

Some remarks about induced QCD

David J. Gross¹

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA

Received 10 August 1992

Migdal and Kazakov have suggested that lattice QCD with an adjoint representation scalar in the infinite coupling limit could induce QCD. I find an exact saddlepoint of this theory for infinite N in the case of a quadratic scalar potential. I discuss some aspects of this solution and also show how the continuum $D=1$ matrix model with an arbitrary potential can be reproduced through this approach.

1. Introduction

Recently Kazakov and Migdal have proposed a lattice gauge model that, on the one hand, is potentially soluble for large N and on the other hand might induce QCD [1]. Their idea was to consider a scalar field, Φ , in the adjoint representation of $SU(N)$, on a lattice of spacing a , coupled to an $SU(N)$ lattice gauge field in the standard fashion,

$$S = \sum_x N \text{Tr} \left(-U(\Phi(x)) + \sum_\mu \Phi(x) U_\mu(x) \Phi(x + \mu a) U_\mu^\dagger(x) \right) \quad (1.1)$$

(μ runs over the 2D lattice vectors on a hypercubic lattice of dimension D). Note that there is no kinetic term for the gauge field. This is equivalent to the infinitely strong coupling limit of the standard lattice action. They argued that if one integrates out the scalar mesons (even in the case of noninteracting scalars with $U(\Phi) = \frac{1}{2} m^2 \Phi^2$), then at distances large compared to a , one would induce in four dimensions an effective gauge interaction, $(N/48\pi^2) [\ln(1/ma)] \text{Tr} F_{\mu\nu} F^{\mu\nu}$. The basic idea is that the infrared slavery of the scalars, at the size of the lattice spacing, produces an effective gauge theory at a larger scale (much larger than the inverse scalar mass), which then produces the usual asymptotically free fixed point theory. Migdal has analyzed the large N saddlepoint equations of this model in great detail [2] and discussed the scaling properties of purported solutions, as well as the equations for the spectrum of states that emerges [3].

There are many problems with this idea. For one the hard gluons are not absent and their contribution will overwhelm that of the scalars at short distances. Their asymptotic freedom is more powerful than the infrared slavery of scalars. It might be possible to overcome this problem by considering a general potential $U(\Phi)$ and fiddling with its parameters to arrive at the QCD fixed point. Another issue is that the above theory possesses a much larger symmetry than the $SU(N)$ gauge symmetry of the usual lattice action. It is not difficult to see that, in D dimensions, it is invariant under $(D-1) \times (N-1)$ extra local $U(1)$ -gauge symmetries. This is because the transformation $U_\mu(x) \rightarrow V_\mu^\dagger(x) U_\mu(x) V_\mu(x + \mu a)$ leaves the action invariant as long as $V_\mu(x)$ is a unitary matrix that commutes with $\Phi(x)$. If $V_\mu(x)$ were independent of μ then this would be the ordinary gauge invariance. Thus we have $D-1$ new gauge symmetries, which are of course isomorphic to the special unitary transforma-

¹ Research supported in part by NSF grant PHY90-19754.

tions that commute with Φ . Thus $V_\mu(x) = D_\mu(x)\Omega(x)$, where $\Omega(x)$ is the unitary matrix that diagonalizes Φ and $D_\mu(x)$ is diagonal.

This extra symmetry is evident if one considers the naive continuum limit of (1.1), by expanding about $U_\mu = 1 + iaA_\mu$, since the action will not depend on the diagonal components of the vector potential (in a basis where Φ is diagonal). In fact, one must fix, in addition to the usual gauge fixing, the extra $(D-1) \times (N-1)$ local symmetries to eliminate these components. This can easily be done.

A subset of this symmetry is the local Z_N symmetry, $U_\mu(x) \rightarrow Z_\mu U_\mu(x) Z_\mu^\dagger$, where Z_μ is an element of the center of the group. This has been noted recently by Kogan et al. [4]. This symmetry alone prevents the Wilson loop from acquiring an expectation value, and should be broken if we are to recover the QCD fixed point from this formulation. Presumably the same is true of the extra continuous symmetries discussed above.

In this paper I shall construct an exact solution of the saddlepoint equations for the case of a quadratic potential. Remarkably, it turns out that the eigenvalue distribution, for large N , is given by a semi-circular law with a mass parameter that depends on m and on the dimension of space-time D . This trivial solution bears no resemblance to continuum QCD. Instead it describes the strong coupling lattice theory – with color singlet mesons.

