FACTORIZATION AND THE PARTON MODEL IN QCD

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We argue that mass-singularities of inclusive cross sections in QCD factor to all orders in perturbation theory as required for a parton model interpretation.

If QCD is the field theory underlying the strong interactions, why does the parton model work? [1] Why is it that the mass singularity logarithms associated with collinear light-quark and gluon emission do not completely invalidate perturbation theory? The answer must be that the distribution and decay functions of the parton model are "renormalized" quantities which already include the large logarithms of naive perturbative calculations. This explanation requires that the semi-inclusive parton cross sections computed in perturbation theory must factor (in the sense of the convolution integrals which define the parton model), so that the large logs can be absorbed. [2]. In inclusive lepton–hadron scattering the required factorization does take place to all orders in perturbation theory (as shown by the operator product expansion).

In this note, we summarize an argument to be presented in more detail in a forthcoming paper [3] that the required factorization actually takes place to all orders in perturbation theory for any semi-inclusive process that admits a parton model interpretation. We will first define the factorization we want in a theory with massless quarks. To develop an all-orders proof of the factorization property in QCD, we specialize to axial gauge. The key feature of axial gauge is that no large logs from collinear emissions arise from interferance between diagrams in which unobserved partons are emitted from different legs ‡1. Technically speaking, we argue that the contribution to the (off-shell) cross section which is two-particle irreducible in the channel of the leg with momentum p^μ is finite as $p^2 \to 0$ ‡2. From thereon

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‡1 A special axial gauge, light-cone gauge, has been used to study lowest-order and/or leading log calculations. See ref. [4]. Similar Coulomb gauge calculations have been performed in ref. [5].
‡2 The importance of the finiteness of the 2PI part for an all-orders proof of factorization was demonstrated by Mueller, ref. [7], for a special case.

the proof hinges on the kinematics of ladder diagrams. We close by giving a practical recipe for parton model calculations in QCD.

We work in the limit of massless quarks and continue the semi-inclusive cross sections off-mass-shell in each observed momentum. Factorization can then be studied one leg at a time [2].

Explicitly, suppose $A_j(p)$ is a semi-inclusive differential cross section involving an incoming j-type parton (j runs over quarks, antiquarks and gluons) with momentum p. $A_j(p)$ is continued off-shell in p^2 and any other observed momenta. In general, $A_j(p)$ may depend on other observed parton momenta and on momenta carried by weak or electromagnetic currents, but for the moment, we will keep all other momenta fixed. We will argue that to all orders in perturbation theory $A_j(p)$ factors as follows:

$$A_j(p) = \sum_k \int_0^1 \frac{d\beta}{\beta} \Gamma_{jk}(\beta, p^2)\, \widetilde{A}_k(\beta p) + O(p^2) , \qquad (1)$$

where all ln p^2 singularities are contained in the factor Γ which can be absorbed into the distribution function. $\widetilde{A}_k(\beta p)$ is evaluated at $p^2 = 0$.

For an observed outgoing leg, the momentum is scaled differently but the result is similar. If $B_j(p')$ is an off-shell semi-inclusive cross section for an outgoing j-type parton with momentum p', it satisfies

$$B_j(p') = \sum_k \int_0^1 \frac{d\beta}{\beta} \Gamma'_{jk}(\beta, p'^2)\, \widetilde{B}_k(p'/\beta) + O(p'^2) , \quad (2)$$

where Γ' contains all ln p'^2 singularities and can be associated with the decay function, and $\widetilde{B}$ is evaluated at $p'^2 = 0$.

The off-mass-shell semi-inclusive cross section has the structure of a forward absorptive amplitude [6] with one zero momentum channel for each observed momentum. For convenience, we will choose all incoming (outgoing) observed parton momenta to be spacelike (timelike). This eliminates some obvious infrared problems and simplifies the kinematics. We define a contribution to the cross section to be two-particle irreducible (2PI) in the p^μ channel if there is no nontrivial way of separating the external legs carrying the momentum p from the other external legs by cutting only two lines.

For us, the significance of the 2PI contribution is that it is free of ln p^2 singularities (in axial gauge). The singularities as $p^2 \to 0$ in individual diagrams are of two types, the collinear or mass-singularity logs which are our primary concern and the soft or infrared logs associated with emission of soft partons which are not necessarily collinear. Neither type of singularity is present in the sum of all 2PI diagrams.

