Journal of Mathematical Economics 16 (1987) 169–189. North-Holland

ON THE DEMAND GENERATED BY A SMOOTH AND CONCAVIFIABLE PREFERENCE ORDERING

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Submitted December 1984, accepted April 1987

It is shown that if a consumer's preference ordering is strictly convex and is representable by means of a concave, twice continuously differentiable utility function, then the partial derivative of a demanded commodity with respect to its price is bounded from above in a neighborhood of a price vector at which the demand fails to be differentiable. In the case of two commodities, if the demand does not possess finite derivatives with respect to prices at a certain point, then the partial 'derivative' of a commodity with respect to its price is equal to minus infinity. The same result holds for n commodities under 'almost every' choice of coordinates in the commodity space. If preferences are weakly convex but the same representation assumption holds, demand may not be single-valued but own-price difference quotients are still bounded from above.

1. Introduction and statement of results

One of the less obvious phenomena in the theory of consumers' behavior (some even call it a paradox) is the occurrence of Giffen goods, i.e., commodities of which more is demanded as their prices go up. No known simple assumption concerning the preferences (of the consumer) rules out this behavior. It was discovered recently [Jordan (1982)] that if the consumer's preference ordering $\succsim$ is representable by means of a *concave* utility function (but $\succsim$ is not necessarily strictly convex), then the demand correspondence, while possibly not lower semi-continuous, nonetheless cannot have 'jumps' in the wrong direction: if p and p^* are sufficiently close price systems, x and x^* are in the sets demanded by p and p^*, respectively, and if for each commodity j, $x_j \geqq x_j^*$ if $p_j > p_j^*$ and $x_j \leqq x_j^*$ if $p_j < p_j^*$, then x_j is close to x_j^* at least for some commodity j (for which $p_j \neq p_j^*$). Of course, this condition is trivially satisfied if demand is single-valued and continuous. In the present paper we show that, under somewhat stronger assumptions, an upper bound can be placed on the own-price difference quotient, so that infinite Giffen effects are not possible. In particular, this shows that the representation of preferences by a concave utility function also has implications for continuous single-valued demand functions.

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*We would like to thank T. Kim for noticing some errors in an earlier version of this paper.

Let us assume that $\succsim$ is a strictly convex, complete, continuous and transitive preference relation defined on an open, convex subset X of R^n – the commodity space. Let p^0 be a price vector such that the maximal element (with respect to $\succsim$) in the budget set is contained in the interior of X. Then there exists a certain open ball B containing p^0 such that the demand function $f: B \rightarrow R^n$ is well defined and continuous.

It is well known [Debreu (1972), Katzner (1968)] that $f(p)$ is not necessarily differentiable, even if $\succsim$ is representable by means of a twice continuously differentiable utility function u (defined on X). It was shown in Debreu (1972) that f is differentiable at p if, and only if, the indifference hypersurface $\{y : y \sim f(p)\}$ has positive Gaussian curvature at $x = f(p)$. It is well known [Fenchel (1956), Kannai (1981)] that if the Gaussian curvature of the indifference surface does not vanish at any point then the utility function u can be chosen to be *concave* (and not just quasi-concave), at least in compact subsets of X. Thus, the difference between concavifiable and non-concavifiable preference orderings is manifest only near points with vanishing Gaussian curvature of the indifference surfaces; this difference should be reflected in the behavior of the demand functions near points where differentiability is lost.

Since $\succsim$ is strictly convex, the Gaussian curvature does not vanish on a dense open subset of every indifference hypersurface. [This follows, see Shaforth (1981), without invoking the rather deep characterization of surfaces with vanishing curvature as developable surfaces [Hicks (1965)].] The continuity of the inverse demand function implies that prices p^0, at which the demand is not differentiable, are limit points of the set of prices p where f is continously differentiable. This approximation is the basis for the results in section 2, below. Theorem 2.1 states that if the demand function fails to be differentiable at a positive price vector p^0, the lim sup of own-price derivatives at nearby prices is bounded above. In the two-commodity case, the lim sup is $-\infty$. More generally, this result is obtained whenever the vector representing the 'commodity' is not perpendicular to a principal direction of the indifference surface (through the demand at p^0) for which the principal curvature vanishes. Example 2.5 indicates that the directional hypothesis is necessary, and that the two-commodity result does not otherwise extend to the n-commodity case.

