# THE EXTERNAL FIELD PROBLEM IN THE LARGE $N$ LIMIT OF QCD

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Several problems in lattice gauge theories such as mean field theory or the few plaquette problem lead to the evaluation of the properties of one link in an external matrix source. This problem is solved here in the large $N$ limit. There are two phases characterized by a single parameter, the average value of the inverse of the modulus of the eigenvalues of the external source. The third derivative of the free energy is discontinuous at the transition point.

*1. Introduction.* A standard formulation of mean field theory for magnetic systems approximates the neighboring spins of a given lattice site by an external field proportional to the magnetization in that field. The basic assumption of this approach is that the fluctuations of any spin $S_i$ about its mean value $S_i$ are small and that one can therefore neglect all the terms in the hamiltonian beyond the ones linear in $S_i - \langle S_i \rangle$. One is then led to consider the elementary problem of independent spins in an external self-consistent field.

The equivalent approach for lattice gauge theories leads one to consider independent link variables in an external source created by the links contained in the adjacent plaquettes [1]. However the independent link problem is not elementary. One must compute partition functions of the form

$$Z = \int \mathrm{d}U \exp N \,\mathrm{tr}(UA^\dagger + AU^\dagger) ,$$

here $U$ is, for example, an $N \times N$ special unitary matrix, and $A$ an arbitrary $N \times N$ matrix source.

In this note we derive the exact form of $Z$ in the large $N$ limit. The form of the solution depends on the magnitude of the matrix $A$. The answer for the weak coupling (large $A$) regime has recently been derived by Brower and Nauenberg [2]. We have derived the result for both the weak and the strong coupling regimes. The parameter which characterizes these regimes is

$$s = \frac{1}{N}\sum_{a=1}^{N} \lambda_a^{-1/2} = \frac{1}{N}\mathrm{Tr}(AA^\dagger)^{-1/2} ,$$

where the $\lambda_a$ are the eigenvalues of $AA^\dagger$. The weak coupling regime corresponds to $s < 2$ (the matrix $A$ is proportional to the inverse of the coupling constant squared), and the strong coupling regime to $s > 2$. At $s = 2$ there is a "phase" transition at which we switch from the weak to strong coupling solutions. The third derivative of the free energy, $-\ln Z$, is discontinuous at $s = 2$. The one plaquette model of Gross and Witten [3] is a special example in which $A = \beta \mathbf{1}$, $s = 1/\beta$, and the transition occurs at $\beta \equiv 1/(g^2 N) = 1/2$.

The results are derived by using the Schwinger–Dyson equations of motion (in ref. [2] a weaker version of these equations was derived). In the large $N$ limit we obtain a coupled system of first order, nonlinear partial differential equations. These equations may be reduced to a nonlinear Riemann–Hilbert problem which, remarkably enough, can be solved exactly.

The result of this calculation may be used to study the problem of several coupled plaquettes or to explore the self-consistent mean field approach to QCD.

*2. An elementary example.* The external field problem for magnetic systems which is analogous to large $N$ SU($N$) lattice gauge theory is an $N$ component com-

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plex classical spin $\boldsymbol{u}$ ($\boldsymbol{u}\cdot\boldsymbol{u}^* = 1$). Here the partition function for an individual spin in an external field, $\boldsymbol{a}$, is given by

$$Z(\boldsymbol{a}\cdot\boldsymbol{a}^*) = K\int \mathrm{d}^N u\, \mathrm{d}^N u^*\, \delta(\boldsymbol{u}\cdot\boldsymbol{u}^* - 1) \times \exp N(\boldsymbol{u}\boldsymbol{a}^* + \boldsymbol{u}^*\boldsymbol{a}) \tag{1}$$

(the normalizing factor $K$ is chosen such that $Z(0) = 1$). The large $N$ behavior of $Z$ can be obtained by several methods; one can for instance introduce a Lagrange multiplier for the constraint, integrate over the $\boldsymbol{u}$ variables, thereby reducing the problem to the evaluation of a one-dimensional integral which may be calculated by the saddle-point method. However this method is not applicable to the matrix problem since the corresponding Lagrange multiplier is itself a matrix. Thus after integrating out the matrix $U$ we are left with $N^2$ coupled degrees of freedom and the saddle-point technique is not applicable. Consequently we shall attempt to calculate $Z$ by considering the large $N$ limit of the equations of motion.