2. The quadratic potential

The model described by (1.1) can be reduced to an integral over N degrees of freedom per site, by integrating out the matrices that diagonalize $\Phi(x)$. This can be done, link by link, in terms of the Itzykson–Zuber integral ($\Delta(\Phi) = \prod_{i < j} [\Phi_i - \Phi_j]$) [5]:

$$I(\Phi, \Psi) \equiv \int \mathcal{D}U \exp(N \text{Tr}[\Phi U \Psi U^\dagger]) = c \frac{\det[\exp(N\Phi_i \Psi_j)]}{\Delta(\Phi)\Delta(\Psi)}. \tag{2.1}$$

The effective action for the eigenvalues is then

$$S = \sum_{x,i} [-NU(\Phi_i(x))] + \sum_{x,i \neq j} \ln |\Phi_i(x) - \Phi_j(x)| + \sum_{x,\mu} \ln [I(\Phi(x), \Phi(x + \mu a))]. \tag{2.2}$$

In the large N limit the integral will be dominated by the translationally invariant saddlepoint for the density of eigenvalues, $\rho(\nu)$, of the matrices Φ . The saddlepoint equation is

$$P \int d\nu \frac{\rho(\nu)}{\Phi_a - \nu} = \frac{1}{2} U'(\Phi_a) - D \lim_{N \rightarrow \infty} \left(\frac{\partial I(\Phi, \Psi)}{\partial \Phi_a} \right) \Big|_{\Phi = \Psi}. \tag{2.3}$$

Migdal has simplified these using the Schwinger–Dyson equations that are satisfied by $I(\Phi, \Psi)$ [2]. These are consequences of the fact that I satisfies $\text{Tr}[(1/N)\partial/\partial\Phi]^k I = \text{Tr}(\Psi)^k I$. The net result is that one derives an equation for the function $V'(z) \equiv \int dz \rho(\nu)/(z - \nu)$, whose imaginary part is $\text{Im } V'(\nu) = -\pi\rho(\nu)$,

$$\text{Re } V'(\lambda) = P \int \frac{d\nu}{2\pi i} \ln \left(\frac{\lambda - (1/2D)U'(\nu) - [(D-1)/D] \text{Re } V'(\nu) + i\pi\rho(\nu)}{\lambda - (1/2D)U'(\nu) - [(D-1)/D] \text{Re } V'(\nu) - i\pi\rho(\nu)} \right). \tag{2.4}$$

Migdal has studied the scaling properties of conjectured solutions of this equation. Here we shall find an exact solution in the case of a quadratic, gaussian, potential, $U(\Phi) = \frac{1}{2} m^2 \Phi^2$. First, we note that this equation simplifies dramatically for $D=1$. This is not surprising since in one dimension the gauge field plays no role, can be reabsorbed into a definition of Φ , and (1.1) reduces to the standard action for a scalar field in one dimension. In particular for the quadratic potential the path integral is a gaussian,

$$Z = \int \prod_n \mathcal{D}\Phi_n \exp\left(-N \sum_n \text{Tr}\left(\frac{1}{2} m^2 \Phi_n^2 - \Phi_n \Phi_{n+1}\right)\right). \tag{2.5}$$

Thus the eigenvalues of Φ will be given by the semi-circular distribution, namely $\pi\rho(\nu) = \sqrt{\mu - \frac{1}{4}\mu^2\nu^2}$, where μ is determined by the mean of the squares of the eigenvalues, $\langle (1/N) \text{Tr}(\Phi^2) \rangle = 1/\mu$. It is therefore sufficient to calculate the expectation value of $(1/N) \text{Tr}(\Phi^2)$, which is given by the one-loop integral

$$\frac{1}{N} \text{Tr}(\Phi^2) = \int_{-\pi}^{\pi} \frac{dp}{2\pi} \frac{1}{m^2 + 2 \cos p} = \frac{1}{\sqrt{m^4 - 4}} \equiv \frac{1}{\mu}. \tag{2.6}$$

It is easy to verify that this solves (2.4), using the fact that

$$V'(z) = \frac{1}{2}\mu z - \sqrt{\frac{1}{4}\mu^2 z^2 - \mu}, \quad \text{Re } V'(v) = \frac{1}{2}\mu v, \quad \text{for } |v| \leq 2/\sqrt{\mu}. \tag{2.7}$$

However, if we return to (2.4), we see that the integral involved is of the same form for any D , as long as $\text{Re } V'(v)$ is linear in v . This suggests that we can find a solution of (2.4) with a semi-circular distribution of eigenvalues for a quadratic potential in any dimension.

