STRUCTURE OF NON-LINEAR REALIZATION IN SUPERSYMMETRIC THEORIES

M. BANDO [a], T. KURAMOTO [a], T. MASKAWA [b] and S. UEHARA [a]
[a] *Department of Physics, Kyoto University, Kyoto 606, Japan*
[b] *Research Institute for Fundamental Physics, Kyoto University, Kyoto 606, Japan*

Received 15 December 1983

We clarify the origin of massless particles which come out when an internal symmetry breaks down preserving supersymmetry. A special feature is that even if original and final symmetries are fixed, there exist many inequivalent non-linear realizations. We present the theorems which make clear the essential structure of the phenomena. The (non-linear) transformation law for the massless particles is accurately defined and it is shown how to construct the supersymmetric non-linear lagrangian.

It is well known that if a global internal symmetry group G breaks down to a subgroup H, there come out massless scalars, the number of which is equal to that of the broken generators, i.e., dim [G/H]. An interesting point is that the low energy theorem determines the effective lagrangian for the massless scalars uniquely once the symmetry groups G and H are given. Systematic study of such non-linear realizations of a symmetry was skillfully done by Coleman et al. [1] about fifteen years ago. Recently supersymmetric theories have been studied from various viewpoints for application to current problems in particle physics. Of course in those models when an internal symmetry breaks down, there come out massless scalars. The number of such massless scalars, however, is not necessarily equal to, but, in general, larger than dim [G/H] [2]. In other words, besides what is called Goldstone bosons there come out unexpected massless scalars (quasi Goldstone bosons). Furthermore some number of massless fermions appear (quasi Goldstone fermions). These phenomena are understandable if one notices the following matter. In a ($N = 1$) supersymmetric model, a massless scalar is a component of a chiral superfield. Thus a Goldstone boson is a component of a chiral superfield (Goldstone superfield). Since a chiral superfield consists of two scalar bosons and one fermion, Goldstone superfields can be classified into two types: (i) The P-type superfield; both of the scalar components are Goldstone bosons. (ii) The M-type superfield; only one of the scalar components is a Goldstone boson and another is a quasi Goldstone boson. Both types of superfields contain quasi Goldstone fermions. Here if all Goldstone superfields are of the P-type, there appear only massless fermions in addition to what are called Goldstone bosons, while if all Goldstone superfields are of the M-type, there appear other massless scalars besides the Goldstone bosons and quasi Goldstone fermions. There exist intermediate cases where some of the Goldstone superfields are of the P-type and the rest of the M-type. An important point is that even if one fixes only a symmetry breaking pattern ($G \rightarrow H$), each number of P-type and M-type superfields is not uniquely determined, in general, and it depends fairly on the dynamics of the original system.

In this letter we present an outline of the structure of non-linear realization in supersymmetric theories. Details will be reported in a separate paper [3]. The number of massless scalars (Goldstone and quasi Goldstone bosons) is shown to be equal to the real dimension of the coset space $G^c/\hat{H}$, where G^c is a complexification of the group G and $\hat{H}$ is some complex group including H^c (a complexification of H). We present some theorems which determine the structure of $\hat{H}$ in a complete form. At the end of this letter we show how to construct supersymmetric non-linear lagrangians.

0.370-2693/84/$ 03.00 © Elsevier Science Publishers B.V. (North-Holland Physics Publishing Division)

First we investigate the origin of massless particles which come out when an internal symmetry group breaks down in supersymmetric theories. Let us consider an effective lagrangian $\mathcal{L}_{\text{eff}}(\phi^A)$ constructed by chiral superfields ϕ^A [‡1], where $\mathcal{L}_{\text{eff}}$ is understood to be integrated out by other irrelevant fields. In general, ϕ^A belong to a reducible representation ρ in a compact Lie group G which is a symmetry group of $\mathcal{L}_{\text{eff}}$. Since we now consider the system in which a group G breaks down to its subgroup H preserving ($N = 1$) supersymmetry, the minimal point ϕ_0^A of the effective potential V_{eff} is a fixed point of H,

$$\rho(\text{H}) \cdot \phi_0 = \phi_0 \,. \tag{1}$$

An important point is that, since ϕ^A is a complex field, we *can extend* the domain of ρ from G to G^c in such a way that the extension may preserve irreducibility of representations and be analytic for the coordinate variables of G^c [3]. We call it "analytic representation" [‡2]. Of course it does not mean that G^c is a symmetry group of $\mathcal{L}_{\text{eff}}$.