The absence of infrared logs is familiar [8] and independent of gauge. It arises from a cancellation between corresponding real and virtual diagrams.

The absence of mass-singularity logs is peculiar to physical gauges [4,5] (see also refs. [9,10]) like axial gauge. The key feature of axial gauge is that one can define effective quark–quark–gluon and three-gluon vertices which vanish as the emission becomes collinear. Power counting shows that the 2PI contribution is free of mass-singularities. The details will be given in a forthcoming paper [3].

Once the 2PI amplitudes have been shown to be free of mass singularities, the argument that factorization takes place involves straightforward kinematics. We will go through it for an incoming leg. We ignore spin and suppress a summation over parton types. This simplifies the notation without leaving out anything essential.

The off-shell absorptive amplitude $A(p)$ satisfies the following integral equation:

$$A(p) = Z(p)I(p) + Z(p)\int d^4k\, K(p, k) Z^{-1}(k) A(k) . \qquad (3)$$

In eq. (3), $I(p)$ is the contribution which is 2PI in the p^μ channel. It depends on all the other observed momenta (as does $A(p)$) but since these are held fixed throughout this analysis we ignore them. $K(p, k)$ is the sum of all contributions to scattering of a parton with momentum p into a parton with momentum k plus anything which is 2PI in the p^μ channel. $K(p, k)$ is defined to include the dressed propagators on the k legs. $Z(p)$ is the sum of self-energy corrections on the external legs. These are important for keeping track of gauge invariance. The second term in eq. (3) is clearly not 2PI because the legs with momenta k^μ can be cut.

Our plan is as follows: we will express the integrated momentum k in terms of variables which are kinematically contrained to run over fixed finite ranges. Then the solution to the integral equation for $A(p)$ can be written down in matrix form:

$$A = Z(1-K)^{-1} I\,. \tag{4}$$

The factorization property that we want is obtained by matrix manipulation.

Consider the integrated momentum k^μ. Write k^μ in terms of a fixed vector m^μ ($m^2 = 0$, $mp > 0$), the initial momentum p^μ and a transverse momentum κ^μ ($m\kappa = p\kappa = 0$, $\boldsymbol{\kappa}^2 = -\kappa^2 \geq 0$) as follows:

$$k^\mu = -\alpha m^\mu + \beta p^\mu + \sqrt{-k^2}\,\kappa^\mu\,, \tag{5a}$$

where

$$\alpha = [-k^2(1-\kappa^2) + \beta^2 p^2]/2\beta m p\,. \tag{5b}$$

We will take $-k^2$, $\boldsymbol{\kappa}$ and β as our independent variables. The kinematics requires

$$1 \geq \beta, \quad 0 \leq -k^2, \quad 0 \leq \ {}^2 \leq 1-\beta\,. \tag{6}$$

The upper limit on $-k^2$ is set by the kinematics (built into the support of K and I) and is some function of the large invariants. We will choose an M^2 which is larger than, but of the order of, this upper limit and take that as the fixed end point of our $-k^2$ integration. This M^2 is the arbitrary scale which must be introduced to factor the $\ln p^2$ singularities. The lower limit of the β integration is also set by the kinematics. In fact, the analysis below is valid only if β is kinematically bounded away from zero, so we will assume $0 < \beta \leqslant 1$.

Next, we use the finiteness of the 2PI cross sections as follows: Define [7]

$$\widetilde{I}(k) \equiv I(k)|_{k^2=p^2=0} = \widetilde{I}(\beta p)|_{p^2=0}\,. \tag{7}$$

and

$$\widetilde{K}(k,k') \equiv K(k,k')|_{k^2=p^2=0} = \widetilde{K}(\beta p, k')|_{p^2=0}\,. \tag{8}$$

In both cases, the second equality follows from eqs. (5), (6). The requirement that β is bounded away from zero is important here. Because of the pole at $\beta = 0$ in eq. (5b), the $k^2 \to 0$ limit is not uniform as $\beta \to 0$. Note also that $\widetilde{K}$ makes sense bacause $K(k,k')$, like $I(k)$, is 2PI in the k^μ channel. However, $\widetilde{K}(k,k')$ is singular as $-k'^2$ goes to zero.