In section 3 we allow the preference relation to be weakly convex, and also weaken the smoothness requirement slightly to the assumption that the utility function u is differentiable and its derivative, Du, is Lipschitz-continuous. In this case, the above mentioned approximation is no longer valid, but Theorem 3.1 states that the Giffen effect remains bounded. An example in section 3 shows that the Lipschitz continuity of Du is essential to the result.

Theorem 3.1 generalizes the n-commodity assertion in Theorem 2.1, using a very different method of proof. The proof of Theorem 2.1 is in the spirit of classical demand theory, comparing the asymptotic income and substitution effects [eq. (2.21), below] as the determinant of the usual bordered Hessian approaches zero. The proof of Theorem 3.1 is more in the geometrical spirit of convex analysis.

2. The strictly convex, C^2 case

Let $\succsim$ be a strictly convex, complete and transitive preference relation defined on an open convex subset X of R^n, and let $\succsim$ be representable by a concave, twice continously differentiable utility function $u: X \to R$. We also assume that u has no maximum on X (i.e., that $\succsim$ is not satiated in X).

Let $\omega \in R^n$ and $I \in R$ be given, and define the (possibly empty-valued) demand correspondence $f: R^n \to X$ by $f(p) = \{x^0 \in X : x^0$ maximizes $\succsim$ in the set $\{x: p(x-\omega) \leqq I\}\}$. The budget constraint $p(x-\omega) \leqq I$ includes as special cases the budget constraints used in partial $(\omega = 0)$ and general $(I = 0)$ equilibrium theory. Let $B = \{p \in R^n : f(p) \neq \phi\}$. It follows that B is an open subset of $R^n \backslash \{0\}$, and that f is a continuous (single-valued) function on B. We shall restrict our attention in the present section to prices in B.

Theorem 2.1. If the demand function is not differentiable at a strictly positive price vector $p^0 \in B$, then

$$\limsup_{p \to p^0} \frac{\partial f_i}{\partial p_i}(p) < +\infty, \qquad 1 \leqq i \leqq n, \tag{2.1}$$

where in (2.1) the limit is taken over those p at which f is differentiable.

If $n=2$, then (2.1) can be strengthened to

$$\lim_{p \to p^0} \frac{\partial f_i}{\partial p_i}(p) = -\infty, \qquad i = 1, 2. \tag{2.2}$$

Using finite differences, we have the following one-sided Lipschitz conditions:

There exists a $K < \infty$ and a neighborhood of p^0, such that if $(p_1, \dots, p_i, \dots, p_n)$ and $(p_1, \dots, p_i + h, \dots, p_n)$ are both in this neighborhood, then

$$f_i(p_1, \dots, p_i + h, \dots, p_n) - f_i(p_1, \dots, p_i, \dots, p_n) \leqq Kh, \qquad 1 \leqq i \leqq n. \tag{2.3}$$

If $n=2$, then (2.3) can be strengthened to: For every $K > 0$ there exists a neighborhood of p^0 such that if (p_1,p_2) and $(p_1,p_2)+he^i$ are both in this neighborhood, where e^i denotes the ith unit vector, then

$$f_i(p+he^i)-f_i(p)\leqq -Kh, \qquad i=1,2. \tag{2.4}$$

At the end of this section we will illustrate the distinction between (2.1) and (2.2) with an example (Example 2.5) of a smooth, concave utility function with positive partial derivatives in an open subset of R^3 with $\partial f_i/\partial p_i$ oscillating near a price p^0 where the demand is not differentiable [thus (2.2) and (2.4) do not hold for $n>2$]. Nevertheless, (2.2) holds 'almost always' for general n, in the following sense.

Define a *package* a of commodities to be a vector $(a_1,\dots,a_n)$ such that one may buy commodities in the proportions $(x_1,\dots,x_n)=(\lambda a_1,\dots,\lambda a_n)$. (The term originates from 'option packages' offered by car dealers; note that the components a_i are not restricted to be non-negative.) The price of a unit of the package will be $p_a=p_1a_1+\dots+p_na_n$. (Thus, the usual commodities are packages for which $a=e^i$ – *the* ith unit vector.) Let us assume, to simplify the notation, that a is a unit vector, and let $a^1,\dots,a^{n-1}$ be an orthonormal system in R^n such that $(a,a^i)=0$ for $1\leqq i\leqq n-1$. Then the inner products pa^i denote the prices of the packages a^i, $1\leqq i\leqq n-1$, and we can study how the demand f_a for the package a varies when we keep pa^i fixed for $1\leqq i\leqq n-1$ and let only p_a vary.

