s < \epsilon^1_0 \text{ for all } 0 < \mu_0 \leq \epsilon^1_{\mu_0}, 0 \leq \lambda_0 \leq \epsilon^0_{\mu_0}.
\]

Proof. Take \( V^k_\epsilon(x) = \sum_{j=1}^k \psi(x) |U^0_j(x - b_{j,1})| \in (H^1_0)^k(\Omega) \) with \( \psi(x) \), a radial function, satisfying
\[
0 \leq \psi \leq 1, \quad \psi = 1 \text{ if } x \in \bigcup_{j=1}^k B\left(b_{j,1}, \frac{r}{2}\right), \quad \psi = 0 \text{ if } x \notin \bigcup_{j=1}^k B(b_{j,1}, r),
\]
\[
|\nabla \psi| \leq \frac{4}{r},
\]
where \( r > 0 \) small enough.

It is obvious that
\[
\pi_k(V^\epsilon) = \left( \frac{\int_\Omega \left( |\nabla V^\epsilon|^2 - \mu_0 \frac{|V^\epsilon|^2}{|\nabla V^\epsilon|^2} - \lambda_0 |V^\epsilon|^2 \right) dx - \sum_{i=1}^k \sum_{j=1}^m \mu_i \int_\Omega \frac{|V^\epsilon|^2}{|\nabla V^\epsilon|^2} dx}{\int_\Omega K(x)|V^\epsilon|^2 dx} \right)^{\frac{m-1}{2}} V^\epsilon
\]
\[
:= t^\epsilon_k V^\epsilon \in \mathcal{N}_c(\mu_0, \mu_1, \lambda_0).
\]

Then
\[
T^s(\pi_k(V^\epsilon)) = \frac{\int_\Omega \psi^s(x) |\nabla \pi_k(V^\epsilon)|^2 dx}{\int_\Omega |\nabla \pi_k(V^\epsilon)|^2 dx}
\]
\[
= \frac{\sum_{j=1}^k \int_{B(b_{j,1}, \epsilon)} \psi^s(x) |\nabla \pi_k(V^\epsilon)|^2 dx}{\sum_{j=1}^k \int_{B(b_{j,1}, \epsilon)} |\nabla \pi_k(V^\epsilon)|^2 dx}.
\]
Proof. \( \mu \) exists and holds.

By (3.3) in [0 < \( \epsilon \) < \( \epsilon \)], uniformly in \( \epsilon \).

\[ \int_{\Omega} |\nabla V_{i}^{2}|^{2} dx = kS^{2} + O(\epsilon^{-2}), \]

\[ \int_{\Omega} K(x)|V_{i}^{2}|^{2} dx = KmS^{2} + O(\epsilon) + O(\epsilon^{4}), \]

\[ \sum_{i=1}^{k} \mu_{i} \int_{\Omega} \frac{|V_{i}^{2}|^{2}}{|x-a|^{2}} dx = \sum_{i=1}^{k} \frac{k\mu_{i}^{2}}{2} \int_{\Omega} |U_{i}^{2}|^{2} dx + o(\epsilon^{2}). \]

\[ \lambda_{0} \int_{\Omega} |V_{i}^{2}|^{2} dx = O(\epsilon^{2}). \]

By (3.3) in [25], we also know that

\[ \int_{\Omega} \frac{|V_{i}^{2}|^{2}}{|x|^{2}} dx \geq \epsilon^{2}, \quad \epsilon \to 0. \]

The above estimates imply that there exists \( t_{1} > 0 \) such that \( t_{1}^{*} \geq t_{1} \) as \( \epsilon \) small enough. Hence

\[ \max_{t \geq t_{1}} J_{\epsilon}(tV_{i}^{2}) \leq \max_{\epsilon \leq 0} \left( \int_{\Omega} \left( \frac{t^{2}}{2} |\nabla V_{i}^{2}|^{2} - \frac{t^{2*}}{2} K(x)|V_{i}^{2}|^{2*} \right) dx \right) \]

\[ - \sum_{i=1}^{k} \mu_{i} \int_{\Omega} \frac{|V_{i}^{2}|^{2}}{|x-a|^{2}} dx - \frac{t_{1}^{2} \lambda_{0}}{2} \int_{\Omega} |V_{i}^{2}|^{2} dx - \frac{t_{1}^{2} \mu_{0}}{2} \int_{\Omega} |V_{i}^{2}|^{2} \frac{1}{|x|^{2}} dx. \]

Since \( N \geq 5 \), \( \sum_{i=1}^{k} \frac{\mu_{i}^{2}}{|x-a|^{2}} > 0 \), \( \alpha_{4} > 2 \), there exist \( \epsilon^{1}_{0} > 0 \), \( \epsilon^{0}_{0} > 0 \) such that (5.1) holds.

Let \( \mathbb{R}^{N} = \bigcup_{i=1}^{k} \mathbb{R}^{N} \) with \( \mathbb{R}^{N} = \{ x = (y, z) \in \mathbb{R}^{2} \times \mathbb{R}^{N-2} : y = |y| \cos \theta, \ sin \theta, \ |x-a| = \theta \leq \frac{\mu^{2}}{k} \} \), \( i = 1, 2, \ldots, k \).