We use the result that

$$I = \int_{-2/\sqrt{\mu}}^{2/\sqrt{\mu}} \frac{dv}{2\pi i} \ln \left(\frac{\lambda - \frac{1}{2}bv + i\sqrt{\mu - \frac{1}{4}\mu^2\nu^2}}{\lambda - \frac{1}{2}bv - i\sqrt{\mu - \frac{1}{4}\mu^2\nu^2}} \right) = \frac{2\mu}{(b^2 - \mu^2)} \left(\lambda - \sqrt{\lambda^2 - \frac{(b^2 - \mu^2)}{\mu}} \right). \tag{2.8}$$

This result is easily established (change variables $\nu = -i(z - 1/z)/\sqrt{\mu}$, integrate once by parts and then the z integral can be done by contour integration). In our case, assuming the semi-circular distribution in (2.7) $b = m^2 + [(D-1)/D]\mu$. On the other hand, from the integral it must be that $b^2 = \mu^2 + 4$. Therefore we derive a quadratic equation for μ , whose solutions are

$$\mu_{\pm}(D) = \frac{m^2(D-1) \pm D\sqrt{m^2 - 4(2D-1)}}{2D-1}. \tag{2.9}$$

In particular for $D=1$, $\mu_+(1) = \mu = \sqrt{m^4 - 4}$, which agrees with the direct calculation of (2.6).

Thus we have found a saddlepoint for any D , as long as the potential is quadratic. It is given by (2.7), with μ given above. If we start with a large mass, $m^2 > m_c^2 = 2D$, then only μ_+ is positive and must be chosen. This solution can be verified to satisfy the large mass expansion discussed by Kazakov and Migdal [1]. I have explicitly checked that a semi-circular distribution of eigenvalues, with

$$\mu = \mu_+ = m^2 - \frac{2D}{m^2} - \frac{2D(2D-1)}{m^6} - \frac{4D(2D-1)^2}{m^{10}} - \dots,$$

satisfies the $1/m$ expansion up to terms of order $(1/m)^{14}$.

The free energy, $F = -(1/N^2 \cdot \text{Vol.}) \ln Z$, can be calculated at the saddlepoint by the following device. We use the fact that for $D=1$ the free energy can be evaluated by gaussian integration to yield

$$F(D=1) = \frac{1}{2} \int_{-\pi}^{\pi} \frac{dp}{2\pi} \ln(m^2 + 2 \cos p) = \frac{1}{2} \ln \left(\frac{m^2 + \sqrt{m^4 - 4}}{2} \right).$$

Then we note that for any D the free energy is given at large N saddlepoint by

$$F(D) = \frac{m^2}{2N} \text{Tr} \Phi_s^2 - \frac{1}{N^2} D^2(\Phi_s) - \frac{D}{N^2} \ln I(\Phi_s, \Phi_s), \tag{2.10}$$

where Φ_s is the saddlepoint scalar field. The first two terms are easily evaluated for a semi-circular distribution of eigenvalues, which allows us to express the logarithm of the Itzykson-Zuber integral, for a semicircular distribution of eigenvalues with parameter μ , as

$$\frac{1}{N^2} \ln I(\mu) = \frac{\sqrt{\mu^2+4}-\mu}{2\mu} - \frac{1}{2} \ln \left(\frac{\sqrt{\mu^2+4}+\mu}{2\mu} \right).$$

It then follows that the free energy, for a semi-circular distribution of eigenvalues with parameter μ is

$$F(D) = \frac{1}{2} \frac{m^2}{2\mu} + \frac{1}{2} \ln \mu - \frac{1}{2} D \left[\sqrt{1 + \frac{4}{\mu^2}} - 1 - \ln \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4}{\mu^2}} \right) \right]. \quad (2.11)$$

It is easily checked that the saddlepoint of (2.11) determines μ to satisfy (2.9).