Theorem 2.2. If the demand function is not differentiable at a price vector $p^0\in B$ *and the vector* a *is not perpendicular to any principal direction [of the indifference surface through* $x(p^0)$*] for which the principal curvature vanishes, then*

$$\lim_{p\to p^0}\frac{\partial f_a}{\partial p_a}(p)=-\infty, \tag{2.5}$$

where the limit is taken over those p *for which* $x=f(p)$ *is differentiable.*

Remark. The formula (2.2) follows from Theorem 2.2 as a special case, as the positivity of the partial derivatives of u at $x(p^0)$ (which follows from the strict positivity of p^0) rules out the possibility that e^i is perpendicular to the indifference curve.

Proof of Theorem 2.1. We start by pointing out that at the demanded point $f(p)$, the budget constraint reads

$$\sum_{i=1}^{n} p_i(f_i-\omega_i)=I, \tag{2.6}$$

and the gradient of u is proportional to the price vector. Denoting the proportionality factor by $\lambda = \lambda(p_1, \ldots, p_n)$, we can write

$$u_i(f(p)) = \lambda p_i, \qquad 1 \leqq i \leqq n. \tag{2.7}$$

(Here and in the sequel we use subscripts to denote partial derivatives; thus, $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$, etc.) By assumption, the gradient of u never vanishes on X. Hence $\lambda \neq 0$, and the fact that $f(p)$ is maximal in $\{x : p(x-\omega) \leqq I\}$ implies that $\lambda > 0$. The strict positivity of p^0 implies that $u_i(x) > 0$, $1 \leqq i \leqq n$ for all x near $x(p^0)$.

Consider now the two-commodity case. In order to investigate the local behavior of the demand in a neighborhood of $p^0 = (p_1^0, p_2^0)$, introduce new coordinates y_1, y_2 near $x_1^0 = f_1(p^0)$, $x_2^0 = f_2(p^0)$, by

$$x_1 = x_1^0 - p_2^0 y_1 + p_1^0 y_2, \qquad x_2 = x_2^0 + p_1^0 y_1 + p_2^0 y_2. \tag{2.8}$$

[Thus (x_1^0, x_2^0) becomes the new origin, y_1 is the coordinate in the direction tangent to the indifference curve through (x_1^0, x_2^0), and y_2 is the normal coordinate.] Then the Hessian matrix of the second derivatives of u with respect to the variables y_1, y_2 is given by

$$H = \begin{bmatrix} \dfrac{\partial^2 u}{\partial y_1^2}, & \dfrac{\partial^2 u}{\partial y_1 \partial y_2} \\[2ex] \dfrac{\partial^2 u}{\partial y_2 \partial y_1}, & \dfrac{\partial^2 u}{\partial y_2^2} \end{bmatrix}$$
$$= \begin{bmatrix} (p_2^0)^2 u_{11} - 2p_1^0 p_2^0 u_{12} + (p_1^0)^2 u_{22}, & -p_1^0 p_2^0 u_{11} + ((p_1^0)^2 - (p_2^0)^2) u_{12} + p_1^0 p_2^0 u_{22} \\ -p_1^0 p_2^0 u_{11} + ((p_1^0)^2 - (p_2^0)^2) u_{12} + p_1^0 p_2^0 u_{22}, & (p_1^0)^2 u_{11} + 2p_1^0 p_2^0 u_{12} + (p_2^0)^2 u_{22} \end{bmatrix}. \tag{2.9}$$

It is well-known [Debreu (1972), Katzner (1968)] that the functions f_1 are differentiable at p_1^0, p_2^0 if and only if $\partial^2 u/\partial y_1^2 \neq 0$ at $y_1 = y_2 = 0$, as the value of $\partial^2 u/\partial y_1^2(0,0)$ is proportional to the curvature [at (x_1^0, x_2^0)] of the indifference curve through (x_1^0, x_2^0), and is equal by (2.9) to the usual bordered Hessian.