The function $Z$ obviously satisfies the equation

$$\partial^2 Z/\partial a_i \partial a_i^* = N^2 Z \tag{2}$$

(repeated indices are always to be summed). The invariance of the measure under unitary transformations of $\boldsymbol{u}$ implies that $Z$ is a function of the single variable

$$\lambda = \boldsymbol{a}\cdot\boldsymbol{a}^* . \tag{3}$$

Eq. (2) can then be rewritten as

$$\lambda Z''(\lambda) + N Z'(\lambda) = N^2 Z(\lambda) . \tag{4}$$

We now use the fact that $\ln Z$ is proportional to $N$ and write

$$Z = \exp N W_N(\lambda) , \tag{5}$$

$$N^{-1}\lambda W_N''(\lambda) + \lambda W_N'^2 + W_N' = 1 . \tag{6}$$

In the large $N$ limit it is consistent to assume that $W_N$ is independent of $N$, and thus the term $\lambda W_N''/N$ can be dropped in eq. (6). Thus

$$\lambda W'^2 + W' = 1 . \tag{7}$$

This equation is easily solved, and since $W(0) = 0$ we obtain

$$W(\lambda) = (1 + 4\lambda)^{1/2} - 1 - \log[\tfrac{1}{2} + \tfrac{1}{2}(1 + 4\lambda)^{1/2}] . \tag{8}$$

*3. Differential equations for the matrix problem.* We now consider the partition function for a unitary link variable [in the large $N$ limit SU($N$) is equivalent to U($N$)] in an external source $A$

$$Z(AA^\dagger) = \int \mathrm{d}U \exp N\, \mathrm{tr}(UA^\dagger + AU^\dagger) . \tag{9}$$

The integral runs over all unitary $N\times N$ matrices and $\mathrm{d}U$ is the invariant (Haar) measure on U($N$) normalized so that $\int \mathrm{d}U = 1$. Using the fact that $UU^\dagger = 1$ we derive

$$\partial^2 Z/\partial A_{ab}\partial A^\dagger_{bc} = \delta_{ac} N^2 Z . \tag{10}$$

These $N^2$ equations are sufficient to determine $Z$ which is a function of $N^2$ variables. Actually due to the invariance of the measure under unitary transformations $Z$ is a function of only $N$ real positive variables, namely the eigenvalues $\lambda_a$ of the matrix $H$

$$H = AA^\dagger . \tag{11}$$

Indeed one can prove this obvious property in the following manner: (i) any matrix $A$ may be written $A = KV$ with $K$ hermitian positive and $V$ unitary, (ii) $K$ may be diagonalized by a unitary transformation $S$, $K = SDS^\dagger$, and if we perform the change of variables $U \to SU'S^\dagger V$ in the integral (9), it is then clear that $Z$ depends only upon $D$. Therefore we obtain

$$\frac{\partial^2 Z}{\partial A_{ab}\partial A^\dagger_{bc}} = \frac{\partial Z}{\partial H_{ac}} + \frac{\partial^2 Z}{\partial H_{ad}\partial H_{nc}} H_{nd} . \tag{12}$$

We can express all derivatives in terms of $\partial Z/\partial\lambda_a$ and $\partial^2 Z/\partial\lambda_a^2$ provided we know how to calculate the first and second derivatives of $\lambda_a$ with respect to the matrix elements of $H$. This last step is given by the well-known formulae of first and second order perturbation theory. The result is ($a$ is here a free index)

$$\lambda_a \frac{\partial^2 Z}{\partial\lambda_a^2} + N\frac{\partial Z}{\partial\lambda_a} + \sum_b{}' \frac{\lambda_b}{\lambda_b - \lambda_a}\left(\frac{\partial Z}{\partial\lambda_b} - \frac{\partial Z}{\partial\lambda_a}\right) = N^2 Z . \tag{13}$$

We again set $Z = \mathrm{e}^{NW}$ ($W$ is now proportional to $N$), so that

$$\frac{1}{N}\lambda_a \frac{\partial^2 W}{\partial\lambda_a^2} + \lambda_a\left(\frac{\partial W}{\partial\lambda_a}\right)^2 + \frac{\partial W}{\partial\lambda_a} + \frac{1}{N}\sum_b{}' \frac{\lambda_b}{\lambda_b - \lambda_a}\left(\frac{\partial W}{\partial\lambda_b} - \frac{\partial W}{\partial\lambda_a}\right) = 1 . \tag{14}$$

