Existence and multiplicity of solutions for critical elliptic equations with multi-polar potentials in symmetric domains

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ABSTRACT

In this paper, we consider the elliptic equations with critical Sobolev exponents and multi-polar potentials in bounded symmetric domains and prove the existence and multiplicity of symmetric positive solutions by using the Ekeland variational principle and the Lusternik–Schnirelmann category theory.

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1. Introduction

The critical elliptic problem has been studied extensively since the initial work [1] by Brézis and Nirenberg. The critical elliptic problem with one Hardy-type potential has also attracted much attention in recent years. We refer the interested readers to a partial list [2–11] and the references therein. Here, we are concerned with the critical elliptic problem with multi-polar (Hardy-type) potentials. In [12], Cao and Han considered the problem...
and proved the existence of positive and sign-changing solutions in bounded smooth domains, where \( 2^* := \frac{2N}{N-2} \) is the critical Sobolev exponent. In [13], problem (1.1) with \( K(x) \equiv 1 \) was investigated by Felli and Terracini, and the existence of positive solutions with the smallest energy was deduced both in \( \mathbb{R}^N \) and in bounded smooth domains. In [14], Felli and Terracini also studied problem (1.1) with symmetric multi-polar potentials in \( \mathbb{R}^N \) when \( K(x) \equiv 1 \) and showed the existence of symmetric positive solutions. Other related results on critical elliptic problems with multiple Hardy-type potentials can be seen in [15–17] and the references therein.

However, as far as we know, there are few results about the multiplicity of solutions for the critical elliptic problem with multi-polar potentials. In this paper, motivated by [14,18], we consider the critical elliptic problem with symmetric multi-polar potentials in bounded symmetric domains and prove the existence and multiplicity of symmetric positive solutions.

More accurately, we are interested in the problem

\[
\begin{aligned}
\mathcal{P}_{\lambda_0, K} & \left\{ \begin{array}{l}
\begin{aligned}
-\Delta u - \mu_0 \frac{u}{|x|^2} - \sum_{l=1}^{m} \sum_{i=1}^{k} \mu_i \frac{u}{|x-a_i|^2} &= \lambda_0 u + K(x)|u|^{2^*-2} u & \text{in } \Omega,

u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\end{array} \right. \\
& \text{for } \Omega \subset \mathbb{R}^N (N \geq 4) \text{ is a } \mathbb{Z}_k \times SO(N-2)-\text{invariant bounded smooth domain},
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N (N \geq 4) \) is a \( \mathbb{Z}_k \times SO(N-2) \)-invariant bounded smooth domain, \( k \geq 3, \mu_0, \lambda_0 \in \mathbb{R}, \mu_i \in \mathbb{R}, \ l = 1, 2, \ldots, m, \) and \( K(x) \) is a \( \mathbb{Z}_k \times SO(N-2) \)-invariant positive bounded function on \( \bar{\Omega} \). Here the domain \( \Omega \) is said to be \( \mathbb{Z}_k \times SO(N-2) \)-invariant if \( (e^{2\pi \sqrt{T}/k}y, Tz) \in \Omega, \forall x = (y, z) \in \Omega \subset \mathbb{R}^2 \times \mathbb{R}^{N-2} \) and \( K(x) \) is said to be \( \mathbb{Z}_k \times SO(N-2) \)-invariant on \( \bar{\Omega} \) if \( K(y, z) = K(e^{2\pi \sqrt{T}/k}y, Tz), \forall x = (y, z) \in \bar{\Omega} \subset \mathbb{R}^2 \times \mathbb{R}^{N-2} \), where \( T \) is any rotation of \( \mathbb{R}^{N-2} \). Note that if \( \Omega \) is \( \mathbb{Z}_k \times SO(N-2) \)-invariant, we can write \( \Omega = \Omega^{(2)} \times B^{(N-2)}(0, R) \subset \mathbb{R}^2 \times \mathbb{R}^{N-2} \), where \( \Omega^{(2)} \) is \( \mathbb{Z}_k \)-invariant in \( \mathbb{R}^2 \) (that is, \( e^{2\pi \sqrt{T}/k}y \in \Omega^{(2)}, \forall y \in \Omega^{(2)} \) and \( B^{(N-2)}(0, R) \) is a ball in \( \mathbb{R}^{N-2} \) centered at the origin with radius \( R \)). The group \( \mathbb{Z}_k \times SO(N-2) \) acts on \( H^1_{\text{loc}}(\Omega) \) as \( u(y, z) \rightarrow u(e^{2\pi \sqrt{T}/k}y, Tz) \). Given \( m \) regular polygons with \( k \) sides, centered at the origin and lying on the plane \( \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N \), we assume that

(A1). \( a^l_i \in \Omega, i = 1, 2, \ldots, k, \) are the vertices of the \( l \)th polygon, \( l = 1, 2, \ldots, m, \)

(A2). \( \Omega \supset B^{(2)}(0, R) \times B^{(N-2)}(0, \tilde{R}) \) with \( R > \max \{ r_l, l = 1, 2, \ldots, m \} \), where \( r_l = |a^1_l| = |a^2_l| = \cdots = |a^k_l| \).

It is easy to see that the above assumptions can be easily satisfied, such as, a domain

\[
\begin{aligned}
\Omega &= B^{(2)}(0, R) \times B^{(N-2)}(0, \tilde{R}) \\
&= \{ (x_1, x_2, x_3, \ldots, x_N) | x_1^2 + x_2^2 < R^2, x_3^2 + \cdots + x_N^2 < \tilde{R}^2 \}
\end{aligned}
\]
with \( a_i^l = \{a_i^l(2), 0\} \in \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N \), \( a_i^l(2) = e^{2(i-1)\pi \sqrt{-1}/k} \{r_l, 0\}, 0 < r_1 < r_2 < \cdots < r_m < R, i = 1, 2, \ldots, k, \) and \( l = 1, 2, \ldots, m. \)

We will prove the existence and multiplicity of \( \mathbb{Z}_k \times \mathbb{S}(N - 2) \)-invariant positive solutions for the problem \((\mathcal{P}_{\lambda_0,K})\).

Before that, we also consider the limiting case of the problem \((\mathcal{P}_{\lambda_0,K})\), that is,

\[
(\mathcal{P}_{\lambda_0,K}^\infty) \begin{cases} 
-\Delta u - \mu_0 \frac{u}{|x|^2} - \sum_{l=1}^m k\mu_l \left( \delta_{S_{r_l}} * \frac{1}{|x|^2} \right) u = \lambda_0 u + K(x)|u|^{2^*-2}u & \text{in } \Omega_B(R, \tilde{R}), \\
u = 0 & \text{on } \partial\Omega_B(R, \tilde{R}),
\end{cases}
\]

where \( S_{r_l} := \{(x,0) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : |x| = r_l\} \), the distribution \( \delta_{S_{r_l}} \in \mathcal{D}'(\mathbb{R}^N) \) supported in \( S_{r_l} \) and defined by, as in [14],

\[
\mathcal{D}'(\mathbb{R}^N)(\delta_{S_{r_l}}, \varphi)_{\mathcal{D}'(\mathbb{R}^N)} := \frac{1}{2\pi r_l} \int_{S_{r_l}} \varphi(x) d\sigma(x), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^N)
\]

with \( d\sigma \) the line element on \( S_{r_l} \), \( \mathcal{D}(\mathbb{R}^N) \) the space of smooth functions with compact support in \( \mathbb{R}^N \), \( \Omega_B(R, \tilde{R}) := B_{r_1}(0, R) \times B_{r_m}(0, \tilde{R}) \) with \( R > \max\{r_l, l = 1, 2, \ldots, m\} \), \( N \geq 4 \) and \( K(x) \) a \( \mathbb{S}(2) \times \mathbb{S}(N-2) \)-invariant positive bounded function on \( \Omega_B(R, \tilde{R}) \). For simplicity of notation, we write \( \Omega_B(R) \) instead of \( \Omega_B(R, \tilde{R}) \) in the sequel. Here \( K(x) \) is said to be \( \mathbb{S}(2) \times \mathbb{S}(N-2) \)-invariant on \( \Omega_B(R) \) if \( K(y, z) = K(|y|, |z|), \forall x = (y, z) \in \Omega_B(R) \subset \mathbb{R}^2 \times \mathbb{R}^{N-2} \). We will prove the existence of \( \mathbb{S}(2) \times \mathbb{S}(N-2) \)-invariant positive solutions for the problem \((\mathcal{P}_{\lambda_0,K}^\infty)\).

The paper is organized as follows. In Section 2, we give some preliminary results. Section 3 is devoted to the existence of one \( \mathbb{S}(2) \times \mathbb{S}(N-2) \)-invariant positive solution for the problem \((\mathcal{P}_{\lambda_0,K}^\infty)\), provided that \( K(x) \) satisfies some growth condition at zero; for details see Theorem 3.1. In Section 4, we show the existence of one \( \mathbb{Z}_k \times \mathbb{S}(N-2) \)-invariant positive solution for the problem \((\mathcal{P}_{\lambda_0,K})\) in Theorem 4.1. Some growth conditions on \( K(x) \) are needed, of course. In Section 5, the multiplicity of \( \mathbb{Z}_k \times \mathbb{S}(N-2) \)-invariant positive solutions for the problem \((\mathcal{P}_{\lambda_0,K})\) is obtained by the Ekeland variational principle and the Lusternik–Schnirelmann category theory, respectively. Here, besides the growth conditions on \( K(x) \), the parameters \( \lambda_0, \mu_0, \mu_l, l = 1, 2, \ldots, m, \) are requested to be close to zero; for details see Theorems 5.1 and 5.9. At last, a nonexistence result is proved in Appendix.

2. Notations and preliminary results

Throughout this paper, positive constants will be denoted by \( C \).

Denote

\[
(H_0^1)^{\text{circ}}(\Omega_B(R)) := \{u(y, z) \in H_0^1(\Omega_B(R)) : u(y, z) = u(|y|, |z|)\},
\]

where \( (y, z) \in \Omega_B(R) \subset \mathbb{R}^2 \times \mathbb{R}^{N-2} \), and

\[
(H_0^1)^k(\Omega) := \{u(y, z) \in H_0^1(\Omega) : u(e^{2\pi \sqrt{-1}ky}, z) = u(y, |z|)\},
\]
where \((y, z) \in \Omega \subset \mathbb{R}^2 \times \mathbb{R}^{N-2}\).

It is known that the nonzero critical points of the energy functional

\[
J_{\text{circ}}(u) := \frac{1}{2} \int_{\Omega_B(R)} \left( \|\nabla u\|^2 - \mu_0 \frac{u^2}{|x|^2} \right) \, dx
- \sum_{l=1}^m \frac{k_l \mu_l}{2} \int_{\Omega_B(R)} \left( \frac{1}{2\pi r_l} \int_{S_{r_l}} \frac{u^2(y)}{|x - y|^2} \, d\sigma(x) \right) \, dy
- \frac{\lambda_0}{2} \int_{\Omega_B(R)} u^2 \, dx - \frac{1}{2^*} \int_{\Omega_B(R)} K(x)|u|^{2^*} \, dx
\]

defined on \((H_0^1)^{\text{circ}}(\Omega_B(R))\), and the energy functional

\[
J_k(u) := \frac{1}{2} \int_{\Omega} \left( \|\nabla u\|^2 - \mu_0 \frac{u^2}{|x|^2} \right) \, dx
- \sum_{l=1}^m \sum_{i=1}^k \frac{\mu_i}{2} \int_{\Omega} \frac{u^2}{|x - u_i|^2} \, dx
- \frac{\lambda_0}{2} \int_{\Omega} u^2 \, dx
- \frac{1}{2^*} \int_{\Omega} K(x)|u|^{2^*} \, dx
\]
defined on \((H_0^1)^k(\Omega)\) are equivalent to the nontrivial weak solutions for the problem \((\mathcal{P}_{\lambda_0,K})\) and \((\mathcal{P}_{\lambda_0,K})\), respectively.

We show a Hardy-type inequality first, which is an improved version of Theorem 1.1 in [14].

**Proposition 2.1.** Let \(\Omega' \subset \Omega_B \subset \mathbb{R}^N (N \geq 3)\), be bounded or not, and \(R > r > 0\). Then, for any \(u \in H_0^1(\Omega')\), the map \(y \mapsto |u(y)|^2 \int_{S_r} \frac{d\sigma(x)}{|x - y|^r} \in L^1(\Omega')\) and

\[
\bar{m} \int_{\Omega'} |u(y)|^2 \left( \frac{1}{2\pi r} \int_{S_r} \frac{d\sigma(x)}{|x - y|^2} \right) \, dy \leq \int_{\Omega'} |\nabla u(y)|^2 \, dy,
\]

where the constant \(\bar{m} := \left( \frac{N-2}{2} \right)^2\) is optimal and not attained.

**Proof.** If \(\Omega' = \Omega_B\) then we can prove as Theorem 1.1 in [14] that,

\[
\left( \frac{N-2}{2} \right)^2 \geq \inf_{u \in H_0^1(\Omega_B(R)) \setminus \{0\}} \frac{\int_{\Omega_B(R)} |\nabla u(y)|^2 \, dy}{\int_{\Omega_B(R)} |u(y)|^2 \left( \frac{1}{2\pi r} \int_{S_r(z)} \frac{d\sigma(x)}{|x - y|^r} \right) \, dy},
\]

and the constant \(\left( \frac{N-2}{2} \right)^2\) is optimal and not attained. Then for any domain \(\Omega' \subset \Omega' \subset \mathbb{R}^N\), the results follow. \(\square\)

**Remark 2.2.** (1) If \(\Omega' = \mathbb{R}^N\), then Proposition 2.1 is reduced to Theorem 1.1 in [14].

(2) For any domain \(\Omega'\) (bounded or not) with \((z, 0) \in \Omega' \subset \mathbb{R}^2 \times \mathbb{R}^{N-2}\), if \(\{(x, 0) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : |x - z| \leq r\} \subset \Omega'\), then it is also easy to prove that

\[
\bar{m} \int_{\Omega'} |u(y)|^2 \left( \frac{1}{2\pi r} \int_{S_r(z)} \frac{d\sigma(x)}{|x - y|^2} \right) \, dy \leq \int_{\Omega'} |\nabla u(y)|^2 \, dy,
\]

where \(S_r(z) := \{(x, 0) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : |x - z| = r\}\), the constant \(\bar{m} = \left( \frac{N-2}{2} \right)^2\) is optimal and not attained.
Denote by $D^{1,2}(\mathbb{R}^N)$ the closure space of $C^\infty_0(\mathbb{R}^N)$ with respect to the norm

$$||u||_{D^{1,2}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{1/2}.$$ 

The limiting problem

$$\begin{cases}
-\Delta u - \mu \frac{u}{|x|^2} = |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N \setminus \{0\}, u \in D^{1,2}(\mathbb{R}^N),
\end{cases} \quad (2.2)$$

where $\mu < \overline{\mu}$, admits a family of solutions

$$U_\epsilon^\mu := C_\mu(N) \left( \frac{\epsilon}{\epsilon^2|x|/(\sqrt{\overline{\mu}} - \sqrt{\mu}) + |x|(\sqrt{\overline{\mu}} + \sqrt{\mu})/\sqrt{\overline{\mu}}} \right)^{\frac{N-2}{2}},$$

with $\epsilon > 0$ and $C_\mu(N) = (\frac{4N(N-\mu)}{N-2})^{\frac{N-2}{4}}$; see [19–22]. Moreover, for $0 \leq \mu < \overline{\mu}$, all solutions of (2.2) take the above form and these solutions minimize

$$S(\mu) := \min_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2}) dx}{(\int_{\mathbb{R}^N} |u|^{2^*} dx)^{2/2^*}},$$

and

$$\int_{\mathbb{R}^N} (|\nabla U_\epsilon^\mu|^2 - \mu \frac{|U_\epsilon^\mu|^2}{|x|^2}) dx = \int_{\mathbb{R}^N} |U_\epsilon^\mu|^2 dx = S(\mu)^{\frac{N}{2^*}}.$$

Note that $S(0) := S$ is the best Sobolev constant.