**Lemma 5.7.** Let \( N \geq 5 \), \( \mu_{0}^{+} + \sum_{i=1}^{m} k_{\Phi_{i}}^{+} < \mu_{0} \). Assume that \( (K_{4}), (K_{5}), (K_{6}) \), and \( (\Phi) \) hold, then there exist \( \epsilon^{2}_{0} > 0 \), \( \epsilon^{1}_{0} > 0 \), \( \epsilon_{0} > 0 \) such that for all \( 0 < \mu_{0} < \epsilon^{2}_{0} \), \( \epsilon_{0} \leq \mu_{l} < \epsilon^{1}_{0} \), \( |\mu_{l}| \leq \mu_{l} \) \( l = 1, \ldots, m, l \neq L \), \( \mu_{0} \leq \lambda_{0} < \epsilon^{1}_{0} \), it holds

\[ \mathcal{E} \leq \eta^{4}. \]

**Proof.** By contradiction we assume that there exist \( \mu_{0}^{+} \to 0, \mu_{l}^{+} \to 0, \lambda_{0}^{+} \to 0 \) and \( u_{n} \in H^{s}(\mu_{0}, \mu_{l}, \lambda_{0}) \) such that

\[ J_{\epsilon}(u_{n}) = \frac{1}{2} \left( \int_{\Omega} |\nabla u_{n}|^{2} - \mu_{0}^{2} |u_{n}|^{2} \right) dx - \sum_{i=1}^{k} \mu_{i} \int_{\Omega} \frac{|u_{n}|^{2}}{|x-a|^{2}} dx \]

\[ - \frac{\lambda_{0}^{2}}{2} \int_{\Omega} |u_{n}|^{2} dx - \frac{1}{2} \int_{\Omega} K(x)|u_{n}|^{2*} dx. \]
\[ c \leq \tilde{c} = \frac{1}{N} \frac{\mathcal{S}}{\mathcal{K}_M^{\frac{d}{2}}} \]

Then it is obvious that \( \{u_n\} \) is bounded. Denote \( u'_1 := u_n|_{\mathcal{R}^N \cap \Omega} \). Now let us consider \( u'_1 \) in \( H^1_0(\mathcal{R}^N \cap \Omega) \).

Up to a subsequence, there exists \( l > 0 \) such that

\[
\lim_{n \to \infty} \int_{\mathcal{R}^N \cap \Omega} |\nabla u'_1|^2 \, dx = \lim_{n \to \infty} \int_{\mathcal{R}^N \cap \Omega} K(x) |u'_1|^2 \, dx = l.
\]

As in Lemma 3.11 in [18], we deduce that \( l = \frac{\mathcal{S}}{\mathcal{K}_M^{\frac{d}{2}}} \) and then

\[
\lim_{n \to \infty} \int_{\mathcal{R}^N \cap \Omega} (K_M - K(x)) |u'_1|^2 \, dx = 0,
\]

which implies a contradiction. \( \square \)

**Lemma 5.8.** Let \( N \geq 5, \mu_0^+ = \sum_{i=1}^{m} \mu_i^+ < \mathcal{T} \). Assume that \( (\mathcal{K}_k), (\mathcal{K}_3), (\mathcal{K}_0), (\mathcal{H}) \) hold and \( 0 < \mu_0 < \min\{\mu_0^+, \mu_0^+ \}, 0 < \mu_0 < \epsilon_{\mu_0}^+, 0 \leq \lambda_0 < \min\{\lambda_{\mu_0}, \epsilon_{\mu_0}^{-}, \lambda_0^{-}\} \). Then for all \( u \in M^t(\mu_0, \mu_1, \lambda_0) \), there exist \( \rho_0 > 0 \) and a differential function \( f : B(0, \rho_0) \subset (H^1)^k(\Omega) \to \mathbb{R} \) such that \( f(0) = 1 \) and \( f(w)(u - w) \in M^t(\mu_0, \mu_1, \lambda_0) \), \( \forall w \in B(0, \rho_0) \). Moreover, for all \( v \in (H^1)^k(\Omega) \),

\[
(f'(0), v) = 2 \int_{\Omega} \left( \nabla uu \cdot \nabla - \mu_0 \frac{uu}{|x|} \right) dx - 2 \sum_{i=1}^{m} \sum_{j=1}^{k} \mu_i \int_{\Omega} \frac{uu}{|x-a|^2} dx - 2 \lambda_0 \int_{\Omega} uv dx - 2^* \int_{\Omega} K(x) |u|^{2^*-2} uv dx.
\]

**Proof.** The proof is standard and we sketch it here. For \( u \in M^t(\mu_0, \mu_1, \lambda_0) \), define a function \( F : \mathbb{R} \times (H^1)^k(\Omega) \to \mathbb{R} \) by

\[
F_u(t, w) := \langle f'_u(t(u-w)), t(u-w) \rangle = t^2 \int_{\Omega} \left( (\nabla(u-w))^2 - \mu_0 \frac{(u-w)^2}{|x|^2} \right) dx - t^2 \sum_{i=1}^{m} \sum_{j=1}^{k} \mu_i \int_{\Omega} \frac{u^2}{|x-a|^2} dx - t^2 \lambda_0 \int_{\Omega} (u-w)^2 dx - t^{2^*} \int_{\Omega} K(x) |u-w|^{2^*} dx.
\]

Then \( F_u(1, 0) = \langle f'_u(u), u \rangle = 0 \) and

\[
\frac{d}{dt} F_u(1, 0) = 2 \int_{\Omega} \left( |\nabla u|^2 - \mu_0 \frac{u^2}{|x|^2} \right) dx - 2 \sum_{i=1}^{m} \sum_{j=1}^{k} \mu_i \int_{\Omega} \frac{u^2}{|x-a|^2} dx
\]

\[
- 2 \lambda_0 \int_{\Omega} u^2 dx - 2^* \int_{\Omega} K(x) |u|^{2^*} dx \neq 0.
\]

By using the implicit function theorem the results follow. \( \square \)
Now we prove the main result in this part.