The first term is as before of order $1/N$ (note that the last term involves a sum of $N$ terms of order one and cannot be dropped). Thus in the large $N$ limit

$$W_a^2 + \frac{1}{N}\sum_b{}' \frac{W_b - W_a}{\lambda_b - \lambda_a} = \frac{1}{\lambda_a}\left(1 - \frac{1}{N}\sum_b W_b\right),$$

$$W_a \equiv \partial W/\partial\lambda_a . \qquad (15)$$

In order to explicitly take the $N \to \infty$ limit it is convenient to introduce the density of eigenvalues of $H$

$$\rho(x) = \frac{1}{N}\sum_{a=1}^{N} \delta(x - \lambda_a) \qquad (16)$$

and to consider $W$ as a functional of $\rho$. We then have

$$W_a = \frac{1}{N}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta W}{\delta\rho(x)}\bigg|_{x=\lambda_a} \qquad (17)$$

and eq. (15) takes the form, for $N = \infty$,

$$W^2(x) + \int \mathrm{d}y\, \rho(y)\frac{W(x) - W(y)}{x - y}$$

$$= \frac{1}{x}\left[1 - \int \mathrm{d}y\, \rho(y)\, W(y)\right], \qquad (18)$$

where $x$ is restricted to the support $[a, b]$ of the measure $\rho$. We can already see from eq. (18) the difference between the weak and the strong coupling regions. These correspond to large or small moments of the measure $\rho$, respectively. Indeed in the strong coupling limit $W(x)$ can be expanded in powers of the moments of $\rho$ and is not singular at $x = 0$. Therefore the integral

$$\alpha = \int \mathrm{d}x\, \rho(x)\, W(x) \qquad (19)$$

must equal one in this region. However, in the weak coupling region there is no reason for $\alpha$ to be equal to one and $W$ in fact is singular at $x = 0$. A standard approach to such singular integral equations is to reduce them to a "Riemann–Hilbert" problem. Therefore we find it useful to define the analytic functions

$$f(z) \equiv \int_a^b \frac{\mathrm{d}x\, \rho(x)}{z - x}, \qquad (20)$$

which is determined by the eigenvalues of $AA^\dagger$, and

$$F(z) \equiv \int_a^b \frac{\mathrm{d}x\, \rho(x)\, W(x)}{z - x}, \qquad (21)$$

which is unknown. The functions $f$ and $F$ are real analytic with cuts running from $a$ to $b$. If we approach the cut $[a, b]$ from above we obtain

$$\mathrm{Im}\, F = -\mathrm{i}\pi W(x)\, \rho(x) = W(x)\, \mathrm{Im}\, f, \qquad (22)$$

and the real part of $F$ can be determined using eq. (18). The problem of determining $W(x)$ is equivalent to the calculation of the function $F(z)$. It is convenient to consider separately the strong and weak "coupling" regimes.

*4. The strong coupling solution.* Here the integral $\alpha$, eq. (19), is equal to one, $W(x)$ is regular at $x = 0$ and the real part of $F$ satisfies the equation

$$\mathrm{Re}\, F = W^2(x) + W(x)\, \mathrm{Re}\, f . \qquad (23)$$

A standard approach to the construction of $F(z)$ would be to eliminate $W(x)$ from eqs. (22), (23), thereby obtaining a quadratic equation relating Re $F$ and Im $F$. However the form of eqs. (22), (23) suggests the following ansatz

$$F(z) = W(z)\, f(z) + W^2(z), \qquad (24)$$

where we assume that $W(z)$ is a real analytic function. However in order to satisfy eqs. (22), (23) we must demand

$$\mathrm{Im}\, W\, \mathrm{Im}\, f = 0, \qquad (25)$$

$$\mathrm{Im}\, W\, (\mathrm{Re}\, f + 2\, \mathrm{Re}\, W) = 0 . \qquad (26)$$

Thus Im $W$ vanishes on the support of $\rho$. Outside the interval $[a, b]$ Im $W$ can be nonzero only if Re $W = -\frac{1}{2}\mathrm{Re}\, f = -\frac{1}{2} f$. These equations must be supplemented by asymptotic conditions as $|z| \to \infty$. These are

$$f(z) \sim 1/z ,$$

since the integral of $\rho$ is by definition equal to one, and

$$F(z) \sim 1/z ,$$

since in this region $\alpha = 1$. Consequently, eq. (24) implies

$$W^2(z) \sim 1/z . \qquad (27)$$

We must now construct the function $W(z)$ which satisfies eqs. (25)–(27). Assume that Im $W \neq 0$ for $-\infty < x < -c$ and consider the function $W(z)(z + c)^{1/2}$ which is analytic in a cut plane from $-\infty$ to $-c$ ($c$ is a positive constant to be determined below). As we approach the cut from above, eq. (26) and the reality of $f$ on the real axis outside $[a, b]$ imply