In terms of the Lie algebra $\mathscr{h} = \{\Lambda_\text{H}\}$ [‡3], eq. (1) is rewritten as

$$\rho(c_\alpha \Lambda_\text{H}^\alpha) \cdot \phi_0 = c_\alpha \rho(\Lambda_\text{H}^\alpha) \cdot \phi_0 = 0 \,, \quad \text{for } c_\alpha \in \mathbf{R} \,. \tag{2}$$

The first equality holds even for an complex number $c_\alpha \in \mathbf{C}$ since we adopt "analytic representation". Hence ϕ_0^A is a fixed point of H^c. Here we define $\hat{\text{H}}$ as

$$\hat{\text{H}} \equiv \{g; g \in \text{G}^\text{c} \text{ and } \rho(g) \cdot \phi_0 = \phi_0\} \,. \tag{3}$$

By definition

$$\hat{\text{H}} \supset \text{H}^\text{c} \,. \tag{4}$$

Let ξ be "representatives" of $\text{G}^\text{c}/\hat{\text{H}}$. Since the operation $\rho(\xi)$ on ϕ_0^A is "effective", the field variables ϕ^A can be represented by ξ and the rest, σ,

$$\phi^A = \phi^A(\xi, \sigma) \,. \tag{5}$$

At the vacuum point ξ and σ are chosen to be zero.

‡1 ϕ^A are either elementary or composite chiral superfields.

‡2 For example, the contragradient representation $(\rho^\text{T})^{-1}$ is used as an extension of the complex conjugate representation ρ^*.

‡3 We use *script* letters $\mathscr{g}$, $\mathscr{h}$ etc. for Lie algebras corresponding to the Lie groups represented by capital Latin letters G, H etc.

The effective potential V_{eff} is given by

$$V_{\text{eff}} = F_\alpha g^{\alpha\beta}(\xi, \sigma) \bar{F}_\beta \,, \quad (\alpha, \beta = \xi, \sigma) \,, \tag{6}$$

where the metric $g^{\alpha\beta}(\xi, \sigma)$ is non-singular at the vacuum point and the F-component F_α of a chiral superfield ξ or σ is represented by the superpotential W as

$$\bar{F}_\alpha = -(g^{-1})^{\alpha\beta} \, \delta W / \delta \phi_\beta \,. \tag{7}$$

Since W is a G-invariant analytic function of ϕ^A, it is actually G^c-invariant thanks to analytic extension of ρ. Therefore W is independent of the variable ξ, which leads to the following equation straightforwardly

$$F_\xi = 0 \,. \tag{8}$$

Since the supersymmetry is unbroken

$$F_\sigma|_{\sigma=0} = 0 \,, \tag{9}$$

thus

$$F_\sigma = (\text{const})\sigma + (\text{higher power of } \xi \text{ and } \sigma) \,. \tag{10}$$

From eqs. (6) and (10) one observes that there is no mass term of ξ [‡4], only interaction terms can appear. If the supersymmetry is broken,

$$F_\sigma|_{\sigma=0} \neq 0 \,, \tag{11}$$

and the quasi Goldstone bosons get masses. We have the following theorem.

Theorem 1. Let ξ be representatives of $\text{G}^\text{c}/\hat{\text{H}}$, then all ξ correspond to massless particles and the number of quasi Goldstone bosons N_Q is given by [‡5]

$$N_a = \dim[\text{G}^\text{c}/\hat{\text{H}}] - \dim[\text{G}/\text{H}] \,. \tag{12}$$

As to the group $\hat{\text{H}}$, we present the following theorem.