The following lemma is from [14].

**Lemma 2.3.** Let $N \geq 4$, $\mu < \overline{\mu}$. If $u \in D^{1,2}(\mathbb{R}^N)$ is a solution for problem (2.2), then there exist positive constants $\kappa_0(u)$ and $\kappa_\infty(u)$ such that

$$u(x) = |x|^{\frac{N-2}{2}(1-\nu_\mu)}[\kappa_0(u) + O(|x|^{\alpha})], \quad \text{as } x \to 0, \quad (2.3)$$

$$u(x) = |x|^{\frac{N-2}{2}(1+\nu_\mu)}[\kappa_\infty(u) + O(|x|^{-\alpha})], \quad \text{as } |x| \to +\infty, \quad (2.4)$$

for some $\alpha \in (0,1)$, where $\nu_\mu = (1 - \frac{4\mu}{(N-2)^2})^{1/2}$. And hence there exists a positive constant $\kappa(u)$ such that

$$\frac{1}{\kappa(u)} U_\mu^1 \leq u(x) \leq \kappa(u) U_\mu^1. \quad (2.5)$$

Denote, for any $u \in D^{1,2}(\mathbb{R}^N)$,

$$u_\epsilon(x) := \epsilon^{-\frac{N-2}{2}} u \left( \frac{x}{\epsilon} \right), \quad \text{for any } \epsilon > 0.$$ 

For any solution $u^\mu \in D^{1,2}(\mathbb{R}^N)$ for problem (2.2), denote $V(x) = \varphi(x)|u^\mu(x)|$ with $\varphi(x)$ satisfying

$$\varphi(x) \in C^\infty_0(B(0,r)), \quad 0 \leq \varphi(x) \leq 1, \varphi(x) \equiv 1 \text{ if } x \in B \left( 0, \frac{r}{2} \right), \quad |\nabla \varphi(x)| \leq C(2.6)$$

where $0 < r < 1$ small enough.

Now we give the following lemma.
Lemma 2.4. Let $N \geq 4$, $\mu < \overline{\mu}$. Then there hold

$$
\int_{B(0,r)} |V|^2 \, dx = \begin{cases} 
O(\epsilon^2) & \text{if } \mu < \overline{\mu} - 1, \\
O(\epsilon^2 \ln \epsilon) & \text{if } \mu = \overline{\mu} - 1, \\
O(\epsilon^2 \sqrt{\mu - \mu}) & \text{if } \mu > \overline{\mu} - 1,
\end{cases} 
$$

(2.7)

$$
\int_{B(0,r)} |V|^2 \, dx = \int_{\mathbb{R}^N} |\mu|^2 \, dx - O(\epsilon^2 \sqrt{\mu - \mu}), 
$$

(2.8)

$$
\int_{B(0,r)} \left( |\nabla V|^2 - \mu \frac{V^2}{|x|^2} \right) \, dx = \int_{\mathbb{R}^N} \left( |\nabla u|^2 - \mu \frac{|u|^2}{|x|^2} \right) \, dx 
+ \begin{cases} 
O(\epsilon^2 \ln \epsilon) & \text{if } \mu = \overline{\mu} - 1, \\
O(\epsilon^2 \sqrt{\mu - \mu}) & \text{if } \mu \neq \overline{\mu} - 1,
\end{cases}
$$

(2.9)

and for any $\xi \in \mathbb{R}^N \setminus \{0\}$,

$$
\int_{B(0,r)} \frac{|V(x)|^2}{|x + \xi|^2} \, dx = \begin{cases} 
\epsilon^2 \int_{\mathbb{R}^N} |\mu|^2 \, dx + o(\epsilon^2) & \text{if } \mu < \overline{\mu} - 1, \\
\kappa^2\kappa_{\infty}(\mu)^2 \epsilon^2 \ln \epsilon + O(\epsilon^2) & \text{if } \mu = \overline{\mu} - 1, \\
C(N, \mu, \epsilon^2) e^{2\sqrt{\mu - \mu}} + o(\epsilon^2 \sqrt{\mu - \mu}) & \text{if } \mu > \overline{\mu} - 1.
\end{cases}
$$

(2.10)

**Proof.** The proofs of (2.7)–(2.9) are essentially similarly to Lemmas A.1 and A.2 in [8].

By using Lemma 2.3,

$$
\int_{B(0,r)} |V|^2 \, dx \leq \epsilon^{-(N-2)} \int_{|x| < r} |\mu \left( \frac{x}{\epsilon} \right) |^2 \, dx = \epsilon^2 \int_{|y| < \frac{r}{\epsilon}} |u(y)|^2 \, dy 
$$

$$
\leq \kappa^2(\mu) \epsilon^2 \int_{|y| < \frac{r}{\epsilon}} |U^1(r)^2 \, dy 
= C^2_{\mu}(N) \kappa^2(\mu) \epsilon^2 \int_{|y| < \frac{r}{\epsilon}} \frac{1}{|y| (\sqrt{\mu - \mu} + |y| (\sqrt{\mu + \mu} - \sqrt{\mu}))} N \, dy 
$$

$$
= C^2_{\mu}(N) \kappa^2(\mu) \omega_N \epsilon \int_{0}^{\frac{r}{\epsilon}} \int_{t^{N-1}}^{t^N} (t^{\sqrt{\mu - \mu} - \sqrt{\mu}} + t^{\sqrt{\mu + \mu} - \sqrt{\mu}}) N \, dt 
$$

$$
= C^2_{\mu}(N) \kappa^2(\mu) \omega_N \epsilon \int_{0}^{\frac{r}{\epsilon}} \int_{t^{N-1}}^{t^N} \frac{1}{t^{\sqrt{\mu - \mu} - \sqrt{\mu}} + t^{\sqrt{\mu + \mu} - \sqrt{\mu}})} N \, dt 
$$

$$
\leq C^2_{\mu}(N) \kappa^2(\mu) \omega_N \epsilon \left( \int_{0}^{1} \frac{t^{N-1}}{t^{N-2}} \, dt + \int_{1}^{\frac{r}{\epsilon}} \frac{t^{N-1}}{t^{N-2} + t^{N-1} \sqrt{\mu}} \, dt \right) 
$$

$$
\leq \begin{cases} 
O(\epsilon^2) & \text{if } \mu < \overline{\mu} - 1, \\
O(\epsilon^2 \ln \epsilon) & \text{if } \mu = \overline{\mu} - 1, \\
O(\epsilon^2 \sqrt{\mu - \mu}) & \text{if } \mu > \overline{\mu} - 1,
\end{cases}
$$

(2.11)
where \( \omega_N \) is the surface measure of the unit sphere in \( \mathbb{R}^N \). So (2.7) is obtained.

Notice that

\[
\int_{|y| < \frac{r}{\varepsilon}} |V|^2 \, dx \leq \int_{|y| < \frac{r}{\varepsilon}} |u^\mu(y)|^2 \, dy = \int_{\mathbb{R}^N} |u^\mu(y)|^2 \, dy - \int_{|y| > \frac{r}{\varepsilon}} |u^\mu(y)|^2 \, dy,
\]

and

\[
\int_{|y| > \frac{r}{\varepsilon}} |u^\mu(y)|^2 \, dy \geq \frac{1}{n^2 \pi (u)} \int_{|y| > \frac{r}{\varepsilon}} |U^1_\mu(y)|^2 \, dy
\]

\[
= \frac{C^2_\mu (N)}{\kappa^2 (u)} \int_{|y| > \frac{r}{\varepsilon}} \frac{1}{(|y|/\sqrt{n}-\sqrt{n}-\mu)/\sqrt{n} + |y|/\sqrt{n}+\sqrt{n}+\mu)} \, |y|^N \, dy
\]

\[
= \frac{C^2_\mu (N)}{\kappa^2 (u)} \omega_N \int_\frac{r}{\varepsilon}^\infty \frac{t^{-N-1}}{(t/\sqrt{n}-\sqrt{n}-\mu)/\sqrt{n} + t/(\sqrt{n}+\sqrt{n}+\mu)} \, dt
\]

\[
\geq \frac{C^2_\mu (N)}{2N \kappa^2 (u)} \omega_N \int_\frac{r}{\varepsilon}^\infty t^{-1-2\sqrt{n}} \, dt
\]

\[
= O(\varepsilon^{2\sqrt{n}}). \quad (2.12)
\]

Then (2.8) follows.

Note that \( u^\mu \in D^{1,2}(\mathbb{R}^N) \) is a solution for problem (2.2), i.e.

\[-\Delta u^\mu - \mu \frac{u^\mu}{|x|^2} = |u^\mu|^2 - 2u^\mu \quad \text{in} \quad \mathbb{R}^N.\]

To prove (2.9), multiplying the above equation by \( \varphi^2(\varepsilon \cdot) u^\mu(\cdot) \) and integrating by parts, it holds

\[
\int_{|y| < \frac{r}{\varepsilon}} \varphi^2(\varepsilon y)|u^\mu(y)|^2 \, dy = \int_{|y| < \frac{r}{\varepsilon}} \left( \varphi^2(\varepsilon y) |\nabla u^\mu(y)|^2 - \mu \frac{\varphi^2(\varepsilon y)|u^\mu(y)|^2}{|y|^2} \right) \, dy
\]

\[+ \int_{|y| < \frac{r}{\varepsilon}} 2\varphi(\varepsilon y) \nabla(\varphi(\varepsilon y)) u^\mu(y) \nabla u^\mu(y) \, dy. \]

Hence

\[
\int_{B(0,r)} \left( |\nabla V|^2 - \mu \frac{V^2}{|x|^2} \right) \, dx = \int_{|y| < \frac{r}{\varepsilon}} \left( \varphi^2(\varepsilon y) |\nabla u^\mu(y)|^2 - \mu \frac{\varphi^2(\varepsilon y)|u^\mu(y)|^2}{|y|^2} \right) \, dy
\]

\[+ \int_{|y| < \frac{r}{\varepsilon}} 2\varphi(\varepsilon y) \nabla(\varphi(\varepsilon y)) u^\mu(y) \nabla u^\mu(y) \, dy
\]

\[+ \int_{|y| < \frac{r}{\varepsilon}} |\nabla(\varphi(\varepsilon y))|^2 |u^\mu(y)|^2 \, dy
\]

\[= \int_{\mathbb{R}^N} \left( |\nabla u^\mu|^2 - \mu \frac{|u^\mu|^2}{|x|^2} \right) \, dx - \int_{|y| > \frac{r}{\varepsilon}} |u^\mu(y)|^2 \, dy
\]

\[+ \int_{\frac{r}{\varepsilon} \leq |y| < \frac{r}{\varepsilon}} \varphi^2(\varepsilon y)|u^\mu(y)|^2 \, dy
\]

\[+ \int_{|y| < \frac{r}{\varepsilon}} |\nabla(\varphi(\varepsilon y))|^2 |u^\mu(y)|^2 \, dy. \]
Similarly to (2.12),
\[
\int_{|y| \geq \frac{r}{2}} |u^\mu(y)|^2 \, dy = O(\epsilon^{2 \sqrt{p - \mu}}),
\]
\[
\int_{\frac{r}{2} \leq |y| < \frac{r}{2}} \varphi^2(\epsilon y)|u^\mu(y)|^2 \, dy = O(\epsilon^{2 \sqrt{p - \mu}}).
\]
Hence (2.9) holds since
\[
\int_{|y| < \frac{r}{2}} |\nabla(\varphi(\epsilon y))|^2 |u^\mu(y)|^2 \, dy \leq C \epsilon^2 \int_{\frac{r}{2} \leq |y| < \frac{r}{2}} |u^\mu(y)|^2 \, dy
\]
\[
= \begin{cases} O(\epsilon^2 \ln \epsilon) & \text{if } \mu = \overline{p} - 1, \\ O(\epsilon^{2 \sqrt{p - \mu}}) & \text{if } \mu \neq \overline{p} - 1, \end{cases}
\]
where the last equality can be obtained similarly to (2.11).

Now we prove (2.10). As in [12],
\[
\int_{B(0,r)} \frac{|V(x)|^2}{|x + \xi|^2} \, dx = \frac{\epsilon^{-(N-2)}}{|\xi|^2} \int_{|x| < r} \left| u^\mu \left( \frac{x}{\epsilon} \right) \right|^2 \, dx + \epsilon^{-(N-2)}
\]
\[
\times \int_{|x| < r} \left( \frac{1}{|x + \xi|^2} - \frac{1}{|\xi|^2} \right) \left| u^\mu \left( \frac{x}{\epsilon} \right) \right|^2 \, dx
\]
\[
+ \epsilon^{-(N-2)} \int_{\frac{r}{2} < |x| < r} \left( \frac{\varphi^2 - 1}{|x + \xi|^2} \right) \left| u^\mu \left( \frac{x}{\epsilon} \right) \right|^2 \, dx
\]
\[:= A_1(\epsilon) + A_2(\epsilon) + A_3(\epsilon).\]

To continue we distinguish three cases: \( \mu < \overline{p} - 1, \mu = \overline{p} - 1 \) and \( \mu > \overline{p} - 1 \).