The famous Fenchel conditions for concavifiability [Fenchel (1956), Katzner (1968)] imply that if $\partial^2 u/\partial y_1^2 = 0$ then $\partial^2 u/\partial y_1 \partial y_2 = 0$. This can easily be seen, in our notation, by letting H operate as a quadratic form on an arbitrary vector (ξ_1, ξ_2). Then the concavity of u implies that

$$\frac{\partial^2 u}{\partial y_1^2}\xi_1^2 + 2\xi_1\xi_2\frac{\partial^2 u}{\partial y_1 \partial y_2} + \xi_2^2\frac{\partial^2 u}{\partial y_2^2} \leq 0. \tag{2.10}$$

But if $\partial^2 u/\partial y_1^2 = 0$, then if $\partial^2 u/\partial y_1 \partial y_2 \neq 0$ we can make the left-hand side of (2.10) positive by choosing $\xi_2 = 1$, $\operatorname{sign}(\xi_1) = \operatorname{sign}(\partial^2 u/\partial y_1 \partial y_2)$ and $|\xi_1|$ large, a contradiction. [Using the determinant of H we could get a more quantitative result, see the proof of (2.1).] The two linear equations $\partial^2 u/\partial y_1^2 = 0$, $\partial^2 u/\partial y_1 \partial y_2 = 0$ in u_{11}^0, u_{12}^0, u_{22}^0 [here we set $u_{ij}(x_1^0, x_2^0) = u_{ij}^0$], can be written as

$$\begin{aligned} &(p_2^0)^2 u_{11}^0 - 2p_1^0 p_2^0 u_{12}^0 + (p_1^0)^2 u_{22}^0 = 0, \\ &-p_1^0 p_2^0 u_{11}^0 + ((p_1^0)^2 - (p_2^0)^2) u_{12}^0 + p_1^0 p_2^0 u_{22}^0 = 0. \end{aligned} \tag{2.11}$$

Multiplying the first equation in (2.11) by p_2^0, the second equation by p_1^0 and subtracting, we get the equation

$$p_2^0((p_2^0)^2 + (p_1^0)^2) u_{11}^0 - p_1^0((p_1^0)^2 + (p_2^0)^2) u_{12}^0 = 0.$$

Hence

$$p_2^0 u_{11}^0 - p_1^0 u_{12}^0 = 0 \tag{2.12}$$

holds whenever the demand functions f_i are not differentiable.

In the two-commodity case considered here, the budget constraint (2.6) reads

$$p_1(f_1 - \omega_1) + p_2(f_2 - \omega_2) = I, \tag{2.13}$$

and the fact that the gradient (u_1, u_2) of u is proportional, at the demanded point, to the price vector (p_1, p_2) implies that

$$p_2 u_1(f_1, f_2) - p_1 u_2(f_1, f_2) = 0. \tag{2.14}$$

Differentiating (2.13) and (2.14) with respect to p_2 (using the chain rule) and denoting differentiation with respect to p_2 by $\cdot$, we get (after rearranging terms):

$$\begin{aligned} (p_2 u_{11} - p_1 u_{12})\dot{f}_1 + (p_2 u_{12} - p_1 u_{22})\dot{f}_2 &= -u_1, \\ p_1 \dot{f}_1 + \qquad\qquad p_2 \dot{f}_2 &= -f_2. \end{aligned} \tag{2.15}$$

Using Cramer's rule, we get from (2.15) that

$$\dot{f}_2 = \frac{-f_2(p_2u_{11}-p_1u_{12})+p_1u_1}{p_2^2u_{11}-2p_1p_2u_{12}+p_1^2u_{22}}. \tag{2.16}$$

If (p_1, p_2) tends to (p_1^0, p_2^0), then the denominator, which is equal to $\partial^2u/\partial y_1^2$ and is non-positive by concavity of u, tends to zero. By (2.12), the first term in the numerator also tends to zero, whereas p_1u_1 tends to the *strictly positive* value of $p_1^0u_1(x_1^0, x_2^0)$. Hence $\dot{f}_2$ tends to $-\infty$ and (2.2) is proved (the proof for $\partial f_1/\partial p_1$ is similar). □