Theorem 2. ($\hat{\text{H}}$-structure theorem). Let ρ be an "analytic representation" and

$$\hat{\mathscr{h}} \equiv \{X; X \in \mathscr{g}^\text{c} \text{ and } \rho(X)\phi_0 = 0\} \,, \tag{13}$$

then $\hat{\mathscr{h}}$ is a direct sum of $\mathscr{h}^\text{c}$ and a nilpotent ideal $\mathscr{r}$,

$$\hat{\mathscr{h}} = \mathscr{h}^\text{c} \oplus \mathscr{r} \,. \tag{14}$$

And all the eigen values of "restriction" of $\mathscr{g}^\text{c}$-adjoint representation on $\mathscr{r}$ vanishes. Thus the

‡4 Private communication by T. Kugo (see ref. [4]).

‡5 The symbol "dim" counts *real* dimension of a manifold.

group $\hat{H}$ is a *semi*-direct product of H^c and R,

$$\hat{H} = H^c \times R . \qquad (15)$$

The proof of this theorem will be given in a separate paper [3], where we will also exhibit all possible candidates for $\hat{\mathscr{h}}$ when G and H are classical groups. One comment is in order: On account of the "physical" condition on $\hat{\mathscr{h}}$ of eq. (13), this theorem provides a more detailed structure than Levi's theorem does [5]. Here we give the following corollary of theorem 1.

Corollary 3

$$N_Q = \dim [G/H] - \dim R . \qquad (16)$$

Next we develop non-linear realization in supersymmetric theories. Once we fix the structure R for a given G/H, non-linear realization can be made by introducing the quasi Goldstone bosons, the number of which is just equal to dim G/H − dim R according to corollary 3. The case of R = {1} requires all the Goldstone superfields to be of the M type (the number of which is just dim G/H), which we call "maximal realization". On the other hand "minimal realization" demands the possible maximal R, where we need the minimal number of M-type superfields. Also we call those between the above extreme cases "intermediate realization". Especially the minimal realization in which only P-type superfields are introduced (i.e. dim R = dim G/H) is called "pure realization". For the case of G and H being classical groups, a pure realization can be made only when [3]

$$G/H = U(N)/U(n_1) \times U(n_2) \times \ldots \times U(n_a)$$

$$\left(\sum_{i=1}^{a} n_i = N\right) , \qquad (17a)$$

$$= O(N)/O(m) \times U(n_1) \times U(n_2) \times \ldots \times U(n_a)$$

$$\left(2\sum_{i=1}^{a} n_i = N - m\right) , \qquad (17b)$$

$$= S_p(2N)/S_p(2m) \times U(n_1) \times U(n_2) \times \ldots \times U(n_a)$$

$$\left(\sum_{i=1}^{a} n_i = N - m\right) . \qquad (17c)$$

Now we show the transformation properties for Goldstone superfields, which can be done basically in the usual method [1]. Every group element $g^c \in G^c$ can be decomposed uniquely into a product of the form [‡6]

$$g^c = k \cdot \hat{h} , \qquad (18)$$

with

$$k \in G^c/\hat{H} , \quad \hat{h} \in \hat{H} . \qquad (19)$$

Thus, for any element $g \in G$ one can write

$$g \cdot k = k'(g, k) \cdot \hat{h}(g, k) , \qquad (20)$$

where $k'(g, k)$ and $\hat{h}(g, k)$ are functions of the indicated variables which are determined by the structure of the group. Then we define the transformation law of goldstone superfields ξ as [‡7]

$$\xi \xrightarrow{g \in G} \xi' = g \xi \hat{h}^{-1}(g, \xi) \qquad (21)$$

One can easily see that the above formula defines correctly the operation of G on ξ. It must be remarked here that the complex conjugate of chiral superfields ξ does not appear in the r.h.s. of eq. (21), i.e., ξ and ξ^* do not mix under the above transformation [‡8]. The transformed field ξ' remains to be a chiral superfield.