For \( \mu < \overline{p} - 1 \), by (2.5) we have that \( \int_{\mathbb{R}^N} |u^\mu|^2 \, dx < \infty \). Then as (3.14) in [12],
\[
A_1(\epsilon) = \frac{\epsilon^2}{|\xi|^2} \int_{\mathbb{R}^N} |u|^2 \, dx + o(\epsilon^2). \quad (2.13)
\]
Noticing (2.5), by using (3.15) and (3.17) in [12], we have
\[
|A_2(\epsilon)| \leq \epsilon^{-(N-2)} \kappa^2(u^\mu) \int_{|x| < r} \left( \frac{1}{|x + \xi|^2} - \frac{1}{|\xi|^2} \right) \left| U^1_{\mu} \left( \frac{x}{\epsilon} \right) \right|^2 \, dx = o(\epsilon^2), \quad (2.14)
\]
\[
|A_3(\epsilon)| \leq \epsilon^{-(N-2)} \kappa^2(u^\mu) \int_{\frac{r}{2} < |x| < r} \frac{|U^1_{\mu} \left( \frac{x}{\epsilon} \right)|^2}{|x + \xi|^2} \, dx = O(\epsilon^{2 \sqrt{p - \mu}}). \quad (2.15)
\]
for \( r > 0 \) small enough, where (2.15) holds for \( \mu < \overline{p} \). Then combining (2.13) with (2.14), (2.15), it gives
\[
\int_{B(0,r)} \frac{|V(x)|^2}{|x + \xi|^2} \, dx = \frac{\epsilon^2}{|\xi|^2} \int_{\mathbb{R}^N} |u|^2 \, dx + o(\epsilon^2) \quad \text{if } \mu < \overline{p} - 1.
\]

For \( \mu = \overline{p} - 1 \), by using (35) in [14],
\[
A_1(\epsilon) = \frac{\epsilon^2}{|\xi|^2} \int_{|y| < \frac{r}{2}} |u^\mu(y)|^2 \, dy = \frac{\epsilon^2}{|\xi|^2} (\kappa^2(u^\mu)) |\ln \epsilon| T
\]
\[ +O(1) = \kappa^2_{\infty}(u^\mu) \frac{e^2 \ln \epsilon}{|\xi|^2} + O(\epsilon^2). \]  
\hspace{2cm} (2.16)

By using (2.5), as (3.22) in [12],
\[ |A_2(\epsilon)| = O(\epsilon^2). \]  
\hspace{2cm} (2.17)

Then (2.16), (2.17) and (2.15) imply that
\[ \int_{B(0,r)} \frac{|V(x)|^2}{|x + \xi|^2} \, dx = \kappa^2_{\infty}(u^\mu) \frac{e^2 \ln \epsilon}{|\xi|^2} + O(\epsilon^2) \quad \text{if } \mu = \overline{\mu} - 1. \]

For \( \mu > \overline{\mu} - 1 \), by using (2.5),
\[
\int_{B(0,r)} \frac{|V(x)|^2}{|x + \xi|^2} \, dx \leq e^{-(N-2)} \int_{|x| < \frac{|\xi|}{2}} \frac{|u^\mu \left( \frac{x}{\epsilon} \right)|^2}{|x + \xi|^2} \, dx \\
\leq \frac{4 \kappa^2(u^\mu) e^2}{|\xi|^2} \int_{|y| < \frac{|\xi|}{2}} |U^1_\mu(y)|^2 \, dy \quad \text{(require } r < \frac{|\xi|}{2}) \\
= \frac{4C^2(N) \kappa^2(u^\mu) \omega_N e^2}{|\xi|^2} \int_0^{\frac{|\xi|}{2}} \int_0^{t^{N-1}} \frac{\mu}{(\sqrt{\mu^2 + \mu^2} + t)^{N-2}} \, dt \\
+ \int_1^{\frac{|\xi|}{2}} \int_0^{t^{N-1}} \frac{\mu}{(\sqrt{\mu^2 + \mu^2} + t)^{N-2}} \, dt \\
\leq \frac{4C^2(N) \kappa^2(u^\mu) \omega_N e^2}{|\xi|^2} \left( \int_0^{1} \frac{t^{N-1}}{t^{N-2} + t^{N-2} - 2\sqrt{\mu^2 + \mu^2}} \, dt + \int_1^{t^{N-1}} \frac{t^{N-1}}{t^{N-2} + t^{N-2} - 2\sqrt{\mu^2 + \mu^2}} \, dt \right) \\
\leq \frac{4C^2(N) \kappa^2(u^\mu) \omega_N e^2}{|\xi|^2} \epsilon^2 \sqrt{\mu^2 - \mu} + o(\epsilon^2 \sqrt{\mu^2 - \mu}) \\
= \frac{C(N, \mu, u^\mu)}{|\xi|^2} \epsilon^2 \sqrt{\mu^2 - \mu} + o(\epsilon^2 \sqrt{\mu^2 - \mu}). \]  
\hspace{2cm} (2.18)

On the other hand,
\[ \int_{B(0,r)} \frac{|V(x)|^2}{|x + \xi|^2} \, dx \geq e^{-(N-2)} \int_{|x| < \frac{|\xi|}{2}} \frac{|u^\mu \left( \frac{x}{\epsilon} \right)|^2}{|x + \xi|^2} \, dx \\
\geq \frac{4 \kappa^2(u^\mu)}{|\xi|^2} \int_{|y| < \frac{|\xi|}{2}} |U^1_\mu(y)|^2 \, dy \quad \text{(require } r < \frac{|\xi|}{2}) \\
= \frac{C(N, \mu, u^\mu)}{|\xi|^2} \epsilon^2 \sqrt{\mu^2 - \mu} + o(\epsilon^2 \sqrt{\mu^2 - \mu}). \]  
\hspace{2cm} (2.19)

Then (2.18) and (2.19) imply that
\[ \int_{B(0,r)} \frac{|V(x)|^2}{|x + \xi|^2} \, dx = \frac{C(N, \mu, u^\mu)}{|\xi|^2} \epsilon^2 \sqrt{\mu^2 - \mu} + o(\epsilon^2 \sqrt{\mu^2 - \mu}) \quad \text{if } \mu > \overline{\mu} - 1. \]
3. Existence of one positive solution for $(\mathcal{P}^\infty_{\lambda_0,K})$

In this section, we show the existence of one positive solution for the problem $(\mathcal{P}^\infty_{\lambda_0,K})$.

Let $N \geq 4$ and $K(x)$ be $\mathbb{SO}(2) \times \mathbb{SO}(N-2)$-invariant on $\Omega_B(R)$ in this section. Denote $\mathcal{D}^{1,2}_{\text{circ}}(\mathbb{R}^N) := \{u(y,z) \in \mathcal{D}^{1,2}(\mathbb{R}^N) : u(y,z) = u(|y|,|z|)\}$, where $(y,z) \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$. For $\mu_0 < \overline{\mu}$, define

$$S_{\text{circ}}(\mu_0) := \inf_{u \in \mathcal{D}^{1,2}_{\text{circ}}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx - \mu_0 \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx}{(\int_{\mathbb{R}^N} |u|^{2^*} dx)^{2/2^*}},$$

(3.1)

which is achieved in $\mathcal{D}^{1,2}_{\text{circ}}(\mathbb{R}^N)$ (see Lemma 6.1 in [14]). It is also easy to know that $S_{\text{circ}}(\mu_0)$ is independent of $\Omega_B(R) \subset \mathbb{R}^N$ in the sense that, for any $\Omega_B(R) \subset \mathbb{R}^N$,

$$S_{\text{circ}}(\mu_0) = \inf_{u \in (H^1_{\text{circ}}(\Omega_B(R)) \setminus \{0\})} \frac{\int_{\Omega_B(R)} |\nabla u|^2 dx - \mu_0 \int_{\Omega_B(R)} \frac{u^2}{|x|^2} dx}{(\int_{\Omega_B(R)} |u|^{2^*} dx)^{2/2^*}}. $$

(3.2)

An assumption on $K(x)$ is as follows.

$$(K_1) \text{ There exists } \alpha_1 > 0 \text{ such that } K(x) = K(0) + O(|x|^\alpha_1) \text{ as } x \to 0 \text{ and }$$

$$\begin{align*}
&\begin{cases}
\alpha_1 > 2 & \text{ if } \mu_0 < \overline{\mu} - 1, \\
\alpha_1 \geq 2 & \text{ if } \mu_0 = \overline{\mu} - 1, \\
\alpha_1 > 2\sqrt{\overline{\mu} - \mu_0} & \text{ if } \overline{\mu} > \mu_0 > \overline{\mu} - 1.
\end{cases}
\end{align*}$$

Set $t^+ = \max\{t,0\}$. We state the main result in this section.

**Theorem 3.1.** Let $N \geq 4$, $\mu_0^+ + \sum_{l=1}^m k\mu_l^+ < \overline{\mu}$ and $(K_1)$ hold. For given $\lambda_0 \in \mathbb{R}$, there exists $\overline{\lambda} \geq 0$, such that if

$$\sum_{l=1}^m \frac{\mu_l^+}{\overline{\lambda}_l} > \overline{\lambda},$$

then the problem $(\mathcal{P}^\infty_{\lambda_0,K})$ admits a $\mathbb{SO}(2) \times \mathbb{SO}(N-2)$-invariant positive solution.

A lemma is crucial.

**Lemma 3.2.** Let $N \geq 4$, $\mu_0^+ + \sum_{l=1}^m k\mu_l^+ < \overline{\mu}$. Assume that $\{u_n\} \subset (H^1_{\text{circ}}(\Omega_B(R))$ is a Palais–Smale $(PS \text{ in short})$ sequence at level $c$ for $J_{\text{circ}}$ restricted to $(H^1_{\text{circ}}(\Omega_B(R))$, that is

$$J_{\text{circ}}(u_n) \to c, \quad J'_{\text{circ}}(u_n) \to 0 \quad \text{in the dual space } ((H^1_{\text{circ}}(\Omega_B(R)))^*.$$

If

$$c < \frac{1}{N} \frac{S_{\text{circ}}^\infty(\mu_0)}{K(0)^{\frac{N-2}{2}}},$$

then $\{u_n\}$ has a converging subsequence in $(H^1_{\text{circ}}(\Omega_B(R))$. 

**Proof.** The proof is omitted since it is standard and similarly to Theorem 4.2 in [23].

Denote
\[
\mathcal{M}_{\text{circ}} := \left\{ u \in (H^1_0)_{\text{circ}}(\Omega_B(R)) : \int_{\Omega_B(R)} \left( |\nabla u|^2 - \mu_0 \frac{u^2}{|x|^2} \right) dx \right. \\
- \sum_{l=1}^{m} k\mu_l \int_{\Omega_B(R)} \left( \frac{1}{2\pi l} \int_{S_{r_l}} \frac{u^2(y)}{|x-y|^2} d\sigma(y) \right) dy \\
\left. = \lambda_0 \int_{\Omega_B(R)} u^2 dx + \int_{\Omega_B(R)} K(x)|u|^2^* dx \right\}
\]

Define
\[
\pi_{\text{circ}} : (H^1_0)_{\text{circ}}(\Omega_B(R)) \setminus \{0\} \to \mathcal{M}_{\text{circ}},
\]
\[
\pi_{\text{circ}}(u) = \frac{\int_{\Omega_B(R)} \left( |\nabla u|^2 - \mu_0 \frac{u^2}{|x|^2} - \lambda_0 u^2 \right) dx - \sum_{l=1}^{m} k\mu_l \int_{\Omega_B(R)} \left( \frac{1}{2\pi l} \int_{S_{r_l}} \frac{u^2(y)}{|x-y|^2} d\sigma(y) \right) dy}{\int_{\Omega_B(R)} K(x)|u|^2^* dx}
\]
for all \( u \in (H^1_0)_{\text{circ}}(\Omega_B(R)) \setminus \{0\} \). Then
\[
J_{\text{circ}}(\pi_{\text{circ}}(u)) = \frac{1}{N} \left( \int_{\Omega_B(R)} \left( |\nabla u|^2 - \mu_0 \frac{u^2}{|x|^2} - \lambda_0 u^2 \right) dx - \sum_{l=1}^{m} k\mu_l \int_{\Omega_B(R)} \left( \frac{1}{2\pi l} \int_{S_{r_l}} \frac{u^2(y)}{|x-y|^2} d\sigma(y) \right) dy \right)^{\frac{N-1}{N}},
\]
for all \( u \in (H^1_0)_{\text{circ}}(\Omega_B(R)) \setminus \{0\} \). Denote
\[
m_{\text{circ}} := \inf_{\mathcal{M}_{\text{circ}}} J_{\text{circ}}.
\]

Now we estimate \( m_{\text{circ}} \).

**Proposition 3.3.** Let \( N \geq 4, \mu_0^+ + \sum_{l=1}^{m} k\mu_l^+ < \overline{\mu} \) and \((\mathcal{K}_1)\) hold. For given \( \lambda_0 \in \mathbb{R} \), there exists \( \underline{\mathcal{L}} \geq 0 \), such that if
\[
\sum_{l=1}^{m} \frac{\mu_l}{r_l^2} > \underline{\mathcal{L}},
\]
then
\[
m_{\text{circ}} < \frac{1}{N} \frac{S_{\text{circ}}^N(\mu_0)}{K(0)^{\frac{N-2}{2}}}.
\]

**Proof.** Assume \( S_{\text{circ}}(\mu_0) \) is attained by some \( u^{\mu_0} \in D^{1,2}_{\text{circ}}(\mathbb{R}^N) \). For simplicity we assume that \( \int_{\mathbb{R}^N} |u^{\mu_0}|^2 \ dx = 1 \). Therefore the function \( v^{\mu_0} = S_{\text{circ}}(\mu_0)^{1/(2^*-2)}|u^{\mu_0}| \) is a nonnegative solution for (2.2). Take \( U_{\text{circ}}(x) = \varphi(x)|u^{\mu_0}(x)| \) with \( \varphi(x) \) satisfying (2.6). Then it follows
Combining the above two equalities and on the proof by the maximum principle.