**Proof of Theorem 5.1.** Let $0 < \mu_0 < \epsilon_{\mu_0} := \min\{\epsilon_{\mu_0}^0, \epsilon_{\mu_0}^1, \epsilon_{\mu_0}^2\}, 0 \leq \mu_L < \epsilon_{\mu_L} := \min\{\epsilon_{\mu_L}^0, \epsilon_{\mu_L}^1\}, |\mu_l| < \epsilon_{\mu_l} (l = 1, \ldots, m, l \neq L), 0 \leq \lambda_0 < \epsilon_{\lambda_0} := \min\{\epsilon_{\lambda_0}^0, \epsilon_{\lambda_0}^1\},$ where $\epsilon_{\mu_0}^0, \epsilon_{\mu_0}^1, \epsilon_{\mu_0}^2, \epsilon_{\mu_L}^0, \epsilon_{\mu_L}^1, \epsilon_{\lambda_0}^0, \epsilon_{\lambda_0}^1$ are given in Lemmas 5.2, 5.6 and 5.7.

Let $\{u_n\} \subset (H^1_0)^k(\Omega)$ be a minimizing sequence for $J_k$ in $M^*(\mu_0, \mu_l, \lambda_0)$, that is, $J_k(u_n) \to m^*$ as $n \to \infty$. We assume $u_n \geq 0$ since $J_k(u_n) = J_k(\text{sup} \{u_n\})$. Then the Ekeland variational principle implies the existence of a subsequence of $\{u_n\}$, denoted also by $\{u_n\}$, such that

$J_k(u_n) \leq m^* + \frac{1}{n}, \quad J_k(w) \geq J_k(u_n) - \frac{1}{n}\|w - u_n\|, \quad \forall w \in M^*(\mu_0, \mu_l, \lambda_0).$

Choose $0 < \rho < \rho_n \equiv \rho_{u_n}$ and $f_n \equiv f_{u_n}$, where $\rho_{u_n}, f_{u_n}$ are given by Lemma 5.8. Set $v_\rho = \rho v$ with $v \in (H^1_0)^k(\Omega)$ and $\|v\|_{H^1_0(\Omega)} = 1$, then $v_\rho \in B(0, \rho_n)$. By using Lemma 5.8, we get $w_\rho = f_\rho(v_\rho)(u_n - v_\rho) \in M^*(\mu_0, \mu_l, \lambda_0)$. As in Theorem 3.13 in [18], it follows $J'_k(u_n) \to 0$ as $n \to \infty$. Therefore $\{u_n\}$ is a PS sequence for $J_k$. Lemma 5.6 gives $m^* < \overline{c}$. Then we end the proof by Lemmas 4.2 and 5.2. \hfill $\Box$.

5.2. Part II

Now we consider the existence of multiple solutions by using the Lusternik–Schnirelmann category theory. The ideas are borrowed from [18, 26, 27].

For $\delta > 0$, set

$E_\delta(K) := \{x \in \Omega \mid \text{dist}(x, \mathcal{C}(K)) \leq \delta\}.$

We need the following.

$(\mathcal{K}_3), b \in \Omega \cap \mathbb{R}^2 \times \{0\}$ for every $b \in \mathcal{C}(K)$.

Note that if $b = (b(2), 0) \in \mathcal{C}(K)$, then $b_l := (e^{2\pi(1-\sqrt{-1})/k}b(2), 0) \in \mathcal{C}(K)$ for every $1 \leq l \leq k$. $(\mathcal{K}_3)'$. There exists $a_\delta > 2$ such that if $b \in \mathcal{C}(K)$, then $K(x) = K(b_l) + O(|x - b_l|^{a_\delta})$ as $x \to b_l$, for every $1 \leq l \leq k$. $(\mathcal{K}_7).$ There exist $R_0$ and $d_0 > 0$ such that $B(0, R_0) \subset \Omega$ and $\sup_{x \in \Omega \setminus |x| > R_0} |K(x)| \leq K_M - d_0$.

Set

$\mathcal{N}_k(\mu_0, \mu_l, \lambda_0) \equiv \{u \in \mathcal{N}_k(\mu_0, \mu_l, \lambda_0) : J_k(u) < \overline{c}\}.$

where $\overline{c}$ is given in Lemma 5.2.