Turning our attention to proving (2.1) for general n, we note first that (2.7) implies that the ratio λ between the gradient of u and the price vector can be written in the form $\lambda = u_i/p_i$ for a certain i such that $p_i \neq 0$ (such an index i exists by non-satiation). Hence λ is a continuously differentiable function of the prices, at least if f is. Setting $-\lambda = \mu$, we can rewrite (2.7) as

$$u_i(f(p)) + \mu p_i = 0, \qquad 1 \leqq i \leqq n. \tag{2.17}$$

We will consider $\partial f_n/\partial p_n$ (the other $\partial f_i/\partial p_i$ can be treated analogously). Differentiating each of the equations in (2.17) with respect to p_n and denoting $\partial/\partial p_n$ by $\cdot$, we get (using the chain rule and $\partial p_i/\partial p_n = \delta_{i,n}$) that

$$\sum_{j=1}^{n} u_{ij}\dot{f}_j + p_i\dot{\mu} = 0, \qquad 1 \leqq i \leqq n-1, \tag{2.18}$$

and

$$\sum_{j=1}^{n} u_{nj}\dot{f}_j + p_n\dot{\mu} = -\mu. \tag{2.19}$$

Differentiating the budget constraint (2.6), we obtain the additional equation

$$\sum_{j=1}^{n} p_j\dot{f}_j = \omega_n - f_n. \tag{2.20}$$

In solving (2.18)–(2.20) for $\dot{f}_n$, we introduce the following notation. Let

$$B_k = \begin{bmatrix} u_{11}, \dots, u_{1k}, & p_1 \\ \vdots & \vdots \\ u_{k1}, \dots, u_{kk}, & p_k \\ p_1, \dots, p_k, & 0 \end{bmatrix}, \qquad H_k = \begin{bmatrix} u_{11}, \dots, u_{1k} \\ \vdots \\ u_{k1}, \dots, u_{kk} \end{bmatrix}.$$

and for every $m \times m$ matrix A let $|A|$ denote the determinant of A and let

$A(i_1,\dots,i_p; j_1,\dots,j_p)$ denote the $(m-p)\times(m-p)$ minor of A obtained by striking out the rows $i_1,i_2,\dots,i_p$ and the columns $j_1,\dots,j_p$, $1\leqq i_1<i_2<\dots<i_p\leqq m$, $1\leqq j_1<j_2<\dots<j_p\leqq m$. In this notation, the usual implicit differentiation of demand [i.e., solving the linear system (2.18), (2.19), (2.20)] yields

$$\dot{f}_n=\frac{-\mu|B_{n-1}|-(\omega_n-f_n)|B_n(n+1;n)|}{|B_n|}$$

$$=-\mu\frac{|B_{n-1}|}{|B_n|}-(\omega_n-f_n)\frac{|B_n(n+1;n)|}{|B_n|}. \tag{2.21}$$

As is well-known [Debreu (1972), Katzner (1968)] and easily seen from (2.21), $\partial f_n/\partial p_n$ exists and is finite if $|B_n|\neq 0$. Note that by quasi-concavity alone we have [Bellman (1960, p. 78) where the corresponding inequalities are derived for bordered positive definite matrices]

$$(-1)^k|B_k|\geqq 0, \qquad 1\leqq k\leqq n. \tag{2.22}$$

Hence the first term in the right-hand side of (2.21) cannot tend to $+\infty$. To handle the second term, we invoke the determinant identity

$$|B_n(n+1;n)|^2=|H_n|\,|B_{n-1}|-|H_{n-1}|\,|B_n|. \tag{2.23}$$

The identity (2.23) is a special case of formula (33) in Gantmacher (1959, p. 21); that formula expresses the minors of the inverse matrix in terms of the minors of the given matrix. Thus, if $C=D^{-1}$ are $(m+1)\times(m+1)$ matrices, then formula (33) in Gantmacher (1959, p. 21) reads in our notation (for the special case of the 2×2 minors obtained by striking out the first $m-1$ rows and columns)

$$|C(1,2,\dots,m-1;\ 1,2,\dots,m-1)|=|D(m,m+1;\ m,m+1)|/|D|.$$

Applying this formula for $D=B_n$ and using the fact that $c_{ij}=(-1)^{i+j}|D(i;j)|/|D|$, we obtain (2.23), at least if B_n is invertible. But then the polynomial identity (2.23) (in the elements $u_{11},\dots,u_{nn},p_1,\dots,p_n$) must hold with no restrictions. A transparent proof of the formula (33) in Gantmacher (1959, p. 21) is presented in Shafroth (1981).