Proof of Theorem 3.1. \(u \in M\),

\[
\int_{\Omega_B(R)} \left( \frac{1}{2\pi r_0} \int_{S_{r_0}} \frac{|U_{\text{circ}}(y)|^2}{|x - y|^2} d\sigma(x) \right) dy = \begin{cases} \\
\kappa_1 \int_{\mathbb{R}^N} |u|^2 dx + O(\varepsilon^2) & \text{if } \mu_0 < \bar{\mu} - 1, \\
\kappa_2 \frac{|\ln \varepsilon|}{r_0^2} + O(\varepsilon^2) & \text{if } \mu_0 = \bar{\mu} - 1, \\
\kappa_2 \frac{|\ln \varepsilon|}{r_0^2} + O(\varepsilon^2) & \text{if } \mu_0 > \bar{\mu} - 1.
\end{cases}
\]

On the other hand, (K1) and (2.8) imply that

\[
\int_{\Omega_B(R)} K(x)|U_{\text{circ}}|^2 dx = K(0) \int_{\Omega_B(R)} |U_{\text{circ}}|^2 dx + \int_{\Omega_B(R)} (K(x) - K(0))|U_{\text{circ}}|^2 dx = K(0) + O(\varepsilon^2) + O(\varepsilon^3).
\]

Combining the above two equalities and Lemma 2.4, for \(\mu_0 < \bar{\mu} - 1\), we have

\[
\begin{aligned}
J_{\text{circ}}(\tau_{\text{circ}}(U_{\text{circ}})) &= \frac{1}{N} \left( \int_{\Omega_B(R)} \left( \nabla U_{\text{circ}} |^2 - \mu_0 \frac{|U_{\text{circ}}|^2}{|x|^2} - \lambda_0 U_{\text{circ}}^2 \right) dx - \sum_{i=1}^m \mu_i \int_{\Omega_B(R)} \left( \frac{U_{\text{circ}}^2}{|x|^2} - 2 \int_{S_{r_0}} \frac{U_{\text{circ}}(y)}{|x - y|^2} d\sigma(x) \right) dy \right) \\
&= \frac{1}{N} \left( \int_{\Omega_B(R)} K(x)|U_{\text{circ}}|^2 dx \right) + O(\varepsilon^2) + O(\varepsilon^3).
\end{aligned}
\]

Hence there exists \(T \geq 0\), such that if

\[
\sum_{i=1}^m \frac{\mu_i}{r_i^2} > T,
\]

then

\[
J_{\text{circ}}(\tau_{\text{circ}}(U_{\text{circ}})) < \frac{1}{N} \frac{S_{\text{circ}}^N(\mu_0)}{K(0) \frac{1}{K(0)}}.
\]

Hence \(m_{\text{circ}} < \frac{1}{N} \frac{S_{\text{circ}}^N(\mu_0)}{K(0) \frac{1}{K(0)}}\).

When \(\mu_0 = \bar{\mu} - 1\) and \(\mu_0 > \bar{\mu} - 1\), the proofs are similar. \(\square\)

**Proof of Theorem 3.1.** Let \(\{u_n\} \subset (H^1_0)^{\text{circ}}(\Omega_B(R))\) be a minimizing sequence for \(J_{\text{circ}}\) on \(M_{\text{circ}}\). By the Ekeland variational principle [24], we can assume \(\{u_n\}\) is a PS sequence.

Proposition 3.3 gives \(m_{\text{circ}} < \frac{1}{N} \frac{S_{\text{circ}}^N(\mu_0)}{K(0) \frac{1}{K(0)}}\). Hence Lemma 3.2 implies that there exists \(u \in M_{\text{circ}}\), such that \(J_{\text{circ}}(u) = J_{\text{circ}}(|u|) = m_{\text{circ}}\). Therefore we end the proof by the maximum principle. \(\square\)
4. Existence of one positive solution for \((\mathcal{P}_{\lambda_0, K})\)

In this section, we show the existence of one positive solution for the problem \((\mathcal{P}_{\lambda_0, K})\).

Denote \(K_M := \max_{x \in \Omega} K(x)\). Let \(N \geq 4\) in this section and assume that \(K(x), \Omega\) are \(\mathbb{Z}_k \times \mathbb{S}_0(N-2)\)-invariant in the sequel. Denote \(\mathcal{D}^{1,2}_k(\mathbb{R}^N) := \{u(y, z) \in \mathcal{D}^{1,2}(\mathbb{R}^N) : u(e^{2 \pi \sqrt{-1} / k} y, z) = u(y, |z|)\}\), where \((y, z) \in \mathbb{R}^2 \times \mathbb{R}^{N-2}\).

As in [14], for any \(\mu < \overline{\mu}\), define

\[
S_k(\mu) := \min_{u \in \mathcal{D}^{1,2}_k(\mathbb{R}^N) \setminus \{0\}} \left( \frac{\int_{\mathbb{R}^N} |\nabla u|^2 - \mu \frac{u^2}{|z|^2} \, dx}{\left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{2/\mu}} \right). \tag{4.1}
\]

It is easy to see that \(S_k(\mu)\) is independent of the \(\mathbb{Z}_k \times \mathbb{S}_0(N-2)\)-invariant domain \(\Omega \subset \mathbb{R}^N\) in the sense that, for any \(\mathbb{Z}_k \times \mathbb{S}_0(N-2)\)-invariant domain \(\Omega\),

\[
S_k(\mu) = \inf_{u \in (H^{1,2}_0(\Omega) \setminus \{0\})} \left( \frac{\int_{\Omega} |\nabla u|^2 - \mu \frac{u^2}{|z|^2} \, dx}{\left( \int_{\Omega} |u|^2 \, dx \right)^{2/\mu}} \right). \tag{4.2}
\]

Two assumptions are needed.

\((K_2)\). There exist \(x_0 \in \Omega \setminus \{0, a_i, i = 1, \ldots, k, l = 1, \ldots, m\}\) and \(\alpha_2 > 2\) such that \(K(x) = K_M + O(|x-x_0|^{\alpha_2})\) as \(x \to x_0\).

Denote \(x_0 := (x_0^{(2)}, x_0^{(N-2)}) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} \cap \Omega\). Since \(K(x)\) is \(\mathbb{Z}_k \times \mathbb{S}_0(N-2)\)-invariant, then for any \(x_{0,j} = (e^{2\pi (j-1) \sqrt{-1} / k} x_0^{(2)}, x_0^{(N-2)}), j = 1, 2, \ldots, k\), there holds \(K(x) = K_M + O(|x-x_{0,j}|^{\alpha_2})\) as \(x \to x_{0,j}\).

\((K_3)\). There exists \(\alpha_3 > 0\) such that \(K(x) = K(a_i^L) + O(|x-a_i^L|^{\alpha_3})\) as \(x \to a_i^L, i = 1, \ldots, k\), with

\[
\begin{cases} 
\alpha_3 > 2 & \text{if } \mu_L < \overline{\mu} - 1, \\
\alpha_3 \geq 2 & \text{if } \mu_L = \overline{\mu} - 1, \\
\alpha_3 > 2\sqrt{\overline{\mu} - \mu_L} & \text{if } \mu_L > \overline{\mu} - 1,
\end{cases}
\]

where \(L(1 \leq L \leq m)\) is a positive natural number such that \(\mu_L < \overline{\mu}\) and

\[
\frac{S_{\frac{N}{2}}^N(\mu_L)}{K(a_i^L)^{\frac{N}{2}}} = \min \left\{ \frac{S_{\frac{N}{2}}^N(\mu)}{K(a_i^L)^{\frac{N}{2}}}, l = 1, \ldots, m \right\}. \tag{4.3}
\]

The main result in this section is as follows.

**Theorem 4.1.** Let \(N \geq 4, \mu_0^+ + \sum_{i=1}^m k \mu_i^+ < \overline{\mu}\).

(i). When

\[
\min \left\{ k \frac{S_{\frac{N}{2}}^N(\mu)}{K(a_i^L)^{\frac{N}{2}}}, k \frac{S_{\frac{N}{2}}^N(\mu_L)}{K(a_i^L)^{\frac{N}{2}}}, \frac{S_{\frac{N}{2}}^N(\mu_0)}{K(0)^{\frac{N}{2}}} \right\} = k \frac{S_{\frac{N}{2}}^N(\mu_0)}{K_M^{\frac{N}{2}}},
\]

assume \((K_2)\) holds, then for given \(\lambda_0 \in \mathbb{R}\), there exists \(\overline{L} \geq 0\), such that if \(\sum_{i=1}^m \sum_{j=1}^k \frac{\mu_i}{|a_i^{(L)} - x_0|^2} > \overline{L}\), the problem \((\mathcal{P}_{\lambda_0, K})\) admits a \(\mathbb{Z}_k \times \mathbb{S}_0(N-2)\)-invariant positive solution.
(ii). When
\[
\min \left\{ k \frac{S_N^M}{K_M^{\frac{N-2}{2}}} , k \frac{S_N^M (\mu_L)}{K(a_L^I)^{\frac{N-2}{2}}} , \frac{S_K^N (\mu_0)}{K(0)^{\frac{N-2}{2}}} \right\} = k \frac{S_N^M (\mu_L)}{K(a_L^I)^{\frac{N-2}{2}}},
\]
assume \((K_4)\) holds, then for given \(\lambda_0 \in \mathbb{R}\), there exists \(\bar{L} \geq 0\), such that if \(\sum_{l=1, l \neq L}^m \sum_{j=1}^k \frac{1}{|a_l^I-a_j^I|^2} + \sum_{l=1, j \neq 1}^k \frac{\mu_l}{|a_l^I-a_j^I|^2} > \bar{L}\), the problem \((\mathcal{P}_{\lambda_0,K})\) admits a \(\mathbb{Z}_k \times \mathbb{O}(N-2)\)-invariant positive solution.

(iii). For \(k\) large enough, assume \((K_1)\) holds, then for given \(\lambda_0 \in \mathbb{R}\), there exists \(\bar{L} \geq 0\), such that if \(\sum_{l=1}^m \frac{k\mu_l}{r_l^2} > \bar{L}\) with \(r_l = |a_l^I| = |a_2^I| = \cdots = |a_k^I|\), the problem \((\mathcal{P}_{\lambda_0,K})\) admits a \(\mathbb{Z}_k \times \mathbb{O}(N-2)\)-invariant positive solution.

The following lemma is important.

**Lemma 4.2.** Let \(N \geq 4, \mu_0^+ + \sum_{l=1}^M k\mu_l^+ < \bar{\mu}\). Assume that \(\{u_n\} \subset (H^1_0)^k(\Omega)\) is a PS sequence at level \(c\) for \(J_k\) restricted to \((H^1_0)^k(\Omega)\), that is
\[
J_k(u_n) \to c, \quad J_k'(u_n) \to 0 \quad \text{in the dual space } ((H^1_0)^k(\Omega))^*,
\]
If
\[
c < \bar{c}(\mu_0, \mu_L) := \frac{1}{N} \min \left\{ k \frac{S_N^M}{K_M^{\frac{N-2}{2}}} , k \frac{S_N^M (\mu_L)}{K(a_L^I)^{\frac{N-2}{2}}} , \frac{S_K^N (\mu_0)}{K(0)^{\frac{N-2}{2}}} \right\},
\]
then \(\{u_n\}\) has a converging subsequence in \((H^1_0)^k(\Omega)\).

**Proof.** We omit the proof here since it is standard and similarly to Theorem 4.1 in [23].

Denote
\[
\mathcal{N}_k(\mu_0, \mu_L, \lambda_0) := \left\{ u \in (H^1_0)^k(\Omega) : \int_{\Omega} \left( |\nabla u|^2 - \mu_0 \frac{u^2}{|x|^2} \right) dx 
- \sum_{l=1}^m \sum_{i=1}^k \mu_l \int_{\Omega} \frac{u^2}{|x-a_l^I|^2} dx \right\}.
\]
Define
\[
\pi_k : (H^1_0)^k(\Omega) \setminus \{0\} \to \mathcal{N}_k(\mu_0, \mu_L, \lambda_0),
\]
\[
\pi_k(u) = \left( \frac{\int_{\Omega} \left( |\nabla u|^2 - \mu_0 \frac{u^2}{|x|^2} - \lambda_0 u^2 \right) dx - \sum_{l=1}^m \sum_{i=1}^k \mu_l \int_{\Omega} \frac{u^2}{|x-a_l^I|^2} dx}{\int_{\Omega} K(x)|u|^2 dx} \right)^{\frac{N-2}{2}} u.
\]
Then
\[
J_k(\pi_k(u)) = \frac{1}{N} \left( \frac{\int_{\Omega} \left( |\nabla u|^2 - \mu_0 \frac{u^2}{|x|^2} - \lambda_0 u^2 \right) dx - \sum_{l=1}^m \sum_{i=1}^k \mu_l \int_{\Omega} \frac{u^2}{|x-a_l^I|^2} dx}{\left( \int_{\Omega} K(x)|u|^2 dx \right)^{\frac{N}{2}}} \right).
\]
for all $u \in (H^1_0)^k(\Omega) \setminus \{0\}$. Denote
\[
 m_k := \inf_{N_k(\mu_0, \mu_1, \lambda_0)} J_k.
\]

**Proposition 4.3.** Let $N \geq 4, \mu_0^+ + \sum_{i=1}^m k \mu_i^+ < \overline{\mu}$.

(I). If the assumptions in (i) of Theorem 4.1 hold, then for given $\lambda_0 \in \mathbb{R}$, there exists $\tilde{L} \geq 0$, such that if \( \sum_{l=1}^m \sum_{j=1}^k \frac{\mu_i}{|a_i^l - x_{0,j}|^2} > \tilde{L} \), there holds
\[
m_k < k \frac{S_N^N}{K_M^{N/2}}.
\]

(II). If the assumptions in (ii) of Theorem 4.1 hold, then for given $\lambda_0 \in \mathbb{R}$, there exists $\tilde{L} \geq 0$, such that if \( \sum_{l=1, l \neq L}^m \sum_{j=1}^k \frac{\mu_i}{|a_i^l - a_j^l|^2} + \sum_{j=1, j \neq L}^k \frac{\mu_j}{|a_i^l - a_j^l|^2} > \tilde{L} \), there holds
\[
m_k < k \frac{S_N^N(\mu_L)}{K(a_i^L)^{N/2}}.
\]

(III). If the assumptions in (iii) of Theorem 4.1 hold, then for given $\lambda_0 \in \mathbb{R}$, there exists $\tilde{L} \geq 0$, such that if \( \sum_{l=1}^m k \frac{\mu_l}{r^l} > \tilde{L} \), there holds
\[
m_k < \frac{1}{N} \frac{S_N^N(\mu_0)}{K(0)^{N/2}}.
\]

**Proof.** (I) For $x_0$ given in (K2), take $U(x) = \sum_{j=1}^k \psi(x) |U_0^\epsilon(x - x_{0,j})|$ with $\psi(x)$ satisfying
\[
0 \leq \psi \leq 1, \quad \psi = 1 \text{ if } x \in \bigcup_{j=1}^k B \left( x_{0,j}, \frac{r}{2} \right), \quad \psi = 0 \text{ if } x \notin \bigcup_{j=1}^k B(x_{0,j}, r),
\]
\[
|\nabla \psi| \leq \frac{4}{r},
\]
where $r > 0$ small enough, $x_{0,j} = (e^{2\pi j - 1/k} x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(N-2)})$, $j = 1, 2, \ldots, k$.