**Theorem 5.9.** Let $N \geq 5, \delta > 0, \mu_0^2 + \sum_{i=1}^m k \mu_i^2 < \mathcal{P}$. If $(\mathcal{K}_3), (\mathcal{K}_3)', (\mathcal{K}_6), (\mathcal{K}_7), (\mathcal{H})$ hold and

$$\sum_{i=1}^m \frac{\mu_i^2}{|d_i^2 - b_i|^2} > 0, \quad \text{for every } b \in \mathcal{C}(K),$$

then there exist $\epsilon_{\mu_0}^0 > 0, \epsilon_{\mu_0}^1 > 0 (l = 1, \ldots, m), \epsilon_{\lambda_0}^0 > 0$ such that for all $0 < \mu_0 < \epsilon_{\mu_0}^0, 0 \leq \mu_L < \epsilon_{\mu_L}^0, |\mu_l| < \epsilon_{\mu_l}^0 (l = 1, \ldots, m, l \neq L), 0 \leq \lambda_0 < \epsilon_{\lambda_0}^0$, the problem $(\mathcal{B}_{0, K})$ admits at least $\text{Cat}_{C_b(K)}\mathcal{C}(K)$ positive solutions which are $\mathcal{Z}_k \times \text{SO}(N - 2)$-invariant.

The proof of the above theorem depends on some lemmas.

**Lemma 5.10.** Let $\mu_0^2 + \sum_{i=1}^m k \mu_i^2 < \mathcal{P}, 0 < \mu_0 \leq \epsilon_{\mu_0}^0, 0 \leq \mu_L \leq \epsilon_{\mu_L}^0$ (where $\epsilon_{\mu_0}^0, \epsilon_{\mu_L}^0$ are constants appearing in Lemma 5.2) and $(\mathcal{K}_3)$ hold. If $\{u_n\} \subset \mathcal{N}_k(\mu_0, \mu_l, \lambda_0)$ satisfies

$J_k(u_n) \to c < \overline{c}, \quad J'_k|_{\mathcal{N}_k(\mu_0, \mu_l, \lambda_0)}(u_n) \to 0,$

then $\{u_n\}$ has a converging subsequence in $(H^1_0)^k(\Omega)$.

**Proof.** By using Lemmas 4.2 and 5.2, the result follows as Lemma 4.1 in [18]. \hfill $\Box$.
Denote $\Omega := (\mathbb{R}^N \cap \Omega) \setminus \partial (\mathbb{R}^N \cap \Omega)$.

**Lemma 5.11.** Let $N \geq 5$, $\mu_0^+ + \sum_{i=1}^{m} k \mu_i^+ < \Pi$. Assume that $(\mathcal{K}_2)$, $(\mathcal{K}_3)'$, $(\mathcal{K}_5)'$ hold and

$$\sum_{i=1}^{m} \sum_{j=1}^{k} \left( \frac{\mu_i}{|a_j - b_i|^2} \right) > 0, \quad \text{for every } b \in C(\Omega).$$

Then there exist $\varepsilon_0^0 > 0$, $\varepsilon_1^0 > 0$ such that if $0 < \mu_0 < \varepsilon_0^0$, $0 \leq \lambda_0 < \varepsilon_1^0$, it holds $\mathcal{N}_k(\mu_0, \mu_1, \lambda_0) \neq \emptyset$. Moreover, for any $\mu_0^+ \to 0$, $\mu_1^+ \to 0$, $\lambda_0 \to 0$ as $n \to \infty$ and $(v_n) \in \mathcal{N}_k(\mu_0^+, \mu_1^+, \lambda_0^+)$, there exist $x_n^1 := (x_n^{1i}, x_n^{1(N-2)}) \in \Omega_1$ and $r_n \in \mathbb{R}^+$ such that $x_n^1 \to x_0^1 = (x_0^{1i}, 0) \in C(\Omega)$, $r_n \to 0$ and

$$v_n^1 = \left( \frac{S}{K_M} \right)^{\frac{N-2}{4}} u_r \left( \frac{\cdot - x_n^1}{r_n} \right) \to 0 \quad \text{in } \mathcal{D}^{1,2}(\Omega_1), \quad \text{as } n \to \infty, \; i = 1, 2, \ldots, k,$$

where $v_n^i = v_n|_{\Omega_1}$, $x_n^i = \left( e^{2\pi \sqrt{-1/k}x_n^{j-1,2}}, x_n^{(N-2)} \right)$, $x_0^1 = \left( e^{2\pi \sqrt{-1/k}x_0^{j-1,2}}, 0 \right)$, and

$$u_r \left( \frac{\cdot - x_n^1}{r_n} \right) = \frac{C_{r, r_n}}{\left( r^2 + \frac{|x_n^1|^2}{r_n^2} \right)^{\frac{N-2}{4}}}$$

with $C_{r, r_n}$ the normalizing constant such that $\|u_r(\frac{\cdot - x_n^1}{r_n})\|_{2^*} = 1$.

**Proof.** Using the arguments of Lemma 5.6, it is easy to get $\mathcal{N}_k(\mu_0, \mu_1, \lambda_0) \neq \emptyset$. Now we prove the second part.