$$\mathrm{Im}[W(x)(x+c)^{1/2}] = -\tfrac{1}{2}f(x)(-x-c)^{1/2} . \tag{28}$$

Using the asymptotic condition, eq. (27), we can write a dispersion relation for $W(z)$

$$W(x)(x+c)^{1/2} = 1 + \frac{1}{2\pi}\int_{-\infty}^{-c} \mathrm{d}y \frac{f(y)(-y-c)^{1/2}}{x-y}, \tag{29}$$

or, expressing $f$ in terms of $\rho$,

$$W(x)(x+c)^{1/2} = 1 - \frac{1}{2}\int_a^b \mathrm{d}y \frac{\rho(y)}{(x+c)^{1/2}+(y+c)^{1/2}} . \tag{30}$$

The constant $c$ is fixed by the requirement that $\alpha = 1$. Using the representation (30) we obtain the constraint

$$\int_a^b \mathrm{d}x \frac{\rho(x)}{(x+c)^{1/2}} = 2 . \tag{31}$$

Note that this condition also eliminates the potential pole of $W^2(z)$ at $z = -c$, which would be inconsistent with eq. (26).

We shall show below that the validity of the strong coupling solution is limited to the range $s \geqslant 2$ where

$$s \equiv \int_a^b \mathrm{d}x \frac{\rho(x)}{x^{1/2}} . \tag{32}$$

Therefore in the strong coupling regime, i.e., $s \geqslant 2$, $c$ must be positive.

In order to calculate the free energy we have to calculate the indefinite integral of $W(x)$ and to integrate functionally with respect to $\rho$. This last step is a priori extremely difficult since the parameter $c$ is also a functional of $\rho$. However the solution

$$\frac{W}{N} = 2\int_a^b \mathrm{d}x\, \rho(x)(x+c)^{1/2}$$
$$-\frac{1}{2}\int_a^b \mathrm{d}x\,\mathrm{d}y\, \rho(x)\rho(y)\log[(x+c)^{1/2}+(y+c)^{1/2}]$$
$$-c-3/4 , \tag{33}$$

is stationary with respect to $c$ and it satisfies eq. (17). The constant $-3/4$ is determined by the infinite coupling limit ($A = 0$). In this limit we find from eq. (31) that $c$ approaches 1/4 and the normalization of eq. (9) is such that $W$ vanishes. In terms of the eigenvalues themselves eq. (33) reads

$$\lim_{N\to\infty}\frac{\log Z}{N^2} = \lim \frac{W}{N} = \frac{2}{N}\sum_{a=1}^{N}(\lambda_a+c)^{1/2}$$
$$-\frac{1}{2N^2}\sum_{a,b}\log[(\lambda_a+c)^{1/2}+(\lambda_b+c)^{1/2}] - c - 3/4 ,$$

$$\frac{1}{N}\sum_a(\lambda_a+c)^{-1/2} = 2 . \tag{33'}$$

A check on this result is the direct strong coupling expansion of $Z$ in powers of $AA^\dagger$ (it is simpler to derive this expansion from eq. (18)). The result is

$$W/N = \sigma_1 - \tfrac{1}{2}\sigma_2 + \tfrac{2}{3}\sigma_3 - \tfrac{1}{4}(5\sigma_4 + 6\sigma_2^2) + \mathrm{O}(\lambda^5) ,$$

where the $\sigma_m$ are the cumulants of the moments of the measure $\rho$:

$$\rho_n = \frac{1}{N}\sum_a \lambda_a^n ; \quad \rho_1 = \sigma_1 , \quad \rho_2 = \sigma_2 + \sigma_1^2 ,$$

$$\rho_3 = \sigma_3 + 3\sigma_1\sigma_2 + \sigma_1^3 , \text{ etc.}$$

It is tedious but straightforward to verify that this expansion may be recovered from the explicit solution (33).

