QCD AND THE STRING MODEL

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Infinitesimal variations of the path-dependent phase factor $P\exp(\mathrm{i}\int_\sigma A_\mu \mathrm{d}z_\mu)$ in gauge theory are studied. They are shown to satisfy differential equations which are equivalent to those for a quantized string if the gauge fields meet certain constraints.

A currently popular belief holds that quantum chromodynamics (QCD) is the basis for the strong interaction of the quarks. This belief is motivated by the characteristic ultraviolet and infrared properties of non-abelian gauge fields, and includes a conjecture concerning the permanent confinement of quarks and gluons. On the phenomenological side, the string model seems to give a good overall description of hadron dynamics, so most of the theories that have been proposed for the confinement mechanism aspire to realize the string as a flux of gauge fields.

In chromodynamics, the standard procedure for such an attempt is to start from the path-ordered phase factor

$$U[\sigma] = p\exp\left(\mathrm{i}\int_\sigma A_\mu \mathrm{d}z_\mu\right),$$

$$A_\mu = g\sum_{a=1}^{8} A_\mu^a \lambda^a/2,$$

which is written as a 3 × 3 unitary matrix, with its legs belonging to the color indices of the end points of a spacelike path σ; in quantum theory it may be regarded as an operator creating or destroying a flux string σ. Thus, for example, the wave function for a meson M is given by

$$\psi_{\alpha\beta}(x|\sigma|y) = \langle 0|\bar{q}_\beta(y)U[\sigma]q_\alpha(x)|M\rangle, \tag{2}$$

where σ runs from x to y; q_α stands for the quark field with Dirac index α, the flavor index having been contracted with that of U. Another useful object is the Wilson loop

$$W[\sigma] = \langle 0|\mathrm{tr}\, U[\sigma]|0\rangle, \tag{3}$$

where σ is a closed loop. As a criterion for confinement, W is an action integral usually evaluated by means of functional integration, so that the restriction to spacelike σ does not apply.

The purpose of the present note [1] is to derive differential equations for U showing its dependence on infinitesimal variations of σ. It will be seen that these equations naturally lead to the Virasoro equations for a quantized string provided that the gauge fields satisfy certain conditions. In other words, we obtain conditions under which a gauge field theory yields the string model [2].

For the sake of mathematical convenience, we will consider keyboard type variations of the path σ. This means that a locally straight segment of σ is deformed like a keyboard where a key of width w is depressed by an amount h in a normal direction. Since σ cannot then be kept always spacelike, we will for the time being treat the fields as c-numbers. We have, then, in an obvious way,

$$\frac{\delta U[\sigma]}{\delta\sigma_{\mu t}(z)} = \mathrm{i}\, U[\sigma']F_{\mu t}(z)U[\sigma''],$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu]. \tag{4}$$

Here $\delta\sigma_{\mu t}(z)$ is the infinitesimal area element (magnitude wh) due to a deformation in the (μt) plane at point z, which divides σ into σ' and σ'', t stands for the tangential and μ for a normal direction.

Let us further define

$$\phi^a(z/\sigma) = U[\sigma']\lambda^a U[\sigma''], \quad a = 1, \ldots, 8,$$

$$\phi^0(z/\sigma) = U[\sigma']\lambda^0 U[\sigma''] = \sqrt{\tfrac{2}{3}}\, U[\sigma]$$

and combine them to form a new matrix

$$\Phi(z/\sigma) = \sum_{a=0}^{8} \lambda^a \phi^a, \tag{6}$$

whose matrix indices belong to the point z on σ We can easily check that

$$D_t\phi^0(z/\sigma) = \frac{\partial}{\partial z_t}\phi^0 = 0,$$

$$D_t\phi^a(z/\sigma) = \frac{\partial}{\partial z_t}\phi^a + gf^{abc}A_t^b\phi^c = 0 \tag{7}$$

or, in a more compact notation,

$$D_t\Phi(z/\sigma) \equiv \frac{\partial\Phi}{\partial z_t} - i[A_t, \Phi] = 0. \tag{8}$$

In other words, $\Phi(z/\sigma)$ is covariantly constant along σ Next we take normal displacements $\delta\sigma_{\mu t}$ centered around z. We find by elementary means

$$D_{\mu t}\phi^0 = -ig\tfrac{1}{2} F_{\mu t}{}^a \phi^a$$

$$D_{\mu t}\phi^a = -ig\tfrac{1}{2}(\sqrt{\tfrac{2}{3}}F_{\mu t}{}^a \phi^0 + d^{abc}F_{\mu t}{}^b \phi^c),$$

or

$$D_{\mu t}\Phi = -\tfrac{1}{2}i(\Phi F_{\mu t} + F_{\mu t}\Phi) \equiv -\tfrac{1}{2}i\{\Phi, F_{\mu t}\}. \tag{9}$$