Consider $v_n^1$ in $H_1^1(\mathbb{R}^N \cap \Omega)$ and set as in Lemma 3.11 in [18] that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N \cap \Omega} |\nabla v_n^1|^2 \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N \cap \Omega} K(x) |v_n^1|^2 \, dx = 1.$$ 

Then $l = \frac{S}{K_M}^{\frac{N-2}{4}}$ and hence

$$\lim_{n \to \infty} \int_{\mathbb{R}^N \cap \Omega} (K_M - K(x)) |v_n^1|^2 \, dx = 0.$$

Set $u_n(x) := \frac{v_n^1}{|v_n^1|_2^{2^*}}$, then $\|u_n(x)\|_{2^*} = 1$ and

$$\int_{\mathbb{R}^N \cap \Omega} |\nabla u_n(x)|^2 \, dx \to S.$$

Therefore

$$\int_{\Omega_1} |u_n(x)|^{2^*} \, dx \to 1, \quad \int_{\Omega_1} |\nabla u_n(x)|^2 \, dx \to S.$$

By using Corollary 4.1 in [28] and a similar proof to Lemma 4.2 in [18], there exist $x_n^1 \in \Omega_1$ and $r_n \in \mathbb{R}^+$ such that $x_n^1 \to x_0^1 = \mathcal{C}_3(\Omega)$, $r_n \to 0$ as $n \to \infty$ and

$$w_n = u_r \left( \frac{\cdot - x_n^1}{r_n} \right) \to 0 \quad \text{in } \mathcal{D}^{1,2}(\Omega_1), \quad \text{as } n \to \infty.$$

Hence

$$v_n^1 = \left( \frac{S}{K_M} \right)^{\frac{N-2}{4}} u_r \left( \frac{\cdot - x_n^1}{r_n} \right) \to 0 \quad \text{in } \mathcal{D}^{1,2}(\Omega_1).$$
By recalling the symmetry of \( v_n \), we end the proof. \( \square \)

To continue, as in [18] we define

\[
\xi(x) := \begin{cases} 
\frac{x}{R_0}, & \text{if } |x| < R_0, \\
\frac{R_0}{|x|}, & \text{if } |x| \geq R_0.
\end{cases}
\]

For any \( 0 \neq u \in (H_0^1)^k(\Omega) \), set

\[
\Theta(u) := \frac{\int_{\Omega} \xi(x)|\nabla u|^2 \, dx}{\int_{\Omega} |\nabla u|^2 \, dx}.
\]

From the proof of Lemma 5.6, it is known that for any \( 0 \neq u \in (H_0^1)^k(\Omega) \), \( t_{\mu_0, \mu_1, \lambda_0}(u) \in \mathcal{N}_k(\mu_0, \mu_1, \lambda_0) \) with

\[
t_{\mu_0, \mu_1, \lambda_0}(u) = \left( \frac{\int_{\Omega} |\nabla u|^2 - \mu_0 \frac{\lambda_0}{|x|^2} - \lambda_0 u^2 \, dx - \sum_{i=1}^m \mu_i \int_{\Omega} \frac{u^2}{|x-\xi_i|^2} \, dx}{\int_{\Omega} K(x)|u|^2 \, dx} \right)^{\frac{n-2}{2}}.
\]

For \( b = (b^{(i)})_i \in \mathbb{R}^2 \times \{0\} \), define \( \Psi_k : \Omega \to (H_0^1)^k(\Omega) \) as

\[
\Psi_k(b)(x) = t_{\mu_0, \mu_1, \lambda_0}(U_{k(\mu_0, \mu_1, \lambda_0)}(x))U_{k(\mu_0, \mu_1, \lambda_0)}(x) := t_{\mu_0, \mu_1, \lambda_0}(U_{k(\mu_0, \mu_1, \lambda_0)}(x), \varphi(x), \text{a radial function, satisfying}
\]

\[
0 \leq \varphi \leq 1, \quad \varphi = 1 \text{ if } x \in \bigcup_{i=1}^k B \left( b_i, \frac{r}{2} \right), \quad \varphi = 0 \text{ if } x \notin \bigcup_{i=1}^k B(b_i, r), \quad |\nabla \varphi| \leq \frac{4}{r}
\]

with \( r > 0 \) small enough, and \( \epsilon(\mu_0, \mu_1, \lambda_0) \to 0 \) as \( \mu_0 \to 0, \lambda_0 \to 0, \mu_1 \to 0 \).

The proof of the following lemma is almost the same as Lemma 5.6.

**Lemma 5.12.** Let \( N \geq 5 \), \( b \in C(K), \mu_0^+ + \sum_{i=1}^m k_{\mu_1} < \tau \). If \((\mathcal{K}_4), (\mathcal{K}_5)', (\mathcal{K}_6)'\) hold and

\[
\sum_{i=1}^m k_{\mu_1} < \tau, \quad \lambda_0 \geq 0,
\]

then there exist \( \tau_{\mu_0}^{1}, \tau_{\mu_1}^{1} \) such that

\[
j_k(\Psi_k(b)) = \max_{t>0} j_k(U_{k(\mu_0, \mu_1, \lambda_0)}(x)) < \tau \quad \text{for all } 0 < \mu_0 < \tau_{\mu_0}^{1}, \quad 0 < \mu_1 < \tau_{\mu_1}^{1}.
\]

By Lemma 5.12 we see that if \( 0 < \mu_0 \leq \tau_{\mu_0}^{1} \), then \( \Psi_k(b) \in \mathcal{N}_k(\mu_0, \mu_1, \lambda_0) \). It is also obvious that \( j_k(\Psi_k(b)) = \tau + o(1) \) as \( \mu_0 \to 0, \mu_1 \to 0, \lambda_0 \to 0 \) and there exist \( c_1 > 0, c_2 > 0 \) such that

\[
c_1 < t_{\mu_0, \mu_1, \lambda_0}U_{k(\mu_0, \mu_1, \lambda_0)}(x) < c_2 \text{ for all } b \in C(K).
\]