For large mass it is easily verified that F is bounded from below, and achieves its minimum at $\mu = \mu_+$. For $D \leq 1$, μ_+ vanishes at $m^2 = 2D$, and thus we can take a continuum limit of the theory. For $D > 1$, μ_+ is always non-vanishing. Instead as $m^2 \rightarrow 2D$ the other branch μ_- vanishes. However, we must approach the critical value of m^2 from below, since only then is μ_- positive. If we examine the free energy in this case we find that it is unbounded from below. Indeed $F(D) \rightarrow_{\mu \rightarrow 0} (m^2 - 2D)/\mu$. The saddlepoint at $\mu = \mu_-$, which vanishes as $m^2 \rightarrow 2D - 0$, is a local *maximum*. Thus although we may construct a continuum limit about this point it might contain tachyonic excitations.

It is not surprising that F is unbounded from below, for $m^2 < 2D$, since the action can be bounded, for translationally invariant Φ , by

$$S = N \cdot \text{Vol.} \cdot \text{Tr} \left(-\frac{1}{2} m^2 \Phi^2 + \sum_{\mu} \Phi U_{\mu} \Phi U_{\mu}^{\dagger} \right) > N \cdot \text{Vol.} \cdot \text{Tr} \left(-\frac{1}{2} m^2 \Phi^2 + D \Phi \Phi \right),$$

which is unbounded from below. Thus, this is not a stable region of the parameters. Unless there exist other, healthier, large N saddlepoints of the action, then there would not appear to be a scaling solution that would give a continuum theory for a gaussian potential for $D > 1$. Nonetheless, the saddlepoint we have found will describe the large N , strong coupling limit of QCD with adjoint matter.

3. Arbitrary potential in one dimension

Now let us consider the one dimensional system for an arbitrary potential, namely the model described by

$$Z = \int \prod_n \mathcal{D}\Phi_n \exp \left(- \sum_n \text{Tr} \left[\frac{1}{2} m^2 \Phi_n^2 + U(\Phi_n) - \Phi_n \Phi_{n+1} \right] \right). \quad (3.1)$$

This model is hard to solve, except in the double scaling limit, where it was solved exactly as long as the spacing between the points is not too big [6]. However, in the continuum limit, the model can be solved in terms of N free fermions in the potential $\frac{1}{2} m^2 \Phi^2 + V(\Phi)$, and the density of eigenvalues is easily calculable in the large N limit. To take the continuum limit we must scale

$$\Phi_n \rightarrow \frac{\varphi(t)}{\sqrt{a}}, \quad m^2 = 2 + a^2 M^2, \quad U(\Phi) = a W(\varphi). \quad (3.2)$$

Then the above partition function reduces, as the lattice spacing $a \rightarrow 0$, to the continuum integral

$$Z = \int \mathcal{D}\varphi(t) \exp \left(-N \int dt \left(\text{Tr} \left\{ \frac{1}{2} M^2 \varphi(t)^2 + W(\varphi(t)) + \frac{1}{2} [\partial_t \varphi(t)]^2 \right\} \right) \right). \quad (3.3)$$

Now, this model is easily solved, following ref. [7], by reducing it to the problem of N free fermions, (the eigenvalues of φ), with the one-body hamiltonian, $H = \frac{1}{2} p^2 + \frac{1}{2} M^2 \lambda^2 + W(\lambda)$, with $1/N$ playing the role of Planck's constant. In the WKB approximation (i.e. $N \rightarrow \infty$ limit)

$$\rho(E) = \frac{1}{N} \frac{dn}{dE} = \int \frac{dx dp}{2\pi} \delta[E - \frac{1}{2}p^2 - \frac{1}{2}M^2x^2 - W(x)] \rightarrow \rho(x) = \sqrt{2[E - \frac{1}{2}M^2x^2 - W(x)]}. \quad (3.4)$$

The Fermi level, E is determined by the condition that $\int_{x_c}^{x_c} dx \rho(x) = 1$.

Let us return to the matrix chain, and write the equations for this case:

$$\begin{aligned} \text{Re } V'(\lambda) &= \int \frac{dv}{2\pi i} \ln \left(\frac{\lambda - (1 + \frac{1}{2}a^2M^2)v - \frac{1}{2}a^{3/2}W'(\sqrt{av}) + i\pi\rho(v)}{\lambda - (1 + \frac{1}{2}a^2M^2)v - \frac{1}{2}a^{3/2}W'(\sqrt{av}) - i\pi\rho(v)} \right), \\ \text{Im } V'(v) &= -\pi\rho(v). \end{aligned} \quad (3.5)$$