5. *The weak coupling solution.* In this regime the integral $\alpha$ is no longer equal to one and the analytic function $W(z)$ will have a pole at the origin in addition to the previous cut. Away from this pole the analyticity conditions (25), (26) are unchanged. Thus the solution is simply

$$W(x)\, x^{1/2} = 1 + \frac{1}{2\pi}\int_{-\infty}^{0} \mathrm{d}y \frac{f(y)(-y)^{1/2}}{x-y}$$
$$= 1 - \frac{1}{2}\int_a^b \mathrm{d}y \frac{\rho(y)}{x^{1/2}+y^{1/2}} . \tag{34}$$

If we calculate from this solution the value of $\alpha = \int_a^b \mathrm{d}x\, \rho(x)\, W(x)$ we find

$$\alpha = \int_a^b \frac{\mathrm{d}x\, \rho(x)}{x^{1/2}} - \frac{1}{4}\left(\int_a^b \frac{\mathrm{d}x\, \rho(x)}{x^{1/2}}\right)^2 = s(1-s/4) . \tag{35}$$

The right-hand side of eq. (35) is bounded by one and equals one at the point $s = 2$. Below we shall show that this weak coupling result is valid for $s \leqslant 2$. As $s$ is increased to two, $\alpha$ increases to one and the weak coupling solution is identical, at $s = 2$, with the strong coupling solution valid for $s \geqslant 2$.

The functional integration with respect to $\rho$ is now elementary and it gives (in agreement with ref. [2])

$$\frac{\log Z}{N^2} = \frac{2}{N}\sum_a \lambda_a^{1/2} - \frac{1}{2N^2}\sum_{a,b}\log(\lambda_a^{1/2} + \lambda_b^{1/2}) - 3/4 . \tag{36}$$

The constant $-3/4$ is determined by the continuity of $W$ at $s = 2$.

*6. The phase transition.* In summary we have found two separate solutions for $W = N^{-1} \ln Z$. The weak coupling solution, eq. (36), is obviously valid for $s \simeq 0$. In fact it precisely coincides, remarkably enough, with the perturbative expansion of $W$, about the weak coupling saddle-point $U = [1/(AA^\dagger)^{1/2}]\,A$, to one loop order. On the other hand the strong coupling solution is clearly valid for $s \gg 1$, since it is analytic in the matrix elements of $A$ for $A \simeq 0$. Both solutions are analytic functions of $s$, for $s \neq 0, \infty$. Therefore there must be a (phase) transition from one to the other at some value of $s$, at which point the two expressions for $W$ must coincide. This occurs at $s = 2$. It is obvious that the strong (weak) coupling solution is correct for $s \geqslant 2$ ($s \leqslant 2$) since it yields a smaller value of the free energy, $-\ln Z$, in this region.

The nature of the "phase" transition can be determined by examining the neighborhood of $s = 2$. Starting from the strong coupling region, $s > 2$, we let the eigenvalues of $AA^\dagger$ increase. As $s \to 2$, $c$ vanishes as

$$c = 2(s-2) \bigg/ \left(\frac{1}{N}\sum_a \lambda_a^{-3/2}\right) + O(s-2)^2 . \tag{37}$$

It is then straightforward to calculate the difference between $W_{\text{strong}}(s)$ and $W_{\text{weak}}(s)$ for $s \simeq 2$, with the result

$$W_{\text{strong}}(s) - W_{\text{weak}}(s) \simeq \text{const.}\,(s-2)^3 . \tag{38}$$

Therefore the free energy as well as its first two derivatives are continuous at $s = 2$, but in complete analogy with ref. [3] there is a discontinuity in the third derivative, i.e., a "third order transition".

These results may be used to calculate the expectation value of products of $U$'s and $U^\dagger$'s in the external field $A$. For instance if we differentiate $W(AA^\dagger)$ with respect to $A$ we derive

$$\langle U_{ab}\rangle = \frac{\partial W}{\partial A^\dagger_{ba}} = \frac{\partial W}{\partial \lambda_d}\,\phi_c^{(d)*}\,\phi_a^{(d)}\,A_{cb} ,$$

where $\phi^{(d)}$ is the eigenvector of $AA^\dagger$ corresponding to the eigenvalue $\lambda_d$. This leads to

$$\langle U_{ab}\rangle = \left[\frac{1}{(c+AA^\dagger)^{1/2}}\left(1 - \frac{1}{2N}\sum_d \frac{1}{(c+\lambda_d)^{1/2} + (c+AA^\dagger)^{1/2}}\right)\right]_{ma} A_{mb} , \tag{39}$$

where $c = 0$ in the weak coupling phase and $c$ is determined by eq. (31) in the strong coupling phase.

Application of these results will appear in subsequent papers.

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