**Lemma 5.13.** Let \( N \geq 5 \), \( \mu_0^+ + \sum_{i=1}^m k_{\mu_1} < \tau, b \in C(K) \) and \((\mathcal{K}_4), (\mathcal{K}_5)', (\mathcal{K}_6)'\) hold. For all \( b \in C(K), \) it follows

\[
|\nabla \Psi_k(b)|^2 \to d\mu = \sum_{i=1}^k \frac{S_{\mu_i}^2}{K_{d, \mu_i}^2} \delta_{b_i}, \quad |\Psi_k(b)|^{2s} \to d\nu = \sum_{i=1}^k \frac{S_{\mu_i}^{2s}}{K_{d, \mu_i}^{2s}} \delta_{b_i},
\]

as \( \mu_0 \to 0, \mu_1 \to 0, \lambda_0 \to 0 \).
Proof. Note that $\Psi_k(b)$ is bounded in $(H^1_0)^k(\Omega)$ and
\[
\int_{\Omega} |\nabla \Psi_k(b)|^2 \, dx = \sum_{i=1}^{K} \int_{(b_i,r)} |\nabla (t_{\mu_0,\mu_1,\lambda_0} \phi(x) |U_0^{(\mu_0,\mu_1,\lambda_0)}(x-b_i))|^2 \, dx,
\]
\[
\int_{\Omega} K(x)|\Psi_k(b)|^{2^*} \, dx = \sum_{i=1}^{K} \int_{(b_i,r)} K(x)|t_{\mu_0,\mu_1,\lambda_0} \phi(x) |U_0^{(\mu_0,\mu_1,\lambda_0)}(x-b_i)|^{2^*} \, dx.
\]
Assume $\mu_0^n \to 0$, $\mu_1^n \to 0$, $\lambda_0^n \to 0$ as $n \to \infty$. Up to a subsequence, we get the existence of $l > 0$ such that
\[
\lim_{n \to \infty} \int_{(b_i,r)} |\nabla (t_{\mu_0^n,\mu_1^n,\lambda_0^n} \phi(x) |U_0^{(\mu_0^n,\mu_1^n,\lambda_0^n)}(x-b_i))|^2 \, dx
\]
\[
= \lim_{n \to \infty} \int_{(b_i,r)} K(x)|t_{\mu_0^n,\mu_1^n,\lambda_0^n} \phi(x) |U_0^{(\mu_0^n,\mu_1^n,\lambda_0^n)}(x-b_i)|^{2^*} \, dx
\]
\[
= l.
\]
Then $l = \frac{\alpha^2}{K_M}$ and
\[
\lim_{n \to \infty} \int_{(b_i,r)} (K_M - K(x))|t_{\mu_0^n,\mu_1^n,\lambda_0^n} \phi(x) |U_0^{(\mu_0^n,\mu_1^n,\lambda_0^n)}(x-b_i)|^{2^*} \, dx = 0.
\]
Set $u_i(x) := \frac{t_{\mu_0^n,\mu_1^n,\lambda_0^n} \phi(x) |U_0^{(\mu_0^n,\mu_1^n,\lambda_0^n)}(x-b_i)}{|t_{\mu_0^n,\mu_1^n,\lambda_0^n} \phi(x) |U_0^{(\mu_0^n,\mu_1^n,\lambda_0^n)}(x-b_i)||^{2^*}}$, then $\|u_i(x)\|_{2^*} = 1$ and
\[
\int_{(b_i,r)} |\nabla u_i(x)|^2 \, dx \to S.
\]
Applying the arguments of Theorem 3.13 in [18], it holds
\[
|\nabla u_i|^2 \to d\mu_i = S\delta_{b_i}, \quad |u_i|^{2^*} \to d\nu_i = \widetilde{S}\delta_{b_i},
\]
which ends the proof. \(\square\)

Lemma 5.14. Let $N \geq 5$, $\mu_0^n + \sum_{i=1}^{m} k_{i} \mu_i^n < \overline{\mu}$ and $(\mathcal{K}_4)$, $(\mathcal{K}_5)'$, $(\mathcal{K}_6)'$, $(\mathcal{K}_7)$ hold. For $\mu_0 \to 0$, $\mu_i \to 0$, $\lambda_0 \to 0$.

1. $\Theta(\Psi_k(b)) = b + o(1)$, uniformly for $b \in B(0, R_0) \cap C(K)$;
2. $\sup \{\text{dist}(\Theta(u), C(K)) : u \in N_k(\mu_0^n, \mu_i, \lambda_0^n)\} \to 0$.

Proof. (1) Let $b \in B(0, R_0) \cap C(K)$. By Lemma 5.13, we have
\[
\Theta(\Psi_k(b)) = \frac{\int_{\Omega} \xi(x)|\nabla \Psi_k(b)|^2 \, dx}{\int_{\Omega} |\nabla \Psi_k(b)|^2 \, dx} = \frac{\int_{\Omega} \xi(x) d\mu}{\int_{\Omega} d\mu} + o(1) = b + o(1),
\]
where $\mu_0 \to 0$, $\mu_i \to 0$, $\lambda_0 \to 0$.