Then (2.10) gives
\[
\int_{\Omega} \frac{|U(x)|^2}{|x - a_i^l|^2} \, dx = \sum_{j=1}^k \frac{\epsilon^2}{|a_i^l - x_{0,j}|^2} \int_{\mathbb{R}^N} |U_0^\epsilon|^2 \, dx + o(\epsilon^2).
\]

On the other hand, the condition (K2) and (2.8) imply that
\[
\int_{\Omega} \nabla(K(x) |U|^2) \, dx = K_M k S_N^N + O(\epsilon^N) + O(\epsilon^{\alpha_2}).
\]

By summing the above two equalities and Lemma 2.4, we have
\[
J_k(\pi_k(U)) = \frac{1}{N} \left( f_{\Omega} \left( \frac{1}{|x - a_i^l|^2} - \mu_i |U|^2 \right) \, dx - \sum_{i=1}^m \sum_{j=1}^k \mu_i f_{\Omega} \left( \frac{|U|^2}{|x - a_i^l|^2} \right) \, dx \right)^N.
\]
Therefore there exists $\bar{L} \geq 0$, such that if $\sum_{l=1}^{m} \sum_{j=1}^{k} \frac{\mu_{l}}{|a_{l} - x_{0,j}|^{2}} > \bar{L}$, then

$$J_{k}(\pi_{k}(U)) < \frac{1}{N} k \frac{S^{N}}{K_{M}^{2}}.$$ 

Hence $m_{k} < \frac{1}{N} k \frac{S^{N}}{K_{M}^{2}}$.

(II) It is easy to see that $\mu_{L} > 0$ in this case since $S \leq S(\mu_{L})$ if $\mu_{L} \leq 0$.

Take $U_{k}(x) = \sum_{j=1}^{k} \varphi(x)|U_{\mu_{L}}^{e}(x - a_{j}^{L})|$ with $\varphi(x)$ satisfying

$$0 \leq \varphi \leq 1, \quad \varphi = 1 \quad \text{if } x \in \bigcup_{j=1}^{k} B\left(a_{j}^{L}, \frac{r}{2}\right), \quad \varphi = 0 \quad \text{if } x \notin \bigcup_{i=1}^{k} B(a_{j}^{L}, r),$$

$$|\nabla \varphi| \leq \frac{4}{r}$$

with $r > 0$ small enough. Then as in Lemma 6.2 in [13], by (2.10), for some positive constants $\kappa_{1}, \kappa_{2}, \kappa_{3}, \kappa_{4}$, it follows

$$\int_{\Omega} \frac{|U_{k}(x)|^{2}}{|x - a_{1}^{L}|^{2}} dx = \int_{\mathbb{R}^{N}} \frac{|U_{\mu_{L}}^{e}(x)|^{2}}{|x|^{2}} dx + O(\epsilon^{2})$$

and

$$\int_{\Omega} \frac{|U_{k}(x)|^{2}}{|x - a_{1}^{L}|^{2}} dx = \int_{\mathbb{R}^{N}} \frac{|U_{\mu_{L}}^{e}(x)|^{2}}{|x|^{2}} dx + O(\epsilon^{N-2})$$

By using Lemma 2.4, for $\mu_{L} < \bar{\mu} - 1$,

$$\int_{\Omega} \left(\nabla U_{k}^{2} - \mu_{0} \frac{U_{k}^{2}}{|x|^{2}} - \lambda_{0} U_{k}^{2}\right) dx - \sum_{l=1}^{m} \sum_{i=1}^{k} \mu_{l} \int_{\Omega} \frac{U_{k}^{2}}{|x - a_{l}^{L}|^{2}} dx$$
= kS^N_\mu L + O(\epsilon^2) \\
- \epsilon^2 k \int_{\mathbb{R}^N} |U_0|^2 dx \left( \sum_{l=1, l \neq L}^m \sum_{j=1}^k \frac{\mu_l}{|a_l^j - a_j^L|^2} + \sum_{j=1, j \neq 1}^k \frac{\mu_L}{|a_1^j - a_j^L|^2} + o(1) \right) \\
+ O(\epsilon^{N-2}) \\
= kS^N_\mu L - \epsilon^2 k \int_{\mathbb{R}^N} |U_0|^2 dx \left( \sum_{l=1, l \neq L}^m \sum_{j=1}^k \frac{\mu_l}{|a_l^j - a_j^L|^2} \\
+ \sum_{j=1, j \neq 1}^k \frac{\mu_L}{|a_1^j - a_j^L|^2} + o(1) \right) + O(\epsilon^2).

On the other hand, the condition \( K_3 \) and (2.8) imply that

\[
\int_{\Omega} K(x)|U_k|^2 dx = K(a_k^L)kS^N_\mu L + O(\epsilon^2|\mu - \mu_L|).
\]

Hence there exists \( \tilde{L} \geq 0 \), such that if \( \sum_{l=1, l \neq L}^m \sum_{j=1}^k \frac{\mu_l}{|a_l^j - a_j^L|^2} + \sum_{j=1, j \neq 1}^k \frac{\mu_L}{|a_1^j - a_j^L|^2} > \tilde{L} \), then

\[
J_k(\pi_k(U_k)) = \frac{1}{N} \left( \int_\Omega \left( \nabla U_0|^2 - \mu_0 \frac{U_0^2}{|x-a_0^L|^2} - \lambda_0 U_0^2 \right) dx - \sum_{l=1}^m \sum_{j=1}^k \mu_l \int_\Omega \frac{U_0^2}{|x-a_l^j|^2} dx \right) \frac{S^N_\mu L}{(f_\Omega K(x)|U_k|^2 dx)^{\frac{N}{N-2}}}
\]

\[
= \frac{1}{N} \left( kS^N_\mu L - \epsilon^2 k \int_{\mathbb{R}^N} |U_0|^2 dx \left( \sum_{l=1, l \neq L}^m \sum_{j=1}^k \frac{\mu_l}{|a_l^j - a_j^L|^2} + \sum_{j=1, j \neq 1}^k \frac{\mu_L}{|a_1^j - a_j^L|^2} + o(1) \right) + O(\epsilon^2) \right) \frac{S^N_\mu L}{(K(a_k^L)kS^N_\mu L + O(\epsilon^2|\mu - \mu_L|))^{\frac{N}{N-2}}}
\]

\[
< \frac{1}{N} \frac{S^N_\mu L}{K(a_k^L)^{\frac{N}{N-2}}}
\]

which gives \( m_k < \frac{1}{N} \frac{S^N_\mu L}{K(a_k^L)^{\frac{N}{N-2}}} \).

When \( \mu_L = \bar{\mu} - 1 \) and \( \mu_L > \bar{\mu} - 1 \), the proofs are similar.

(III) For \( k \) large enough, we have

\[
\min \left\{ k \frac{S^N_\mu L}{K(a_k^L)^{\frac{N}{N-2}}}, k \frac{S^N_\mu L}{K(a_k^L)^{\frac{N}{N-2}}}, \frac{S^N_{\mu_0}}{K(0)^{\frac{N}{N-2}}} \right\} = \frac{S^N_{\mu_0}}{K(0)^{\frac{N}{N-2}}}.
\]

Take \( U_{\circ} = \varphi(x)|U_{\circ}| \) with \( \varphi(x) \) satisfying (2.6) and argue as in Proposition 3.3, for \( \mu_0 < \bar{\mu} - 1 \),

\[
J_k(\pi_k(U_{\circ})) = \frac{1}{N} \left( \int_\Omega \left( \nabla U_{\circ}^2 - \mu_0 \frac{U_{\circ}^2}{|x-a_0^L|^2} - \lambda_0 U_{\circ}^2 \right) dx - \sum_{l=1}^m \sum_{j=1}^k \mu_l \int_\Omega \frac{U_{\circ}^2}{|x-a_l^j|^2} dx \right) \frac{S^N_{\mu_0}}{(f_\Omega K(x)|U_{\circ}|^2 dx)^{\frac{N}{N-2}}}
\]
On the other hand, Theorem 7.3 in [14] indicates \(\lim_{k \to +\infty} S_k(\mu_0) = S_{\text{circ}}(\mu_0)\). Then if \(k\) large enough,

\[
m_k < \frac{1}{N} \frac{S_N^N(\mu_0)}{K(0)^{\frac{N-2}{2}}}.\]

When \(\mu_0 = \bar{\mu} - 1\) and \(\mu_0 > \bar{\mu} - 1\), the proofs are similar. \(\square\)

**Proof of Theorem 4.1.** Let \(\{u_n\} \subset (H_0^1(\Omega))^k\) be a minimizing sequence for \(J_k\) on \(\mathcal{N}_k(\mu_0, \mu_1, \lambda_0)\). Then by using Proposition 4.3 and Lemma 4.2, the results can be obtained as in the proof of Theorem 3.1. \(\square\)

5. Existence of multiple positive solutions for \((\mathcal{P}_{\lambda_0, K})\)

This section is devoted to the multiplicity of positive solutions for the problem \((\mathcal{P}_{\lambda_0, K})\). The results here consist of two parts. We state them as follows respectively.

We always assume \(N \geq 5\) and \(3 \leq k < \left(\frac{K(0)}{K(\bar{\mu})}\right)^{\frac{N-2}{2}}\) in this section.

5.1. Part I

Denote \(\mathcal{C}(K) = \{b \in \Omega | K(b) = \max_{x \in \Omega} K(x)\}\). We state some assumptions first.

\((K_1)\) \(K(x) \in \mathcal{C}(\bar{\Omega}), K_M = \max_{x \in \Omega} K(x) > \max\{K(0), K(a_i^1), \ i = 1, \ldots, k, \ l = 1, \ldots, m\}\).

\((K_5)\) The set \(\mathcal{C}(K)\) is finite and \(b \in \Omega \cap \mathbb{R}^2 \times \{0\}\) for every \(b \in \mathcal{C}(K)\), say \(\mathcal{C}(K) = \{b_{i,s}, 1 \leq i \leq k, 1 \leq s \leq \frac{1}{K} \text{Card}(\mathcal{C}(K))\}\), where \(b_{i,s} = (b_{i,s}^{(2)}, 0) = (e^{2\pi(i-1)/k}b_{i,s}^{(1)}, 0) \in \mathbb{R}^2 \times \{0\}\).

\((K_6)\) There exists \(\alpha_1 > 2\) such that if \(b_{i,s} \in \mathcal{C}(K)\), then \(K(x) = K(b_{i,s}) + O(|x - b_{i,s}|^{\alpha_1})\) as \(x \to b_{i,s}\).

\((H)\) The first eigenvalue of operator \(-\Delta - \mu_0 \frac{1}{|x|^2} - \sum_{l=1}^{m} \sum_{i=1}^{k} \mu_l \frac{1}{|x-a_l|^2}\) is positive, that is, there exists \(\lambda_0' > 0\) such that

\[
\int_{\Omega} \left(\frac{\nabla u^2}{|x|^2} - \mu_0 \frac{u^2}{|x|^2}\right) \, dx - \sum_{l=1}^{m} \sum_{i=1}^{k} \mu_l \int_{\Omega} \frac{u^2}{|x-a_l|^2} \, dx \geq \lambda_0' \int_{\Omega} u^2 \, dx, \quad \forall u \in H_0^1(\Omega).
\]

Let \(L\) be the positive natural number appeared in (4.3).
Theorem 5.1. Let $N \geq 5, \mu_0^+ + \sum_{l=1}^{m} k \mu_l^+ < \bar{\mu}$. If $(\mathcal{K}_4), (\mathcal{K}_5), (\mathcal{K}_6), (\mathcal{H})$ hold and for every $1 \leq s \leq \frac{1}{k} \text{Card}(\mathcal{C}(K))$, 
\[
\sum_{l=1}^{m} \sum_{j=1}^{k} \frac{\mu_l}{|a_l^j - b_{l,j,s}|^2} > 0,
\]
then there exist $\epsilon_{\mu_0} > 0, \epsilon_{\mu_l} > 0 (l = 1, \ldots, m), \epsilon_{\lambda_0} > 0, \text{such that for all } 0 < \mu_0 < \epsilon_{\mu_0}, 0 \leq \mu_L < \epsilon_{\mu_L}, |\mu_l| < \epsilon_{\mu_l} (l = 1, \ldots, m, l \neq L), 0 \leq \lambda_0 < \epsilon_{\lambda_0}$, the problem $(\mathcal{P}_{\lambda_0,K})$ admits a positive solution which is $\mathbb{Z}_k \times \text{SO}(N-2)$-invariant.

To prove the above theorem, we follow the arguments of [18].

By using Lemma 4.2, we have immediately the following lemma.

Lemma 5.2. If $\mu_0^+ + \sum_{l=1}^{m} k \mu_l^+ < \bar{\mu}$ and $(\mathcal{K}_4)$ hold, then there exist $\epsilon_{\mu_0}^0 > 0, \epsilon_{\mu_L}^0 > 0$ such that 
\[
\frac{k}{N^2} \leq \min \{ \frac{(1 - \frac{m}{N}) N - 1}{K(a_L^i)^{\frac{N}{M}}}, \frac{(1 - \frac{m}{N}) N - 1}{K(0)^{\frac{N}{M}}} \}
\]
and 
\[
\bar{c}(\mu_0, \mu_L) = \bar{c} := \frac{1}{N} \frac{S^N}{k \frac{N}{M}}
\]
for all $0 < \mu_0 \leq \epsilon_{\mu_0}^0, 0 \leq \mu_L \leq \epsilon_{\mu_L}^0$.

Choose $r_0 > 0$ small enough such that 
\[
B(b_{i,s}, r_0) \cap B(b_{j,t}, r_0) = \emptyset \text{ for all } i \neq j \text{ or } s \neq t, 1 \leq i, j \leq k, 1 \leq s, t \leq \frac{1}{k} \text{Card}(\mathcal{C}(K)),
\]
and denote, for any $1 \leq s \leq \frac{1}{k} \text{Card}(\mathcal{C}(K))$, 
\[
T^s(u) := \frac{\int_{\Omega} \psi^s(x)|\nabla u|^2 dx}{\int_{\Omega} |\nabla u|^2 dx}, \quad \psi^s(x) = \min \{1, |x - b_{i,s}|, i = 1, \ldots, k\}.
\]
As in [18], if $u \neq 0$ and $T^s(u) \leq \delta$, then 
\[
\int_{\Omega} \frac{r_0 \int_{\Omega \setminus \bigcup_{i=1}^{k} B(b_{i,s}, r_0)} |\nabla u|^2 dx}{3} \leq \int_{\Omega} |\nabla u|^2 dx.
\]
Therefore we obtain immediately the following lemma.