The first equation has the same content as eq. (4). These equations show that the first order variations of U are of order wh. We would like next to go to higher order variations at the same point and in the same direction μ by iterating eq. (9), but care must be taken because in general there will be terms of order wh^2, w^2h^2, etc. Let us thus parametrize a finite variation by w and h. An inspection of the structure of the functions ϕ shows that we are dealing with a relation of the form

$$d\Phi(h, w)/dh = K(h, w)\,\Phi(h, w), \tag{10}$$

for fixed w. Here K is a matrix acting on the vector Φ, and may be throught of as a Taylor series in w.

$$K(h, w) = K_0(h) + wK_1(h) + w^2K_2(h) + \ldots \tag{11}$$

We have, then,

$$w^{-1}(d/dh - K_0)\,\Phi = (K_1 + wK_2 + \ldots)\Phi \tag{12}$$

and iterating this twice,

$$w^{-2}(d/dh - K_0)^2 = [(K_1 + wK_2' + \ldots)^2$$
$$+ w^{-1}K_1' + K_2' \ldots]\,\Phi + [w^{-1}K_1' + (K_1^2 + K_2')$$
$$+ O(w)]\,\Phi. \tag{13}$$

As $w \to 0$, eq. (12) has a limit

$$\lim_{w\to 0} w^{-1}(d/dh - K_0)\,\Phi = K_1\Phi, \tag{14}$$

which clearly corresponds to eq. (9). From eq. (12) we further obtain

$$w^{-1}(d/dh - K_0)^2$$
$$= [w(K_1 + wK_2 + \ldots)^2 + K_1' + wK_2' + \ldots], \tag{15}$$

so that

$$\lim_{w\to 0} w^{-1}(d/dh - K_0)^2\Phi = K_1'\,\Phi. \tag{16}$$

If $K_1' = 0$, on the other hand, we have

$$\lim_{w\to 0} w^{-2}(d/dh - K_0)^2 = (K_1^2 + K_2')\Phi. \tag{17}$$

This would correspond to an iteration of the operation (14), but there is an extra term on the right hand side which does not follow from the latter. It is therefore necessary to know $K(h, w)$ up to terms of order $[w^2]$.

We will apply our procedure to $D_\mu D_{\mu t}\Phi$ and $D_{\mu t}D_{\mu t}\Phi$, where a summation over μ is implied. First we evaluate eq. (11) explicitly by expanding the phase factor, and find

$$D_\mu\Phi = -w\tfrac{1}{2}i\{F_{\mu t}, \phi\} - \tfrac{1}{8}w^2[D_tF_{t\mu}, \Phi] + O(w^3), \tag{18}$$

so that

$$D_\mu D_\mu \Phi = -w\tfrac{1}{2}\mathrm{i}\{D_\mu F_{\mu t}, \Phi\} - \tfrac{1}{8}w^2[D_\mu D_t F_{t\mu}, \Phi]$$
$$-\tfrac{1}{4}w^2\{F_{\mu t}, \{F_{\mu t}, \Phi\}\} + O(w^3)$$
$$= w\tfrac{1}{2}\mathrm{i}\{J_t, \Phi\} - \tfrac{1}{8}w^2[D_t J_t, \Phi]$$
$$-\tfrac{1}{4}w^2\{F_{\mu t}, \{F_{\mu t}, \Phi\}\} + O(w^3),$$

$$J_\nu \equiv -D_\mu F_{\mu\nu}. \qquad (19)$$

The second form follows because

$$D_\mu D_t F_{t\mu} = [D_\mu, D_t] F_{t\mu} + D_t D_\mu F_{t\mu}$$
$$= -\mathrm{i}[F_{\mu t}, F_{t\mu}] + D_t J_t = D_t J_t \qquad (20)$$

We have thus

$$D_\mu D_{\mu t}\Phi = \lim_{w\to 0} w^{-1} D_\mu D_\mu \Phi = -\tfrac{1}{2}\mathrm{i}\{F_{\mu t}, \Phi\}. \qquad (21)$$

If, on the other hand, the condition

$$J_t = 0, \qquad (22)$$

is satisfied, then

$$D_{\mu t} D_{\mu t}\Phi = \lim_{w\to 0} w^{-2} D_\mu D_\mu \Phi = -\tfrac{1}{4}\{F_{\mu t}, \{F_{\mu t}, \Phi\}\}. \qquad (23)$$

Whether by accident or not, this is the same result that would be obtained by naively iterating eq. (9) It also follows from taking variations at two neighboring points z and z' and letting $z' \to z$. Therefore Φ enjoys a smoothness property under the condition (22), that is, if there are no quark currents along the string.