Now rescale the variables so that $\lambda = \tilde{\lambda}/\sqrt{a}$, $v = \tilde{v}/\sqrt{a}$, $\rho(v) = \sqrt{af(\tilde{v})}$, $V'(\lambda) \equiv aF(\tilde{\lambda})$. In terms of these the equations read

$$\text{Re } F(\tilde{\lambda}) = \frac{1}{a} \int \frac{d\tilde{v}}{2\pi i} \ln \left(\frac{\tilde{\lambda} - \tilde{v} - \frac{1}{2}a^2M^2\tilde{v} - \frac{1}{2}a^2W'(\tilde{v}) + i\pi f(\tilde{v})}{\tilde{\lambda} - \tilde{v} - \frac{1}{2}a^2M^2\tilde{v} - \frac{1}{2}a^2W'(\tilde{v}) - i\pi f(\tilde{v})} \right), \quad (3.6)$$

$$\text{Im } F(\tilde{v}) = -\pi f(\tilde{v}). \quad (3.6)$$

Expanding (3.6) in powers of a we find

$$\text{Re } F(\tilde{\lambda}) = \int d\tilde{v} \frac{f(\tilde{v})}{\tilde{\lambda} - \tilde{v}} + \frac{1}{2}a^2 \int d\tilde{v} \frac{[M^2\tilde{v} + W'(\tilde{v})]f(\tilde{v})}{(\tilde{\lambda} - \tilde{v})^2} - \frac{1}{3}\pi^2 a^2 \int d\tilde{v} \frac{f^3(\tilde{v})}{(\tilde{\lambda} - \tilde{v})^3} + O(a^3). \quad (3.7)$$

Now as $a \rightarrow 0$ we see that these equations are trivially satisfied for *any* function $f(\tilde{v})$. We now demand that the term of order a^2 vanish. Integrating by parts the last term this condition can be written as

$$\frac{1}{2}a^2 \int \frac{d\tilde{v}}{(\tilde{\lambda} - \tilde{v})^2} \{ [M^2\tilde{v} + W'(\tilde{v})]f(\tilde{v}) + \pi^2 f^2(\tilde{v})f'(\tilde{v}) \}.$$

Consequently $M^2\tilde{v} + W'(\tilde{v}) + \pi^2 f(\tilde{v})f'(\tilde{v}) = 0$, an equation that we can solve for f :

$$\pi f(\tilde{v}) = \sqrt{2E - M^2\tilde{v}^2 - 2W(\tilde{v})}. \quad (3.8)$$

E is a constant that can be fixed from the normalization of ρ . This agrees with the fermionic solution (3.4).

4. Conclusions

We have found that infinite coupling QCD, with scalar adjoint matter, is dominated for large N by a very simple saddlepoint. The eigenvalues of the scalar field are distributed according to the semi-circular law. This is quite remarkable, especially as the effective scalar action is certainly not a gaussian. It will be very interesting to evaluate the fluctuations about this saddlepoint – i.e. to work out the effective lagrangian describing the confined mesons in this lattice gauge theory. We note that for zero gauge coupling the eigenvalues also have a semi-circular distribution. In this case μ will be given by the analog, in dimension D , of (2.6). The fact that the eigenvalue distribution is semi-circular for both zero and infinite coupling makes one wonder whether this could be the case for finite coupling as well!

What have we learned about induced QCD? We have certainly learned that the program cannot work for a gaussian potential. This is consistent with the analysis of the scaling solutions of the general equations [8]. So if the idea is to work one must consider non-trivial potentials and hope that the non-asymptotically free scalar

interactions do indeed induce QCD. Although I do not see why this could not happen the physical mechanism is mysterious.

References

- [1] V. Kazakov and A. Migdal, Induced QCD at large N , Princeton preprint PUPT-1322 (May 1992).
- [2] A. Migdal, Exact solution of induced lattice gauge theory at large N , Princeton preprint PUPT-1323 (June 1992).
- [3] A. Migdal, $1/N$ expansion and particle spectrum in induced QCD, Princeton preprint PUPT-1332 (June 1992).
- [4] I. Kogan, G. Semenoff and N. Weiss, Induced QCD and hidden local Z_N symmetry, preprint UBCTP 92-022 (June 1992).
- [5] C. Itzykson and J. Zuber, *J. Math. Phys.* 21 (1980) 411.
- [6] D. Gross and I. Klebanov, *Nucl. Phys. B* 334 (199) 475.
- [7] E. Brézin, C. Itzykson, G. Parisi and J. Zuber, *Commun. Math. Phys.* 59 (1978) 35.
- [8] A. Migdal, private communication.