Lemma 5.3. If $u \in H_0^1(\Omega)$ such that $T^s(u) \leq \delta$, then 
\[
\int_{\Omega} |\nabla u|^2 dx \geq 3 \int_{\Omega \setminus \bigcup_{i=1}^{k} B(b_{i,s}, r_0)} |\nabla u|^2 dx.
\]

Corollary 5.4. If $u \in H_0^1(\Omega), u \neq 0$ such that $T^s(u) \leq \delta, T^t(u) \leq \delta$, then $s = t$.

It is easy to prove that if $(\mathcal{H})$ holds and $0 \leq \lambda_0 < \lambda_0', \mu_0^+ + \sum_{l=1}^{m} k \mu_l^+ < \bar{\mu}$, then there exists $c > 0$ such that 
\[
||u||_{H_0^1(\Omega)} \geq c, \quad \text{for any } u \in N_k(\mu_0, \mu_l, \lambda_0).
\]
Definition 5.5. For any \(1 \leq i \leq k, 1 \leq s \leq \frac{1}{k} \text{Card}(\mathcal{C}(K))\), consider the set
\[
M^s(\mu_0, \mu_1, \lambda_0) := \{ u \in \mathcal{N}_k(\mu_0, \mu_1, \lambda_0) : T^s(u) < \delta \}
\]
and its boundary
\[
\Gamma^s(\mu_0, \mu_1, \lambda_0) := \{ u \in \mathcal{N}_k(\mu_0, \mu_1, \lambda_0) : T^s(u) = \delta \}.
\]
Define
\[
m^s := \inf \{ J_k : u \in M^s(\mu_0, \mu_1, \lambda_0) \}, \quad \eta^s := \inf \{ J_k : u \in \Gamma^s(\mu_0, \mu_1, \lambda_0) \}.
\]

Lemma 5.6. Let \(N \geq 5, \mu_0^1 + \sum_{i=1}^{m} k \mu_i^1 < \bar{\rho}\). If \((K_4), (K_5), (K_6)\) hold and
\[
\sum_{i=1}^{m} \sum_{j=1}^{k} \frac{\mu_i}{|a_j - b_{j,s}|^2} > 0, \quad 1 \leq s \leq \frac{1}{k} \text{Card}(\mathcal{C}(K)),
\]
then \(M^s(\mu_0, \mu_1, \lambda_0) \neq \emptyset\) and there exist \(\epsilon_0 > 0, \epsilon_0 > 0\) such that
\[
m^s < \bar{\epsilon} \quad \text{for all} \quad 0 < \mu_0 \leq \epsilon_0^1, 0 \leq \lambda_0 \leq \epsilon_0^0.
\]

Proof. Take \(V^s(x) = \sum_{j=1}^{k} \varphi(x)|U^s_0(x - b_{j,s})| \in (H^1_0)^k(\Omega)\) with \(\varphi(x)\), a radial function, satisfying
\[
0 \leq \varphi \leq 1, \quad \varphi = 1 \quad \text{if} \quad x \in \bigcup_{j=1}^{k} B \left( b_{j,s}, \frac{r}{2} \right), \quad \varphi = 0 \quad \text{if} \quad x \notin \bigcup_{j=1}^{k} B \left( b_{j,s}, r \right),
\]
\[
|\nabla \varphi| \leq \frac{4}{r},
\]
where \(r > 0\) small enough.

It is obvious that
\[
\pi_k(V^s) = \left( \frac{\int_{\Omega} \left( |\nabla V^s|^2 - \mu_0 |V^s|^2 - \lambda_0 |V^s|^2 \right) dx}{\int_{\Omega} K(x) |V^s|^2 dx} - \sum_{i=1}^{m} \sum_{j=1}^{k} \mu_i \int_{\Omega} \frac{|V^s|^2}{|x - a_j|^2} dx \right)^{\frac{N-2}{2}} V^s
\]
\[
:= t_s^s V^s \in \mathcal{N}_k(\mu_0, \mu_1, \lambda_0).
\]

Then
\[
T^s(\pi_k(V^s)) = \frac{\int_{\Omega} \psi^s(x)|\nabla \pi_k(V^s)|^2 dx}{\int_{\Omega} |\nabla \pi_k(V^s)|^2 dx}
\]
\[
= \sum_{j=1}^{k} \frac{1}{\int_{B(b_{j,s},r)} \psi^s(x)|\nabla \pi_k(V^s)|^2 dx}{\sum_{j=1}^{k} \int_{B(b_{j,s},r)} |\nabla \pi_k(V^s)|^2 dx}.
\]
The above estimates imply that there exists $\epsilon_0$ independent of $\mu_0, \mu_1, \lambda_0$ such that if $0 < \epsilon < \epsilon_0$, then $\pi_k(V^s_\epsilon) \in M^\ast(\mu_0, \mu_1, \lambda_0)$. By using Lemma 2.4, as in Proposition 4.3,

$$
\int_\Omega |\nabla V^s_\epsilon|^2 dx = kS^N + O(\epsilon^{N-2}),
$$

$$
\int_\Omega K(x)|V^s_\epsilon|^{2^*} dx = KMkS^N + O(\epsilon^N) + O(\epsilon^t),
$$

$$
\sum_{i=1}^m \sum_{\lambda=0}^{k-1} \mu_i \int_\Omega \frac{|V^s_\epsilon|^2}{|x-a_i|^2} dx = \lambda_0 \int_\Omega |V^s_\epsilon|^2 dx = O(\epsilon^2).
$$

By (3.3) in [25], we also know that

$$
\int_\Omega \frac{|V^s_\epsilon|^2}{|x|^2} dx \geq c\epsilon^2, \quad \text{as } \epsilon \to 0.
$$

The above estimates imply that there exists $t_1 > 0$ such that $t^2_\epsilon \geq t_1$ as $\epsilon$ small enough. Hence

$$
\max_{t \geq t_1} J_k(tV^s_\epsilon) \leq \max_{t \geq 0} \left\{ \int_\Omega \left( \frac{t^2_\epsilon}{2} |\nabla V^s_\epsilon|^2 - \frac{t^2_\epsilon}{2} K(x)|V^s_\epsilon|^{2^*} \right) dx \right. \right.
$$

$$
- \sum_{i=1}^m \sum_{\lambda=0}^{k-1} \frac{t^2_\epsilon \mu_i}{2} \int_\Omega \frac{|V^s_\epsilon|^2}{|x-a_i|^2} dx \right.
$$

$$
- \frac{t^2_\epsilon \lambda_0}{2} \int_\Omega |V^s_\epsilon|^2 dx - \frac{t^2_\epsilon \mu_0}{2} \int_\Omega \frac{|V^s_\epsilon|^2}{|x|^2} dx.
$$

Since $N \geq 5$, $\sum_{i=1}^m \sum_{\lambda=0}^{k-1} \frac{\mu_i}{|a_i-b_{i,j}|} > 0, \alpha_4 > 2$, then there exist $\epsilon^{\mu_0}_0 > 0, \epsilon^{\lambda_0}_0 > 0$ such that (5.1) holds.

Let $\mathbb{R}^N = \bigcup_{i=1}^k \mathbb{R}^N_i$ with $\mathbb{R}^N_i := \{ x = (y, z) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : y = |y|(\cos\theta, \sin\theta), (i-1)2\pi < \theta < i2\pi \}, i = 1, 2, \ldots, k$.

Lemma 5.7. Let $N \geq 5, \mu^+_0 + \sum_{i=1}^m k_i^+ \mu^+_l < \pi$. Assume that $(K_1), (K_5), (K_6)$ and $(H)$ hold, then there exist $\epsilon^{\mu_0}_0 > 0, \epsilon^{\mu_1}_l > 0, \epsilon^{\mu_0}_0 > 0$ such that for all $0 < \mu_0 < \epsilon^{\mu_0}_0, 0 \leq \mu_l < \epsilon^{\mu_1}_l, |\mu_l| < \epsilon^{\mu_1}_l (l = 1, \ldots, m, l \neq I), 0 \leq \lambda_0 < \epsilon^{\lambda_0}_0$, it holds

$$
\bar{\epsilon} < \eta^s.
$$
Proof. By contradiction we assume that there exist $\mu_0^n \to 0$, $\mu_i^n \to 0$, $\lambda_0^n \to 0$ and $u_n \in \Gamma^s(\mu_0, \mu_1, \lambda_0)$ such that

$$J_{k,n}(u_n) := \frac{1}{2} \left( \int_{\Omega} \left| \nabla u_n \right|^2 - \mu_0^n \frac{\left| u_n \right|^2}{|x|^2} \right) dx - \sum_{l=1}^{m} \sum_{i=1}^{k} \mu_i^n \frac{1}{2} \int_{\Omega} \frac{\left| u_n \right|^2}{|x-a_i^n|^2} dx - \frac{\lambda_0^n}{2} \int_{\Omega} \left| u_n \right|^2 dx - \frac{1}{2^*} \int_{\Omega} K(x) \left| u_n \right|^{2^*} dx$$

$$\to c \leq \bar{c} = \frac{1}{N} \frac{S_+^N}{K_M}.$$ 

Then it is obvious that $\{u_n\}$ is bounded. Denote $u_n^\Omega := u_n|_{\Omega \cap \Omega}$. Now let us consider $u_n^\Omega$ in $H_0^1(\mathbb{R}^N \cap \Omega)$. Up to a subsequence, there exists $l > 0$ such that

$$\lim_{n \to \infty} \int_{\Omega \cap \Omega} \left| \nabla u_n^\Omega \right|^2 dx = \lim_{n \to \infty} \int_{\Omega \cap \Omega} K(x) \left| u_n^\Omega \right|^2 dx = l.$$

As in Lemma 3.11 in [18], we deduce that $l = \frac{S_+^N}{K_M}$ and then

$$\lim_{n \to \infty} \int_{\Omega \cap \Omega} (K_M - K(x)) \left| u_n^\Omega \right|^2 dx = 0,$$

which implies a contradiction. \hfill \Box

Lemma 5.8. Let $N \geq 5, \mu_0^+, \sum_{l=1}^{m} k \mu_l^+ < \bar{\mu}$. Assume that $(K_4), (K_5), (K_6), (H)$ hold and $0 < \mu_0 < \min\{\epsilon_{\mu_0}, \epsilon_{\mu_0}^2\}$, $0 \leq \mu_L < \epsilon_{\mu_L}$, $|\mu_l| < \epsilon_{\mu_l}$ ($l = 1, \ldots, m, l \neq L$), $0 \leq \lambda_0 < \min\{\epsilon_{\lambda_0}, \epsilon_{\lambda_0}^1, \lambda_0^\mu\}$. Then for all $u \in M^s(\mu_0, \mu_1, \lambda_0)$, there exist $\rho_u > 0$ and a differential function $f : B(0, \rho_u) \subset (H_0^1)^k(\Omega) \to \mathbb{R}$ such that $f(0) = 1$ and $f(u)(u - w) \in M^s(\mu_0, \mu_1, \lambda_0), \forall w \in B(0, \rho_u)$. Moreover, for all $v \in (H_0^1)^k(\Omega)$,

$$\langle f'(0), v \rangle = -2 \int_{\Omega} \left( \nabla u \nabla v - \mu_0 \frac{u v}{|x|^2} \right) dx - \sum_{l=1}^{m} \sum_{i=1}^{k} \mu_l \int_{\Omega} \frac{u v}{|x-a_i^n|^2} dx - 2 \int_{\Omega} \mu_0 u v dx - 2^* \int_{\Omega} \mu_0^{1/2} u v dx - 2^* \int_{\Omega} K(x) |u|^2 v dx.$$ 

Proof. The proof is standard and we sketch it here. For $u \in M^s(\mu_0, \mu_1, \lambda_0)$, define a function $F : \mathbb{R} \times (H_0^1)^k(\Omega) \to \mathbb{R}$ by

$$F_u(t, w) := \langle J'_k(t(u - w)), t(u - w) \rangle = t^2 \int_{\Omega} \left( \left| \nabla (u - w) \right|^2 - \mu_0 \frac{(u - w)^2}{|x|^2} \right) dx - t^2 \sum_{l=1}^{m} \sum_{i=1}^{k} \mu_l \int_{\Omega} \frac{(u - w)^2}{|x-a_i^n|^2} dx - t^2 \lambda_0 \int_{\Omega} (u - w)^2 dx - t^2 \int_{\Omega} K(x) |u - w|^2 dx.$$ 

Then $F_u(1, 0) = \langle J'_k(u), u \rangle = 0$ and

$$\frac{d}{dt} F_u(1, 0) = 2 \int_{\Omega} \left( \nabla u^2 - \mu_0 \frac{u^2}{|x|^2} \right) dx - 2 \sum_{l=1}^{m} \sum_{i=1}^{k} \mu_l \int_{\Omega} \frac{u^2}{|x-a_i^n|^2} dx.$$
\[ -2\lambda_0 \int_{\Omega} u^2 dx - 2^* \int_{\Omega} K(x)|u|^2 dx \neq 0. \]

By using the implicit function theorem the results follow. \( \square \)

Now we prove the implicit function theorem the results follow.

**Proof of Theorem 5.1.** Let \( 0 < \mu_0 < \epsilon_{\mu_0} := \min\{\epsilon_{\mu_0}^0, \epsilon_{\mu_0}^1, \epsilon_{\mu_0}^2\} \), \( 0 \leq \mu_L < \epsilon_{\mu_L} := \min\{\epsilon_{\mu_0}^0, \epsilon_{\mu_0}^1, \epsilon_{\mu_0}^2\} \), \( |\mu_l| < \epsilon_{\mu_l} \) \( (l = 1, \ldots, m, l \neq L) \), \( 0 \leq \lambda_0 < \epsilon_{\lambda_0} := \min\{\epsilon_{\lambda_0}^0, \epsilon_{\lambda_0}^1, \epsilon_{\lambda_0}^2\} \), where \( \epsilon_{\mu_0}, \epsilon_{\mu_0}^1, \epsilon_{\mu_0}^2, \epsilon_{\mu_L}, \epsilon_{\mu_0}, \epsilon_{\lambda_0}^0, \epsilon_{\lambda_0}^1, \epsilon_{\lambda_0}^2 \) are given in Lemmas 5.2, 5.6 and 5.7.