Suppose, now, that $F_{\mu t}F_{\mu t} = C$. Then $U = \sqrt{\tfrac{3}{2}}\,\phi^0$ satisfies

$$\left(\frac{\delta}{\delta\sigma_{\mu t}} \frac{\delta}{\delta_{\sigma_{\mu t}}} + C\right) U = 0. \qquad (24)$$

We will show that eq. (24) is equivalent to the equations for a quantized string. In the string model [3], the world sheet of a string is parametrized by internal time (τ) and space (ρ) coordinates. The path σ under consideration corresponds to a cross section τ = const. The action function is taken to be $1/2\pi\alpha'$ times the area of the sheet, α' being the Regge trajectory slope. The lagrangian can be linearized if (τ, ρ) are chosen to be conformal coordinates. This amounts to

$$\frac{\partial z_\lambda}{\partial\tau}\frac{\partial z_\lambda}{\partial\rho} = 2\pi\alpha\, \Pi_\lambda \frac{\partial z_\lambda}{\partial\rho} = 0, \qquad (25a)$$

$$\left(\frac{\partial z_\lambda}{\partial\tau}\right)^2 + \left(\frac{\partial z_\lambda}{\partial\rho}\right)^2 = (2\pi\alpha')^2 \Pi_\lambda{}^2 + \left(\frac{\partial z_\lambda}{\partial\rho}\right)^2 = 0, \qquad (25b)$$

for all points on the sheet. Here Π_λ is the canonical momentum conjugate to z_λ. Eq. (24a) means that only normal displacements of points on a string are physical, whereas eq. (24b) serves as dynamical equations for them In quantum theory, these equations must be regarded as constraints on a wave function χ. When the string is approximated by discrete points $\{\rho_n\}$ with interval $\Delta\rho$, the canonical commutation relations are realized by

$$\Pi_{\lambda n} = (-\mathrm{i}/\Delta\rho)\partial/\partial z_{\lambda n}, \qquad (26)$$

and eqs. (24) may be interpreted to read

$$\Pi_{tn}\chi = 0, \qquad (27a)$$

$$\left[\frac{1}{(y'_\lambda)^2}(\Pi_\lambda)^2 + \left(\frac{1}{2\pi\alpha'}\right)^2\right]_n$$
$$= \left[-\sum_{\mu\perp t}\left(\frac{1}{\Delta z_t}\frac{\partial}{\partial y_\mu}\right)^2 + \left(\frac{1}{2\pi\alpha'}\right)^2\right]_n \chi = 0 \qquad (27b)$$

We see that eq. (27) is identical with eq. (24) if $C = -(1/2\pi\alpha')^2$. Actually, eqs. (27) are somewhat different from standard equations for a quantized string. In the latter, eqs. (27) are replaced by a set of Virasoro equations which are Fourier transforms of eqs. (25). As is well known, these equations have the difficulties of tachyons and the loss of Lorenz invariance. It is an open question whether our form has any advantages or not in this respect.

In order to deal with the wave function (2) of a meson, we must also consider variations of the end points. Suppose the quark fields obey the Dirac equation

$$(\gamma_\mu D_\mu + m)\, q = 0. \qquad (28)$$

By way of definition, we will let a variation of end point x to $x + \Delta x$ be accompanied by addition of a string between the two points. The wave function Φ then satisfies

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)_{\alpha\gamma} \Phi_{\gamma\beta} = \Phi_{\alpha\gamma}\left(\gamma_\mu \frac{\overleftarrow{\partial}}{\partial y_\mu} - m\right) = 0 \quad (29)$$

Summarizing, we have seen that a meson wave function satisfies eqs. (24) and (28), the former being equivalent to string equations, provided that there are no quark currents along $\sigma(J_t = 0)$, and

$$F_{\mu t}F_{\mu t} = -(\tfrac{1}{2}\pi\alpha')^2 1$$

that is,

$$(g^2/6)F_{\mu t}{}^a F_{\mu t}{}^a = -(\tfrac{1}{2}\pi\alpha')^2, \quad (30)$$

$$F_{\mu t}{}^a F_{\mu t}{}^b d^{abc} = 0,$$

the fields being treated as c-numbers. The same situation holds true for the Wilson loop (3).

In a q-number theory, there arises the question of operator ordering. Although the path σ is meant to be spacelike, the timelike variations of the kind considered here inevitably violates the rule, and the noncommutativity of F_{0t} and A_t must be taken into account. Thus, for example,

$$[F_{0t}^a, \phi^b] = d^{abc}\phi^c\delta^2(0). \quad (31)$$

Now, by definition, Φ is path-ordered, so eq. (4) is still valid in the operator sense. Furthermore, a closer examination shows that eq. (9) is also all right in spite of the operator reordering that has occurred in going from eq. (4) to eq. (9). All our equations may therefore be interpreted as q-number relations.