2. Take $\mu_0^n \to 0$, $\mu_i^n \to 0$, $\lambda_0^n \to 0$ as $n \to \infty$ and $\{v_n\} \in N_k(\mu_0^n, \mu_i^n, \lambda_0^n)$. By Lemma 5.11, there exist $x_i^n = (x_i^{(1)}, x_i^{(2)}, x_i^{(3)}) \in \Omega$, and $r_i \in \mathbb{R}^+$ such that $x_i^n \to x_i^n = (x_i^{(1)}, x_i^{(2)}) \in C(K)$, $r_i \to 0$ and
\[
v_i^n = \left( S \left( \frac{\alpha^2}{K_M} \right) - \frac{2}{\lambda_0^n} \right) \frac{x_i^n}{r_i} \to 0 \quad \text{in} \ D^{1,2}(\Omega_i), \quad \text{as} \ n \to \infty, \quad i = 1, 2, \ldots, k.
\]
Since $\Theta(u)$ is continuous, then
\[
\Theta(v_n) = \frac{\int_{\Omega} \xi(x)|\nabla v_n|^2 \, dx}{\int_{\Omega} |\nabla v_n|^2 \, dx} = \frac{\int_{\Omega} \xi(x) \left| \nabla u_r \left( \frac{x-x_n}{\tau_n} \right) \right|^2 \, dx}{\int_{\Omega} \left| \nabla \left( \frac{x-x_n}{\tau_n} \right) \right|^2 \, dx} + o(1) = \xi(x_0^+) + o(1).
\]
Noticing that $x_0^+ \in B(0, R_0)$ leads to $\xi(x_0^+) = x_0^+$, the result desired is true. \hfill $\square$

**Proof of Theorem 5.9.** We follow the arguments of Theorem 4.5 in [18] and Theorem A in [27].

Given $\delta > 0$, by using Lemmas 5.12 and 5.14, there exist $\epsilon_{\mu_0}^* > 0$, $\epsilon_{\mu_1}^* > 0$ ($l = 1, \ldots, m$), $\epsilon_{\lambda_0}^* > 0$ such that for all $0 < \mu_0 < \epsilon_{\mu_0}^*$, $0 \leq \mu_L < \epsilon_{\mu_L}^*$, $|\mu_l| < \epsilon_{\mu_l}^*$ ($l = 1, \ldots, m, l \neq L$), $0 \leq \lambda_0 < \epsilon_{\lambda_0}^*$, we have $\Psi_k(b) \in \mathcal{H}_K(\mu_0, \mu_L, \lambda_0)$ for any $b \in \mathcal{C}(K)$ and
\[
|\Theta(\Psi_k(b)) - b| < \delta, \quad \forall b \in B(0, R_0) \cap \mathcal{C}(K), \quad \text{and} \quad \Theta(u) \in \mathcal{C}_k(K), \quad \forall u \in \mathcal{H}_k(\mu_0, \mu_L, \lambda_0).
\]
Define $\mathcal{H}(t, x) := x + t(\Theta(\Psi_k(b)) - b)$ with $(t, x) \in [0, 1] \times \mathcal{C}(K)$. Then $\mathcal{H}$ is continuous, $\mathcal{H}([0, 1] \times \mathcal{C}(K)) \subset \mathcal{C}(K)$ and $\Theta \circ \mathcal{H}$ is homotopic to the inclusion $\mathcal{C}(K) \hookrightarrow \mathcal{C}(K)$ by Lemma 5.10. It is enough to prove $\text{Cat}(\mathcal{H}_K(\mu_0, \mu_L, \lambda_0)) \geq \text{Cat}_{\mathcal{C}_k(K)}(K)$, which can be obtained as Theorem 4.5 in [18], and therefore the problem $(\mathcal{H}_{\mu_0, \lambda_0})$ admits at least $\text{Cat}_{\mathcal{C}_k(K)}(K)$ solutions which are $\mathbb{Z}_k \times \mathcal{O}(N-2)$-invariant.

It is also easy to show that any of these solutions has a fixed sign, as Theorem 4.5 in [18] and Theorem 1.1 in [29]. \hfill $\square$

**Remark 5.15.** (1) The problem considered here involves symmetry and multiple solutions are obtained, which are different from [12,13,15], where the symmetry is not been considered and existence, but not multiplicity, of solutions is obtained. In [14], Felli and Terracini considered (1.1) with symmetric multi-polar potentials in $\mathbb{R}^N$ when $K(x) \equiv 1$ and proved the existence of positive solutions, while the domain we considered here is bounded and $K(x)$ is positive bounded.

(2) In [30], Felli and Schneider considered the problem related to the Caffarelli–Kohn–Nirenberg type inequality and showed the symmetry-breaking phenomenon in the inequality, namely the existence of non-radial minimizers, which motivates us to study the symmetry-breaking phenomenon of problem (1.1) in the near future. In [31], Badiale and Rolando also proved the existence of positive radial solutions for the semilinear elliptic problem with singular potentials, where both sub-critical and super-critical nonlinearities, rather than critical nonlinearities, are considered.

(3) It can be seen that when $\lambda_0 \neq 0$ and the domain $\Omega$ is bounded, there exist nontrivial solutions for the problem $(\mathcal{H}_{\mu_0, \lambda_0})$. However, if we consider the problem in $\mathbb{R}^N$, then $\lambda_0 \neq 0$ may lead to the nonexistence of nontrivial solutions. Please see Appendix for a nonexistence result.