Let \( \{u_n\} \subset (H^1_0)^k(\Omega) \) be a minimizing sequence for \( J_k \) in \( M^a(\mu_0, \mu_l, \lambda_0) \), that is, \( J_k(u_n) \rightarrow m^a \) as \( n \rightarrow \infty \). We assume \( u_n \geq 0 \) since \( J_k(u_n) = J_k(|u_n|) \). Then the Ekeland variational principle implies the existence of a subsequence of \( \{u_n\} \), denoted also by \( \{u_n\} \), such that

\[ J_k(u_n) \leq m^a + \frac{1}{n}, \quad J_k(w) \geq J_k(u_n) - \frac{1}{n}||w - u_n||, \quad \forall w \in M^a(\mu_0, \mu_l, \lambda_0). \]

Choose \( 0 < \rho < \rho_n \equiv \rho_{u_n} \) and \( f_n \equiv f_{u_n} \), where \( \rho_{u_n}, f_{u_n} \) are given by Lemma 5.8. Set \( v_\rho = pv \) with \( v \in (H^1_0)^k(\Omega) \) and \( ||v||_{H^1_0(\Omega)} = 1 \), then \( v_\rho \in B(0, \rho_n) \). By using Lemma 5.8, we get \( w_\rho = f_n(v_\rho)(u_n - v_\rho) \in M^a(\mu_0, \mu_l, \lambda_0) \). As in Theorem 3.13 in [18], it follows \( J_k'(u_n) \rightarrow 0 \) as \( n \rightarrow \infty \). Therefore \( \{u_n\} \) is a PS sequence for \( J_k \). Lemma 5.6 gives \( m^a < \tilde{c} \). Then we end the proof by Lemmas 4.2 and 5.2. \( \square \)

### 5.2. Part II

Now we consider the existence of multiple solutions by using the Lusternik–Schnirelmann category theory. The ideas are borrowed from [18,26,27].

For \( \delta > 0 \), set

\[ C_\delta(K) := \{x \in \Omega | \text{dist}(x, C(K)) \leq \delta\}. \]

We need the following.

\( (K_5) \). \( b \in \Omega \cap \mathbb{R}^2 \times \{0\} \) for every \( b \in C(K) \).

Note that if \( b = (b^{(2)}, 0) \in C(K) \), then \( b_i := (e^{2\pi(i-1)\sqrt{-1}/kb^{(2)}}, 0) \in C(K) \) for every \( 1 \leq i \leq k \).

\( (K_6) \). There exists \( \alpha_5 > 2 \) such that if \( b \in C(K) \), then \( K(x) = K(b_i) + O(|x - b_i|^{\alpha_5}) \) as \( x \rightarrow b_i \), for every \( 1 \leq i \leq k \).

\( (K_7) \). There exist \( R_0 \) and \( d_0 > 0 \) such that \( B(0, R_0) \subset \Omega \) and \( \sup_{x \in \Omega, |x| > R_0} |K(x)| \leq K_M - d_0 \).

Set

\[ \mathcal{N}_K(\mu_0, \mu_l, \lambda_0) := \{u \in \mathcal{N}_K(\mu_0, \mu_l, \lambda_0) : J_k(u) < \overline{c}\}, \]

where \( \overline{c} \) is given in Lemma 5.2.
Theorem 5.9. Let \( N \geq 5, \delta > 0, \mu_0^+ + \sum_{i=1}^m k \mu_i^+ < \overline{\nu} \). If \((\mathcal{K}_4), (\mathcal{K}_5)', (\mathcal{K}_6)', (\mathcal{K}_7), (\mathcal{H})\) hold and
\[
\sum_{l=1}^m k \sum_{i=1}^k \frac{\mu_l}{|a_l^2 - b_i|^2} > 0, \quad \text{for every } b \in \mathcal{C}(K),
\]
then there exist \( \epsilon_0' > 0, \epsilon_{\mu_l} > 0 \) \((l = 1, \ldots, m), \epsilon_{\lambda_0} > 0 \) such that for all \( 0 < \mu_0 < \epsilon_0', 0 \leq \mu_L < \epsilon_{\mu_l}', |\mu_l| < \epsilon_{\mu_l}', l = 1, \ldots, m, l \neq L \), \( 0 \leq \lambda_0 < \epsilon_{\lambda_0}' \), the problem \((\mathcal{P}_{\lambda_0,K})\) admits at least \( \text{Cat}_{\mathcal{C}_i(K)}\mathcal{C}(K) \) positive solutions which are \( \mathbb{Z}_k \times SO(N - 2) \)-invariant.

The proof of the above theorem depends on some lemmas.

Lemma 5.10. Let \( \mu_0^+ + \sum_{i=1}^m k \mu_i^+ < \overline{\nu}, 0 < \mu_0 \leq \epsilon_{\mu_0}^0, 0 \leq \mu_L \leq \epsilon_{\mu_L}^0 \) (where \( \epsilon_{\mu_0}^0, \epsilon_{\mu_L}^0 \) are constants appearing in Lemma 5.2) and \((\mathcal{K}_4)\) hold. If \( \{u_n\} \in \mathcal{N}_k(\mu_0, \mu_1, \lambda_0) \) satisfies
\[
J_k(u_n) \rightarrow c < \tilde{c}, \quad J_k|\mathcal{N}_k(\mu_0, \mu_1, \lambda_0)(u_n) \rightarrow 0,
\]
then \( \{u_n\} \) has a converging subsequence in \( (H^1_0)^k(\Omega) \).

Proof. By using Lemmas 4.2 and 5.2, the result follows as Lemma 4.1 in [18]. \( \square \)

Denote \( \Omega_i := (\mathbb{R}^N_i \cap \Omega) \setminus \partial(\mathbb{R}^N_i \cap \Omega) \).

Lemma 5.11. Let \( N \geq 5, \mu_0^+ + \sum_{i=1}^m k \mu_i^+ < \overline{\nu} \). Assume that \((\mathcal{K}_4), (\mathcal{K}_5)', (\mathcal{K}_6)'\) hold and
\[
\sum_{l=1}^m k \sum_{i=1}^k \frac{\mu_l}{|a_l^2 - b_i|^2} > 0, \quad \text{for every } b \in \mathcal{C}(K).
\]
Then there exist \( \epsilon_{\mu_0}^n > 0, \epsilon_{\lambda_0}^n > 0 \) such that if \( 0 < \mu_0 < \epsilon_{\mu_0}^n, 0 \leq \lambda_0 < \epsilon_{\lambda_0}^n \), it holds \( \mathcal{N}_k(\mu_0, \mu_1, \lambda_0) \neq \emptyset \). Moreover, for any \( \mu_0^n \rightarrow 0, \mu_1^n \rightarrow 0, \lambda_0^n \rightarrow 0 \) as \( n \rightarrow \infty \) and \( \{v_n\} \in \mathcal{N}_k(\mu_0^n, \mu_1^n, \lambda_0^n) \), there exist \( x_n^i := (x_n^{i,(2), (N-2)}, \mu_1^n) \in \Omega_i \) and \( \rho_n \in \mathbb{R}^+ \) such that \( x_n^i \rightarrow x_n^i = (x_0^{i,(2), (N-2)}, 0) \in \mathcal{C}(K), \rho_n \rightarrow 0 \) and
\[
v_n^i = \left( \frac{S}{K M} \right)^{\frac{N-2}{2}} u_r \left( \frac{-x_n^i}{\rho_n} \right) \rightarrow 0 \quad \text{in } D^{1,2}(\Omega_i), \text{ as } n \rightarrow \infty, i = 1, 2, \ldots, k,
\]
where \( u_r \) is normalized constant such that \( ||u_r||_2^* = 1 \).

Proof. Using the arguments of Lemma 5.6, it is easy to get \( \mathcal{N}_k(\mu_0, \mu_1, \lambda_0) \neq \emptyset \). Now we prove the second part.

Consider \( v_n^i \) in \( H_0^1(\mathbb{R}^N_i \cap \Omega) \) and set as in Lemma 3.11 in [18] that
\[
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N_i \cap \Omega} |\nabla v_n^i|^2 dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N_i \cap \Omega} K(x)|v_n^i|^2 dx = l.
\]
Then \( l = \frac{S^{\frac{N}{2}}}{K_M^{\frac{N-2}{2}}} \) and hence

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N \cap \Omega} (K_M - K(x))|v_n^{1}|^{2*} \, dx = 0.
\]

Set \( w_n(x) := \frac{v_n}{\|v_n\|_{L^*}} \), then \( \|w_n(x)\|_{L^*} = 1 \) and

\[
\int_{\mathbb{R}^N \cap \Omega} |\nabla w_n(x)|^{2} \, dx \to S.
\]

Therefore

\[
\int_{\Omega_1} |w_n(x)|^{2*} \, dx = 1, \quad \int_{\Omega_1} |\nabla w_n(x)|^{2} \, dx \to S.
\]

By using Corollary 4.1 in [28] and a similar proof to Lemma 4.2 in [18], there exist \( x_n^{1} \in \Omega_1 \) and \( r_n \in \mathbb{R}^+ \) such that \( x_n^{1} \to x_0 \in C_{\delta}(K) \), \( r_n \to 0 \) as \( n \to \infty \) and

\[
w_n - u_r \left( \frac{\cdot - x_n^{1}}{r_n} \right) \to 0 \quad \text{in} \quad D^{1,2}(\Omega_1), \quad \text{as} \quad n \to \infty.
\]

Hence

\[
v_n^{1} - \left( \frac{S}{K_M} \right)^{\frac{N-2}{2}} u_r \left( \frac{\cdot - x_n^{1}}{r_n} \right) \to 0 \quad \text{in} \quad D^{1,2}(\Omega_1).
\]

By recalling the symmetry of \( v_n \), we end the proof. \( \square \)

To continue, as in [18] we define

\[
\xi(x) := \begin{cases} 
x & \text{if } |x| < R_0, \\
R_0 \frac{x}{|x|} & \text{if } |x| \geq R_0.
\end{cases}
\]

For any \( 0 \neq u \in (H^1_{0}(\Omega))^k \), set

\[
\Theta(u) := \frac{\Omega \xi(x)|\nabla u|^2 \, dx}{\Omega |\nabla u|^2 \, dx}.
\]

From the proof of Lemma 5.6, it is known that for any \( 0 \neq u \in (H^1_{0}(\Omega))^k \), \( t_{\mu_0, \mu_1, \lambda_0}(u) \in \mathcal{N}_k(\mu_0, \mu_1, \lambda_0) \) with

\[
t_{\mu_0, \mu_1, \lambda_0}(u) = \left( \frac{\Omega \left( |\nabla u|^2 - \mu_0 \frac{u^2}{|x|^2} - \lambda_0 u^2 \right) \, dx - \sum_{l=1}^{k} \sum_{i=1}^{m_l} \mu_l \int_{\Omega} \frac{u^2}{|x-a_l^i|^2} \, dx}{\Omega K(x)|u|^2 \, dx} \right)^{\frac{N-2}{4}}.
\]

For \( b = (b^{(2)}, 0) \in \mathbb{R}^2 \times \{0\} \), define \( \Psi_k : \Omega \to (H^1_{0}(\Omega))^k \) as

\[
\Psi_k(b)(x) = t_{\mu_0, \mu_1, \lambda_0}(U_{\epsilon(\mu_0, \mu_1, \lambda_0)}(x))U_{\epsilon(\mu_0, \mu_1, \lambda_0)}(x) := t_{\mu_0, \mu_1, \lambda_0}U_{\epsilon(\mu_0, \mu_1, \lambda_0)}(x),
\]
Lemma 5.13. It is also obvious that all

Lemma 5.12. where \( l > 2 \) and \( \varphi(x) \), a radial function, satisfying

\[
0 \leq \varphi \leq 1, \quad \varphi = 1 \quad \text{if} \quad x \in \bigcup_{i=1}^{k} B\left(b_i, \frac{r}{2}\right), \quad \varphi = 0 \quad \text{if} \quad x \notin \bigcup_{i=1}^{k} B(b_i, r),
\]

\[
|\nabla \varphi| \leq \frac{4}{r}
\]

with \( r > 0 \) small enough, and \( \epsilon(\mu_0, \mu_1, \lambda_0) \to 0 \) as \( \mu_0 \to 0, \lambda_0 \to 0, \mu_1 \to 0 \).

The proof of the following lemma is almost the same as Lemma 5.6.

Lemma 5.12. Let \( N \geq 5, b \in C(K), \mu_0^+ + \sum_{l=1}^{m} k\mu_l^+ < \overline{\mu} \). If \((K_4), (K_5)', (K_6)' \) hold and

\[
\sum_{l=1}^{m} \sum_{i=1}^{k} |\mu_l^+ - b_i|^2 > 0, \quad \lambda_0 \geq 0,
\]

then there exist \( \tilde{c}_{\mu_0}^1, \tilde{c}_{\mu_L}^1 \) such that

\[
J_k(\Psi_k(b)) = \max_{t>0} J_k(\Psi(t \epsilon(\mu_0, \mu_1, \lambda_0))) < \tilde{c} \quad \text{for all} \quad 0 < \mu_0 \leq \tilde{c}_{\mu_0}^1, \quad 0 \leq \mu_L \leq \tilde{c}_{\mu_L}^1.
\]

By Lemma 5.12 we see that if \( 0 < \mu_0 \leq \tilde{c}_{\mu_0}^1, 0 \leq \mu_L \leq \tilde{c}_{\mu_L}^1 \), then \( \Psi_k(b) \in \mathcal{N}_k(\mu_0, \mu_1, \lambda_0) \).

It is also obvious that \( J_k(\Psi_k(b)) = \tilde{c} + o(1) \) as \( \mu_0 \to 0, \mu_1 \to 0, \lambda_0 \to 0 \) and there exist \( c_1 > 0, c_2 > 0 \) such that \( c_1 < t_{\mu_0, \mu_1, \lambda_0} U_0(\mu_0, \mu_1, \lambda_0)(x) < c_2 \) for all \( b \in C(K) \).

Lemma 5.13. Let \( N \geq 5, \mu_0^+ + \sum_{l=1}^{m} k\mu_l^+ < \overline{\mu}, b \in C(K) \) and \((K_4), (K_5)', (K_6)' \) hold. For all \( b \in C(K) \), it follows

\[
|\nabla \Psi_k(b)|^2 \to d\mu = \sum_{i=1}^{k} \frac{S^{N/2}}{S^{N/2}} \delta_{b_i}, \quad |\Psi_k(b)|^2 \to d\nu = \sum_{i=1}^{k} \frac{S^{N/2}}{S^{N/2}} \delta_{b_i},
\]

as \( \mu_0 \to 0, \mu_1 \to 0, \lambda_0 \to 0 \).