Finally define

$$\Delta I \equiv I - \widetilde{I}\,, \qquad \Delta K \equiv K - \widetilde{K}\,, \tag{9}$$

and we can rewrite eq. (3) as follows:

$$A = Z(1-K)^{-1}\widetilde{A} + Z(1-\Delta K)^{-1}\Delta I\,, \tag{10}$$

where

$$\widetilde{A} = \widetilde{I} + \widetilde{K}(1-\Delta K)^{-1}\Delta I\,. \tag{11}$$

Here $\widetilde{K}$ only appears operating on a ΔI or ΔK which goes to zero as $\widetilde{K}$ blows up, so eq. (11) is well defined. Furthermore, eq. (10) is exactly the factorization we want. The second term on the right hand side vanishes as $p^2 \to 0$ by the definition of ΔI and ΔK. In the first term on the right hand side, $\widetilde{A}$ is a function of the integrated momentum k, but because of eqs. (7) and (8) it only depends on the rescaled leg momentum βp^μ and on the fixed momenta. So we can do the $-k^2$ and $\boldsymbol{\kappa}$ integrations. Define

$$\Gamma(\beta, p^2) = \int \mathrm{d}(-k^2)\, \mathrm{d}^2\kappa\, \tfrac{1}{2}(-k^2) Z(1-K)^{-1}\,. \tag{12}$$

Then eq. (10) has the form of eq. (1), in which the factorization is by a one dimenisonal convolution integral.

The important point is that the finiteness of the 2PI part has allowed us to factor the $\ln p^2$ singularities in a way which is obviously independent of the process. The factor Γ in eqs. (1) and (12) depends only on the external ladders, not on the 2PI core which defines the process. In particular, inclusive electroproduction can be analyzed in this way. Comparing eq. (1) with the result of the usual operator product expansion analysis, we see that if M^2 is taken to be the renormalization point, Γ can be taken to be the standard exponential integral of the anomalous dimension matrix, while $\widetilde{A}$ is related to the coefficient functions. Γ is folded with a "bare" parton distribution function to get the "renormalized" distribution functions which are actually measured in lepton–hadron scattering. The above analysis ensures that the connection between bare and renormalized distribution functions is the same in all parton processes.

The analysis of decay functions is analogous. The kinematics is used as in eqs. (5)–(8) to reduce a four-momentum integral to a one-dimensional convolution integral. Again, the key is finiteness of the 2PI part.

Although axial gauge is very helpful in organizing an all-orders proof of factorization, it is not at all necessary for practical calculations. Knowing that factorization takes place, one can calculate the cross section in any convenient gauge and identify $\widetilde{A}$ by comparing the final result with eqs. (1) and (2) [2].

We can use our analysis to justify and improve the parton model for any process for which the β's of the incoming parton momenta are bounded away from zero. This constraint is satisfied for lepton–hadron scattering, inclusive or semi-inclusive, for hadron–hadron scattering into high-mass μ-pairs, and for hadron–hadron scattering into large-transverse-momentum final states or final states containing heavy quark bound states. But it is not satisfied, for example, for the hadron–hadron total cross section which gets an important contribution from arbitrarily soft partons.

If the β constraint is satisfied, we take the convolution integral which connects the parton distribution or decay functions with the parton cross sections to have the same form as the factorization in eqs. (1), (2). Thus for an incoming hadron with momentum p^μ, we need a distribution integral

$$\sum_j \int \frac{d\beta}{\beta} F_j(\beta, M^2) \ , \tag{13}$$

multiplying a cross section which depends on the rescaled momentum βp^μ. Similarly for an outgoing hadron with momentum p'^μ, we need a decay integral

$$\sum_{j'} \int \frac{d\beta'}{\beta'} D_{j'}(\beta', M^2) \ ,$$

multiplying a cross section depending on p'/β'. All variables which are independent of these rescalings are held fixed. For example, in inclusive electroproduction, which depends on $\nu = pq/m$ and $Q^2 = -q^2$, ν is rescaled while Q^2 is held fixed. Thus, the improved parton model description of electroproduction is simple at fixed Q^2 but not at fixed ν. We can choose $M^2 = Q^2$ and eliminate all large logs.

In purely hadronic processes there are no dimensional invariants which are independent of rescaling. But there is some dimensional quantity μ which is introduced to bound the β's of the initial momenta away from zero and we take M^2 of the order of μ^2. For example, in large p_T scattering, μ is the minimum transverse momentum. In heavy hadron production, μ is the heavy quark mass [3].

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