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**Appendix**

Consider the quasilinear elliptic problem
\[
\begin{cases}
-\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = \lambda |u|^{p-2}u + K(x) |u|^{p^*-2}u & \text{in } \mathbb{R}^N, \\
u \in W^{1,p}(\mathbb{R}^N),
\end{cases}
\]  

where $-\Delta_p u := -\text{div}(|\nabla u|^{p-2} \nabla u)$, $1 < p < N$, $\lambda \in \mathbb{R}$, $\mu < (\frac{N-p}{p})^p$, $p^* := \frac{Np}{N-p}$ is the critical Sobolev exponent, $1 < q < p^*$, $K(x) \in C^1(\mathbb{R}^N)$ satisfying $|K|_{\infty} < \infty$.

When $p = 2$, $K(x) \equiv 1$, the nonexistence of nontrivial solutions for problem (A.1) can be seen in [32]. Here we state the nonexistence result for problem (A.1) motivated by [11].
Theorem A. Assume one of the following three cases holds:

1. \( \lambda > 0, \langle x, \nabla K \rangle \leq 0 \), for a.e. \( x \in \mathbb{R}^N \),
2. \( \lambda < 0, \langle x, \nabla K \rangle \geq 0 \), for a.e. \( x \in \mathbb{R}^N \),
3. \( \lambda = 0, \langle x, \nabla K \rangle > 0 \), for a.e. \( x \in \mathbb{R}^N \),
   or \( \lambda = 0, \langle x, \nabla K \rangle < 0 \), for a.e. \( x \in \mathbb{R}^N \).

Then if \( u \in W^{1,p}(\mathbb{R}^N) \) is a weak solution for problem (A.1), there holds \( u \equiv 0 \).

Proof. Denote \( f(x,u) = \mu \frac{|u|^{p-2} u}{|x|^p} + \lambda |u|^{p-2} u + K(x)|u|^{p-2} u \). Then problem (A.1) can be rewritten as

\[
\begin{align*}
-\Delta_p u &= f(x,u) \quad \text{in } \mathbb{R}^N, \\
\{ u \in W^{1,p}(\mathbb{R}^N). 
\end{align*}
\]  

(A.2)

It is easy to see that, for any \( \omega \in \mathbb{R}^N \setminus \{0\} \), there exists \( C(\omega) > 0 \) such that \( |f(x,u)| \leq C(\omega)(1 + |u|^{p-1}), \forall x \in \omega, u \in \mathbb{R} \). Then as Claim 5.5 in [11], \( u \in C^1(\mathbb{R}^N \setminus \{0\}) \cap W^{2,2}_{loc}(\mathbb{R}^N \setminus \{0\}) \) following from Lemmas 2.1 and 2.2 in [33], Corollary 1.1 in [34] and Theorem 1, Proposition 1 in [35].

Now we prove \( u \in L^q(\mathbb{R}^N) \). Take a cut-off function (see Claim 5.3 in [11]) \( h \in C^\infty(\mathbb{R}) \) satisfying \( h(t) \equiv 0, h(0) \equiv 1, 0 \leq h \leq 1 \). Given \( \epsilon > 0 \) small enough, define \( \eta_\epsilon \) as: \( \eta_\epsilon(x) = h(|x|/\epsilon) \) if \( |x| \leq 3\epsilon \), \( \eta_\epsilon(x) = h(1/\epsilon|x|) \) if \( |x| \geq 2\epsilon \), and \( \eta_\epsilon(x) \equiv 1 \) elsewhere. Then it is obvious that \( \eta_\epsilon \in C^\infty(\mathbb{R}^N \setminus \{0\}) \). Since \( u \) is a weak solution for problem (A.1),

\[
\int_{\mathbb{R}^N} \left( |\nabla u|^{p-2} (\nabla u \cdot \nabla (\eta_\epsilon u)) - \mu \eta_\epsilon \frac{|u|^{p-2}}{|x|^p} |u|^p - \lambda \eta_\epsilon |u|^p \right) dx - \int_{\mathbb{R}^N} K(x) \eta_\epsilon |u|^p dx = 0.
\]

Then by the Hardy inequality, the Sobolev inequality and the Hölder inequality,

\[
|\lambda| \int_{\mathbb{R}^N} \eta_\epsilon |u|^p dx = -\int_{\mathbb{R}^N} K(x) \eta_\epsilon |u|^p dx \\
+ \int_{\mathbb{R}^N} |\nabla u|^{p-2} \left( \nabla u \cdot \nabla (\eta_\epsilon u) - \mu \eta_\epsilon \frac{|u|^{p-2}}{|x|^p} |u|^p \right) dx \\
\leq C \int_{\mathbb{R}^N} |\nabla u|^{p-1} |\nabla \eta_\epsilon| |u| dx \\
\leq C,
\]

where the constant \( C > 0 \) is independent of \( \epsilon \). Letting \( \epsilon \to 0 \), we have \( u \in L^q(\mathbb{R}^N) \).

Therefore, by Claim 5.3 in [11],

\[
\lambda N \left( \frac{1}{p^*} - \frac{1}{q} \right) \int_{\mathbb{R}^N} |u|^p dx - \frac{1}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} \langle x, \nabla K \rangle dx = 0.
\]

which implies \( u \equiv 0 \). \( \square \)

When \( K(x) \equiv 1, \lambda \neq 0 \), Theorem A extends the result in [32] to the \( p \)-Laplace case.

References