Proof. Note that \( \Psi_k(b) \) is bounded in \( (H^1_0)^k(\Omega) \) and

\[
\int_{\Omega} |\nabla \Psi_k(b)|^2 dx = \sum_{i=1}^{k} \int_{B(b_i, r)} |\nabla (t_{\mu_0, \mu_1, \lambda_0}) \varphi(x) [U_0^{(\mu_0, \mu_1, \lambda_0)}(x - b_i)]|^2 dx,
\]

\[
\int_{\Omega} K(x)|\Psi_k(b)|^2^* dx = \sum_{i=1}^{k} \int_{B(b_i, r)} K(x)|t_{\mu_0, \mu_1, \lambda_0} \varphi(x) [U_0^{(\mu_0, \mu_1, \lambda_0)}(x - b_i)]|^2^* dx.
\]

Assume \( \mu_0^0 \to 0, \mu_1^0 \to 0, \lambda_0^0 \to 0 \) as \( n \to \infty \). Up to a subsequence, we get the existence of \( l > 0 \) such that

\[
\lim_{n \to \infty} \int_{B(b_i, r)} |\nabla (t_{\mu_0^0, \mu_1^0, \lambda_0^0}) \varphi(x) [U_0^{(\mu_0^0, \mu_1^0, \lambda_0^0)}(x - b_i)]|^2 dx
\]

\[
= \lim_{n \to \infty} \int_{B(b_i, r)} K(x)|t_{\mu_0^0, \mu_1^0, \lambda_0^0} \varphi(x) [U_0^{(\mu_0^0, \mu_1^0, \lambda_0^0)}(x - b_i)]|^2^* dx
\]

\[
= l.
\]
Then \( l = \frac{-S}{K_M} \) and

\[
\lim_{n \to \infty} \int_{B(b,r)} (K_M - K(x))|t_{\mu_0}^{n,\mu_1} \lambda_n \varphi(x)|U^{(\mu_0,\mu_1,\lambda_n)}_0(x - b_1)|2^* dx = 0.
\]

Set \( w_1(x) := \frac{t_{\mu_0}^{n,\mu_1} \lambda_n \varphi(x)U^{(\mu_0,\mu_1,\lambda_n)}_0(x - b_1)}{\|t_{\mu_0}^{n,\mu_1} \lambda_n \varphi(x)U^{(\mu_0,\mu_1,\lambda_n)}_0(x - b_1)\|_2}, \) then \( \|w_1(x)\|_2 = 1 \) and

\[
\int_{B(b,r)} |\nabla w_1(x)|^2 dx \to S.
\]

Applying the arguments of Theorem 3.13 in [18], it holds

\[
|\nabla w_1|^2 \to d\bar{u}_1 = S\bar{b}_1, \quad |w_1|^2 \to d\tilde{v}_1 = \tilde{b}_1,
\]

which ends the proof. \( \square \)

**Lemma 5.14.** Let \( N \geq 5, \mu_0^+ + \sum_{i=1}^m k \mu_i^+ < \bar{\mu} \) and \((K_4), (K_5)', (K_6)', (K_7)\) hold. For \( \mu_0 \to 0, \mu_1 \to 0, \lambda_0 \to 0, \)

1. \( \Theta(\Psi_k(b)) = b + o(1), \) uniformly for \( b \in B(0, R_0) \cap C(K); \)
2. sup\{dist\( (\Theta(u), C(K)) : u \in \overline{H}_k(\mu_0, \mu_1, \lambda_0) \} \to 0.

**Proof.** (1) Let \( b \in B(0, R_0) \cap C(K). \) By Lemma 5.13, we have

\[
\Theta(\Psi_k(b)) = \frac{\int_{\Omega} \xi(x)|\nabla \Psi_k(b)|^2 dx}{\int_{\Omega} |\nabla \Psi_k(b)|^2 dx} = \frac{\int_{\Omega} \xi(x) d\bar{\mu}}{\int_{\Omega} d\bar{\mu}} + o(1) = b + o(1),
\]

as \( \mu_0 \to 0, \mu_1 \to 0, \lambda_0 \to 0. \)

(2) Take \( \mu_0^n \to 0, \mu_1^n \to 0, \lambda_1^n \to 0 \) as \( n \to \infty \) and \( \{v_n\} \in \overline{H}_k(\mu_0^n, \mu_1^n, \lambda_0^n) \). By Lemma 5.11, there exist \( x_0^i = (x_0^{(2)}_n, x_0^{(N-2)}_n) \in \Omega_1 \) and \( r_1 \to 0, r_2 \to 0 \) such that \( x^n_i \to x^i_0 \)

\[
v_i^i - \left( \frac{S}{K_M} \right)^{N^2} \mu_0^{(i)} \left( \frac{x^n_i}{r_n} \right) \to 0 \quad \text{in} \quad D^{1,2}(\Omega_1), \quad \text{as} \quad n \to \infty, \quad i = 1, 2, \ldots, k.
\]

Since \( \Theta(u) \) is continuous, then

\[
\Theta(v_n) = \frac{\int_{\Omega} \xi(x)|\nabla v_n|^2 dx}{\int_{\Omega} |\nabla v_n|^2 dx} = \frac{\int_{\Omega} \xi(x) \left| \nabla u_r \left( \frac{x - x^n_i}{r_n} \right) \right|^2 dx}{\int_{\Omega} \left| \nabla u_r \left( \frac{x - x^n_i}{r_n} \right) \right|^2 dx} + o(1) = \xi(x^i_0) + o(1).
\]

Noticing that \( x^n_0 \in B(0, R_0) \) leads to \( \xi(x^n_0) = x^i_0 \), the result desired is true. \( \square \)

**Proof of Theorem 5.9.** We follow the arguments of Theorem 4.5 in [18] and Theorem A in [27].

Given \( \delta > 0, \) by using Lemmas 5.12 and 5.14, there exist \( \epsilon_{\mu_0}'' > 0, \epsilon_{\mu_1}'' > 0 \) \((l = 1, \ldots, m), \epsilon_{\lambda_0}'' > 0 \) such that for all \( 0 < \mu_0 < \epsilon_{\mu_0}'' \), \( 0 \leq \mu_L < \epsilon_{\mu_0}'' \), \( |\mu_1| < \epsilon_{\mu_1}'' \), \( l = 1, \ldots, m, l \neq i \),
\( \lambda_0 \leq \lambda_0 < \epsilon_0' \)

\( L \), \( 0 \leq \lambda_0 < \epsilon_0' \), we have \( \Psi_k(b) \in \mathcal{N}_k(\mu_0, \mu_1, \lambda_0) \) for any \( b \in C(K) \) and

\[
|\Theta(\Psi_k(b)) - b| < \delta, \quad \forall \ b \in B(0, R_0) \cap C(K),
\]

and\[
\Theta(u) \in C_\delta(K), \quad \forall \ u \in \mathcal{N}_k(\mu_0, \mu_1, \lambda_0).
\]

Define \( H(t,x) := x + t(\Theta(\Psi_k(b)) - b) \) with \( (t,x) \in [0,1] \times C(K) \). Then \( H \) is continuous, \( H([0,1] \times C(K)) \subset C_\delta(K) \) and \( \Theta \circ H \) is homotopic to the inclusion \( C(K) \hookrightarrow C_\delta(K) \).

By Lemma 5.10, it is enough to prove \( \text{Cat}(\mathcal{N}_k(\mu_0, \mu_1, \lambda_0)) \geq \text{Cat}_{C_\delta(K)}C(K) \), which can be obtained as Theorem 4.5 in [18], and therefore the problem \( (\mathcal{P}_{\lambda_0,K}) \) admits at least \( \text{Cat}_{C_\delta(K)}C(K) \) solutions which are \( \mathbb{Z}_k \times \mathbb{S}(N-2) \)-invariant.

It is also easy to show that any of these solutions has a fixed sign, as Theorem 4.5 in [18] and Theorem 1.1 in [29]. \( \Box \)

**Remark 5.15.** (1) The problem considered here involves symmetry and multiple solutions are obtained, which are different from [12,13,15], where the symmetry is not been considered and existence, but not multiplicity, of solutions is obtained. In [14], Felli and Terracini considered (1.1) with symmetric multi-polar potentials in \( \mathbb{R}^N \) when \( K(x) \equiv 1 \) and proved the existence of positive solutions, while the domain we considered here is bounded and \( K(x) \) is positive bounded.

(2) In [30], Felli and Schneider considered the problem related to the Caffarelli–Kohn–Nirenberg type inequality and showed the symmetry-breaking phenomenon in the inequality, namely the existence of non-radial minimizers, which motivates us to study the symmetry-breaking phenomenon of problem (1.1) in the near future. In [31], Badiale and Rolando also proved the existence of positive radial solutions for the semilinear elliptic problem with singular potentials, where both sub-critical and super-critical nonlinearities, rather than critical nonlinearities, are considered.

(3) It can be seen that when \( \lambda_0 \neq 0 \) and the domain \( \Omega \) is bounded, there exist nontrivial solutions for the problem \( (\mathcal{P}_{\lambda_0,K}) \). However, if we consider the problem in \( \mathbb{R}^N \), then \( \lambda_0 \neq 0 \) may lead to the nonexistence of nontrivial solutions. Please see Appendix for a nonexistence result.

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**Appendix**

Consider the quasilinear elliptic problem

\[
\begin{align*}
-\Delta_p u - \mu \frac{|u|^{p-2} u}{|x|^p} &= \lambda |u|^{q-2} u + K(x)|u|^{p^* - 2} u \quad \text{in} \mathbb{R}^N, \\
\end{align*}
\]

(A.1)
where \(-\Delta_p u := -\text{div}(|\nabla u|^{p-2} \nabla u), 1 < p < N, \lambda \in \mathbb{R}, \mu < (\frac{N-p}{p})^p, p^* := \frac{Np}{N-p}\) is the critical Sobolev exponent, \(1 < q < p^*, K(x) \in C^1(\mathbb{R}^N)\) satisfying \(|K|_{\infty} < \infty\).

When \(p = 2, K(x) \equiv 1\), the nonexistence of nontrivial solutions for problem (A.1) can be seen in [32]. Here we state the nonexistence result for problem (A.1) motivated by [11].

**Theorem A.** Assume one of the following three cases holds:

1. \(\lambda > 0\), \((x, \nabla K) \leq 0\), for a.e. \(x \in \mathbb{R}^N\);
2. \(\lambda < 0\), \((x, \nabla K) \geq 0\), for a.e. \(x \in \mathbb{R}^N\);
3. \(\lambda = 0\), \((x, \nabla K) > 0\), for a.e. \(x \in \mathbb{R}^N\),
   or \(\lambda = 0\), \((x, \nabla K) < 0\), for a.e. \(x \in \mathbb{R}^N\).

Then if \(u \in W^{1,p}(\mathbb{R}^N)\) is a weak solution for problem (A.1), there holds \(u \equiv 0\).

**Proof.** Denote \(f(x, u) = \mu \frac{|u|^{p-2} u}{|x|^p} + \lambda |u|^{q-2} u + K(x)|u|^{p^* - 2} u\). Then problem (A.1) can be rewritten as

\[
\begin{align*}
-\Delta_p u &= f(x, u) \quad \text{in } \mathbb{R}^N, \\
u &= W^{1,p}(\mathbb{R}^N).
\end{align*}
\] (A.2)

It is easy to see that, for any \(\omega \in \mathbb{R}^N \setminus \{0\}\), there exists \(C(\omega) > 0\) such that \(|f(x, u)| \leq C(\omega)(1 + |u|^{p-1})\), \(\forall x \in \omega, u \in \mathbb{R}\). Then as Claim 5.5 in [11], \(u \in C^1(\mathbb{R}^N \setminus \{0\}) \cap W^{2,1}_{\text{loc}}(\mathbb{R}^N \setminus \{0\})\) following from Lemmas 2.1 and 2.2 in [33], Corollary 1.1 in [34] and Theorem 1, Proposition 1 in [35].

Now we prove \(u \in L^q(\mathbb{R}^N)\). Take a cut-off function (see Claim 5.3 in [11]) \(h \in C^\infty(\mathbb{R})\) satisfying \(h(t) \equiv 1\) for \(|t| \leq 1\), \(h(t) \equiv 0\) for \(|t| \geq 2\), \(0 \leq h \leq 1\). Given \(\epsilon > 0\) small enough, define \(\eta_\epsilon\) as:

\(\eta_\epsilon(x) = h(|x|/\epsilon)\) if \(|x| \leq 3\epsilon, \eta_\epsilon(x) = h(1/\epsilon|x|)\) if \(|x| \geq 1/\epsilon\), and \(\eta_\epsilon(x) \equiv 1\) elsewhere. Then it is obvious that \(\eta_\epsilon \in C^\infty(\mathbb{R}^N \setminus \{0\})\). Since \(u\) is a weak solution for problem (A.1),

\[
\int_{\mathbb{R}^N} \left( |\nabla u|^{p-2} (\nabla u \cdot \nabla (\eta_\epsilon u)) - \mu |\eta_\epsilon|^p + \lambda \eta_\epsilon |u|^p \right) dx - \int_{\mathbb{R}^N} K(x) \eta_\epsilon |u|^{p^*} dx = 0.
\]

Then by the Hardy inequality, the Sobolev inequality and the Hölder inequality,

\[
|\lambda| \int_{\mathbb{R}^N} \eta_\epsilon |u|^p dx \leq \int_{\mathbb{R}^N} K(x) \eta_\epsilon |u|^{p^*} dx
\]

\[
+ \int_{\mathbb{R}^N} |\nabla u|^{p-2} \left( |\nabla u| - \mu |\eta_\epsilon|^p \right) dx
\]

\[
\leq C \int_{\mathbb{R}^N} |\nabla u|^p + \int_{\mathbb{R}^N} |\nabla u|^{p-1} |\nabla \eta_\epsilon| |u| dx
\]

\[
\leq C,
\]

where the constant \(C > 0\) is independent of \(\epsilon\). Letting \(\epsilon \to 0\), we have \(u \in L^q(\mathbb{R}^N)\).

Therefore, by Claim 5.3 in [11],

\[
\lambda N \left( \frac{1}{p} - \frac{1}{q} \right) \int_{\mathbb{R}^N} |u|^p dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^{p^*} \langle x, \nabla K \rangle dx = 0,
\]

which implies \(u \equiv 0\). \(\square\)

When \(K(x) \equiv 1, \lambda \neq 0\), Theorem A extends the result in [32] to the \(p\)-Laplace case.
References