We now turn to discussions of the results obtained. The first order eqs. (1) and (9) are straightforward, and no assumptions or dynamics are involved. In going to second order variations, however, new restrictions come in. This is because higher order variations are in general shape dependent; they are not functions of the area wh only. The meaning of derivatives is such that the limits are taken in the order: $h \to 0$, then $w \to 0$. This amounts to regarding the string as the continuum limit of a discrete system.

More important problems are those concerning the interpretation of eqs. (22) and (29). Since we are actually thinking of Φ as a quantum operator or a functional average, it is not legitimate to take out a factor as a c-number. So our conditions must be taken in an approximate sense. Now, the condition $J_t = 0$ means absence, or neglect, of quark currents $\sim\bar{q}\gamma_t\lambda^a q(z)$ along σ. When this term is inserted in $\Phi[x,y]$, the latter can be broken down to a product $\sim\Phi[x,z]\,\Phi[z,y]$, which represents the breaking and joining of strings. These processes must be ignored, or else treated as a perturbation.

It is more difficult to give eq. (30) a precise meaning. A naive interpretation suggests that we are dealing with a string of electric-type flux. If the tangent to the string is in the z-direction at a point, eq. (30) reads (in the Lorentz metric)

$$(g^2/6\pi)\,(E_z^2 - B_x^2 - B_y^2) = (1/2\pi\alpha')^2. \quad (32)$$

This form is an invariant of the little group of the tangent vector.

For a meson wave function $\langle 0|\mathrm{tr}\,\Phi|M\rangle \sim \langle 0|\phi^0|M\rangle$, we might argue that the right-hand side of eq. (23) can be approximated as

$$\tfrac{1}{4}\langle 0|\mathrm{tr}(F_{\mu t}F_{\mu t}\Phi + 2F_{\mu t}\Phi F_{\mu t} + \Phi F_{\mu t}F_{\mu t})|M\rangle$$

$$= \tfrac{1}{4}\langle 0|\mathrm{tr}(2F_{\mu t}F_{\mu t}\Phi + 2\Phi F_{\mu t}F_{\mu t} + C_1^2\Phi)|M\rangle$$

$$\sim C\,\langle 0|\mathrm{tr}\,\Phi|M\rangle,$$

$$C = \tfrac{1}{6}(\langle 0|\mathrm{tr}\,F_{\mu t}F_{\mu t}|0\rangle + \langle M|\mathrm{tr}\,F_{\mu t}F_{\mu t}|M\rangle) + \tfrac{1}{4}C_1^2,$$

$$C_1 = (16/3)\delta^2(0), \quad (33)$$

where we have made use of eq. (31). This must now be equated with $-(1/2\pi\alpha')^2$. The infinite constant makes an interpretation difficult, but it might be dropped, like all other infinities. If we identify $\langle M|F_{\mu t}F_{\mu t}|M\rangle$ with the flux tube contribution and set $\langle 0|F_{\mu t}F_{\mu t}|0\rangle = 0$, we obtain eq. (32) except for a factor 1/2 on the left-hand side. Under this assumption, let us estimate the area of the flux tube, setting $|E_z| = E$, $B_{x,y} = 0$ in the rest frame. We have three relations

$$(g^2/12\pi)\,E^2 = (1/2\pi\alpha')^2, \quad (34a)$$

$$(1/2)E^2 a = 1/2\pi\alpha', \quad (34b)$$

$$Ea = g/\sqrt{3}. \quad (34c)$$

Eqs. (32b) and (32c) are, respectively, the energy of the flux tube per unit length, and the total flux due to a quark source. Perhaps fortuitously, the three equations are consistent with each other, and give

$$a = g^2 \pi \alpha'/3 \approx (0.5 \text{ fermi})^2, \tag{35}$$

for the typical values $\alpha' = 1 \text{ GeV}^{-2}$, $g^2/4\pi = 0.5$.

Another simplifying assumption regarding eq. (33) might be $\langle 0|F_{\mu t}F_{\mu t}|0\rangle = \langle M|F_{\mu t}F_{\mu t}|M\rangle$, leading back to eq. (32). This would correspond to the hypothesis that the vacuum is populated with fluctuating gluon fields which are nothing but nascent closed strings. It is also conceivable that the fluctuations are due to other topological configurations like instantons, merons [4], and monopoles [5]. These latter, however, are all of the magnetic type, and thus seem to conflict, at least superficially, with the negative sign required for C. In this connection, we note that the vacuum expectation value $\langle 0|F_{\mu\nu}F_{\mu\nu}|0\rangle$ found by Shifman et al. [6] is also positive, but has a magnitude comparable to ours.

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