Finally we explain how to construct G-invariant lagrangians. One observes from eq. (21) that a "constant" group element g can be easily cancelled by taking the bilinear form $\xi^\dagger \xi$

$$\xi^\dagger \xi \xrightarrow{g \in G} \xi'^\dagger \xi' = (\hat{h}^{-1})^\dagger \xi^\dagger g^\dagger g \xi \hat{h}^{-1} = (\hat{h}^{-1})^\dagger \xi^\dagger \xi \hat{h}^{-1} . \qquad (22)$$

However $\hat{h}^{-1}(g, \xi)$ and $(\hat{h}^{-1})^\dagger (g, \xi^*)$ cannot be easily cancelled because $(\hat{h}^{-1})^\dagger$ depends on the chiral superfield ξ^* which is *different* from ξ in $\hat{h}^{-1}$. The following three are the only possible recipes of lagrangians:

(i) A-type: This works only when there exists such an analytic representation ρ_0 of G^c that the restric-

‡6 Strictly speaking, eq. (18) holds in some neighborhood of the identity of G^c.

‡7 To be precise, this equation should be written by introducing some unitary representation ρ of the group G and its "analytic representation of G^c as

$$\rho(\xi') = \rho(g) \rho(\xi) \rho(\hat{h}^{-1}) .$$

‡8 Of course, "complex conjugate (*)" and "hermite conjugate (†)" are defined only after taking some representation of the group.

tion to the subgroup $\hat{H}$ contains a trivial representation. Let e_a's be bases of the representation space where $\rho_0(\hat{H})$ is trivial,

$$\rho_0(\hat{H})e_a = e_a \, . \quad (23)$$

Observing that

$$\rho_0(\xi')e_a = \rho_0(g)\rho_0(\xi)\rho_0(\hat{h}^{-1})e_a = \rho_0(g)\rho_0(\xi)e_a \, , \quad (24)$$

we obtain a candidate for the lagrangian,

$$[f(e_a^\dagger \rho_0(\xi^\dagger \xi)e_b)]_D \, , \quad (25)$$

where f is an arbitrary function and $[\;]_D$ means taking the D-component of a superfield.

(ii) B-type: This is a generalization of Zumino's one [2] and works in every case. We introduce projections $\eta_i (i = 1, 2, ...)$ which satisfy the following conditions

$$\text{(a)} \quad \eta_i^2 = \eta_i \quad (\text{no sum for } i) \, , \quad (26\text{a})$$

$$\text{(b)} \quad \eta_i \text{ are diagonalizable} \, , \quad (26\text{b})$$

$$\text{(c)} \quad \rho(\hat{H})\eta_i = \eta_i \rho(\hat{H})\eta_i \quad (\text{no sum for } i) \, , \quad (26\text{c})$$

where ρ is a representation of G^c. Then we get the following candidate for the lagrangian

$$\sum_i c_i [\ln \det_{\eta_i} [\rho(\xi^\dagger \xi)]]_D \, , \quad (27)$$

where the c_i's are constant parameters and $\det_{\eta_i}$ denotes a determinant defined in the subspace which is projected out by η_i from the whole of the representation space. Note that

$$\ln \det_\eta [\rho(\xi^\dagger \xi)] \xrightarrow{g \in G} \ln \det_\eta [\rho(\xi'^\dagger \xi')]$$
$$= \ln \det [\eta \rho(\xi'^\dagger \xi')\eta]$$
$$= \ln \det [\eta \rho((h^{-1})^\dagger)\rho(\xi^\dagger \xi)\rho(\hat{h}^{-1})\eta]$$
$$= \ln \det [\xi \rho((\hat{h}^{-1})^\dagger)\eta \rho(\xi^\dagger \xi)\eta \rho(\hat{h}^{-1})\eta]$$
$$= \ln \det [\eta \rho((\hat{h}^{-1})^\dagger)\eta \eta \rho(\xi^\dagger \xi)\eta \eta \rho(\hat{h}^{-1})\eta]$$
$$= \ln \det_\eta \rho(\xi^\dagger \xi) + \ln \det_\eta \rho(\hat{h}^{-1})$$
$$+ \ln \det_\eta \rho((\hat{h}^\dagger)^{-1}) \, , \quad (28)$$

where the suffix i is suppressed.