We also need the following well-known estimate.

Proposition 2.3. If M is a bound for the entries in H_n (i.e., $|u_{ij}|\leqq M$ for

$1\leqq i, j\leqq n$) and N is a bound for $\|p\|^{-2}=(p_1^2+\cdots+p_k^2)^{-1}$, $k=1,\dots,n$, p near p^0, then

$$(-1)^k|H_k|\leqq(-1)^k MN|B_k|, \qquad 1\leqq k\leqq n. \tag{2.24}$$

[Note that each of the sides of (2.24) is non-negative.]

Proof of Proposition 2.3. For fixed k, let $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_k\geqq 0$ denote the eigenvalues (counted with multiplicities) of the positive semi-definite matrix $-H_k$, and let $\lambda_1^*\geqq\lambda_2^*\geqq\cdots\geqq\lambda_{k-1}^*$ denote the eigenvalues of $-H_k$ restricted to the subspace (of R^k) orthogonal to $(p_1,\dots,p_k)$. Then the maximum–minimum property of eigenvalues [Courant and Hilbert (1953), Gantmacher (1959); see Courant and Hilbert (p. 33) for a simple geometric interpretation] implies that

$$\lambda_1\geqq\lambda_1^*\geqq\lambda_2\geqq\cdots\geqq\lambda_{k-1}^*\geqq\lambda_k \tag{2.25}$$

[compare also Fenchel (1956, p. 501)]. But $|-H_k|=\lambda_1\dots\lambda_k$, and it is well-known [Fenchel (1956, p. 500)] that the determinant of the bordered matrix $|-B_k|$ is equal to $-\lambda_1^*\dots\lambda_{k-1}^*\|p\|^2$ [compare also Debreu (1972); one way of obtaining this is by diagonalizing the $k\times k$ submatrix H_k of B_k]. Moreover, $\lambda_1\leqq M$ (every eigenvalue is a weighted average of the matrix elements). Hence

$$\begin{aligned}(-1)^k|H_k|&=|-H_k|=\lambda_1\lambda_2\dots\lambda_k\leqq\lambda_1\lambda_1^*\dots\lambda_{k-1}^*=-\lambda_1|-B_k|\,\|p\|^{-2}\\&=(-1)^k|B_k|\cdot\lambda_1\|p\|^{-2}\leqq(-1)^k MN|B_k|\end{aligned}$$

and (2.24) is proved.

Estimating $(-1)^n|H_n|$ and $(-1)^{n-1}|H_{n-1}|$ in (2.23) by means of (2.24), we obtain the bound

$$|B_n(n+1;n)|^2\leqq -2MN|B_{n-1}|\,|B_n|. \tag{2.26}$$

[As a matter of fact we could get (2.26) (comparing signs) without the factor 2.]

We see from (2.21) that if $|B_{n-1}|$ remains bounded away from zero (as happens, e.g., if $n=2$) then $\dot{f}_n$ must tend to $-\infty$ as p approaches a non-differentiability point. If, however, both $|B_{n-1}|$ and $|B_n(n+1;n)|$ tend to zero, and if $|B_n(n-1;n)/|B_n|$ is unbounded from above (so that it tends to $+\infty$ on a certain sequence of prices) then by (2.26) $(-B_{n-1}|/|B_n|)^{1/2}$ tends to $+\infty$ in the same speed, so that the second term in the right-hand side is dominated by the first, and (2.1) follows.

A very careful analysis involving the set of zeroes of $|B_n|$ might yield (2.3).