(iii) C-type: A field dependent group element $\hat{h}^{-1}$ in eq. (21) can be cancelled in the following term which is an extension of the projection operators in usual non-supersymmetric theories:

$$P_i \equiv (\xi \eta_i)[\xi^\dagger \xi]_{\eta_i}^{-1}(\eta_i \xi^\dagger) \quad (\text{no sum for } i) \, , \quad (29)$$

where $[\;]_{\eta_i}^{-1}$ means the inverse defined in the subspace projected out by η_i. It is transformed under the group G as

$$P \xrightarrow{g \in G} P' = (\xi' \eta)[\xi'^\dagger \xi']_\eta^{-1}(\eta \xi'^\dagger)$$
$$= g\xi \hat{h}^{-1}\eta[(\hat{h}^{-1})^\dagger \xi^\dagger \xi \hat{h}^{-1}]_\eta^{-1}\eta(\hat{h}^{-1})^\dagger \xi^\dagger g^\dagger$$
$$= g(\xi\eta)(\eta \hat{h}^{-1}\eta)[(\hat{h}^{-1})^\dagger \xi^\dagger \xi \hat{h}^{-1}]_\eta^{-1}(\eta(\hat{h}^{-1})^\dagger \eta)(\eta \xi^\dagger)g^\dagger$$
$$= g(\xi\eta)[\hat{h}^{-1}]_\eta [(\hat{h}^{-1})^\dagger \xi^\dagger \xi \hat{h}^{-1}]_\eta^{-1}[(\hat{h}^\dagger)^{-1}]_\eta (\eta \xi^\dagger)g^\dagger$$
$$= gPg^\dagger \, , \quad (30)$$

where the suffix i is suppressed. Hence the following is a candidate for the lagrangian,

$$[f(\mathrm{Tr}(P_i P_j), \mathrm{Tr}(P_i P_j P_k), ...)]_D \, , \quad (31)$$

where f is an arbitrary function of multi-variables. Note that

$$P_i^2 = P_i \, , \quad \mathrm{Tr}\, P_i = (\text{const}) \, , \quad (32)$$

thus we can set $i \neq j, i \neq j \neq k$, etc. for the arguments in eq. (31).

Some comments are in order: (i) Three kinds of the invariants in eqs. (25), (27) and (31) are inequivalent to each other, in general. (ii) In the B-type all the independent lagrangians are exhausted by taking the fundamental representation as ρ in eq. (27). (iii) The invariant in eq. (27) with the use of general ρ is "equivalent" to that with the fundamental ρ_f since the difference is expressed by that in eqs. (25) and (31). (iv) In pure realization, only the B-type recipe presents supersymmetric non-linear lagrangians. In other words, the arbitrariness of the functions in eqs. (25) and (31) is originated by the introduction of quasi Goldstone bosons.

We have summarized the non-linear realization in supersymmetric theories. Some examples and details are reported in a separate paper [3]. Also we present concrete expressions of all the possible candidates for R for the case of G and H being classical groups there [3].

References

[1] S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 (1969) 2239.
[2] P. Fayet and S. Ferrara, Phys. Rep. 32 (1977) 249; B. Zumino, Phys. Lett. 87B (1979) 203.
[3] M. Bando, T. Kuramoto, T. Maskawa and S. Uehara, in preparation.
[4] T. Kugo, I. Ojima and T. Yanagida, Munich preprint MPI-PAE/PTh 74/83;
C. Lee and H.S. Sharatchandro, Munich preprint MPI-PAE/PTh 54/83;
W. Lerche, Munich preprint MPI-PAE/PTh 59/83.
[5] C. Hochshild, The structure of Lie groups (Holden-Day, 1965).