We prefer to use the following seemingly simpler approach. For $\varepsilon>0$ sufficiently small, the utility function $u_\varepsilon(x)$ given by

$$u_\varepsilon(x)=u(x)-\varepsilon\sum_{j=1}^{n}x_j^2 \tag{2.27}$$

is strictly concave, and $u_{\varepsilon,i}(x)\neq 0$ for x in a neighborhood of $x^0=f(p^0)$. Then $f_{\varepsilon,i}(p)\to f_i(p)$ as $\varepsilon\to 0$ [where $f_\varepsilon(p)$ is the demand of the consumer who is maximizing u_ε]. With $H_{\varepsilon,k}$ and $B_{\varepsilon,k}$ denoting the Hessians and the bordered Hessians of u_ε, respectively, we get from (2.21) the formula

$$\dot{f}_{\varepsilon,n}=-\mu_\varepsilon\frac{|B_{\varepsilon,n-1}|}{|B_{\varepsilon,n}|}-(\omega_n-f_{\varepsilon,n})\frac{|B_{\varepsilon,n}(n+1;n)|}{|B_{\varepsilon,n}|}. \tag{2.21ε}$$

Letting M and N now denote bounds on $|u_{\varepsilon,i,j}|$ and $(\sum_{i=1}^{k}p_i^2)^{-1}$, respectively, in a sufficiently small (but fixed) neighborhood of p^0, we can clearly choose M and N independently of ε (for $\varepsilon>0$ sufficiently small). By (2.26) $\|B_{\varepsilon,n}(n+1;n)\|\leqq c\|B_{\varepsilon,n-1}\|^{1/2}\|B_{\varepsilon,n}\|^{1/2}$ (when $\|A\|$ denotes the absolute value of the determinant of the matrix A, and c is independent of ε if ε is sufficiently small). For $\varepsilon>0$, $|B_{\varepsilon,n}|\neq 0$ and $f_{\varepsilon,n}$ is (classically) continuously differentiable. By (2.7) $-\mu_\varepsilon$ is bounded away from zero in a neighborhood of x^0, say $-\mu_\varepsilon>l>0$, independently of ε. Similarly, $|\omega_n-f_{\varepsilon,n}|\leqq e$ where e is a positive constant. Hence

$$\dot{f}_{\varepsilon,n}\leqq l\frac{|B_{\varepsilon,n-1}|}{|B_{\varepsilon,n}|}+e\frac{\|B_{\varepsilon,n}(n+1,n)\|}{\|B_{\varepsilon,n}\|}. \tag{2.28}$$

Setting $\|B_{\varepsilon,n}(n+1,n)\|/\|B_{\varepsilon,n}\|=t$ and noting that $|B_{\varepsilon,n-1}|$ and $|B_{\varepsilon,n}|$ have different signs, we deduce from (2.28) that

$$\dot{f}_{\varepsilon,n}\leqq-(l/c^2)t^2+et\leqq e^2c^2/(4l). \tag{2.29}$$

Hence, for $\varepsilon>0$, we conclude [by integrating (2.29)] that

$$f_{\varepsilon,n}(p_1,\dots,p_{n-1},p_n+h)-f_{\varepsilon,n}(p_1,\dots,p_n)\leqq Kh$$

if $(p_1,\dots,p_n)$ and $(p_1,\dots,p_{n-1},p_n+h)$ are in a (fixed) neighborhood of p^0, with a finite constant K independent of ε. Letting ε tend to zero, we obtain (2.3).

If $n=2$ then $|B_{\varepsilon,n-1}|=-p_1^2$. Hence (2.26) yields in this case that $\|B_{\varepsilon,n}(n+1;n)\|\leqq c\|B_{\varepsilon,n}\|^{1/2}$, and (2.28) implies

$$\dot{f}_{\varepsilon,2}\leqq-\frac{lp_1^2}{|B_{\varepsilon,n}|}+\frac{ec}{\|B_{\varepsilon,n}\|^{1/2}}. \tag{2.30}$$

Given any $K>0$, we can find a sufficiently small neighborhood V of p^0 and $\varepsilon_0>0$ such that the right-hand side of (2.30) is less than $-K$ for $p\in V$, and $0<\varepsilon<\varepsilon_0$. Hence

$$f_{\varepsilon,2}(p_1,p_2+h)-f_{\varepsilon,2}(p_1,p_2)\leqq -Kh$$

for (p_1,p_2), (p_1,p_2+h) in V, and $0<\varepsilon<\varepsilon_0$. Taking the limit as $\varepsilon\to 0$, we obtain (2.4).

Proof of Theorem 2.2. Note first that

$$p=p_a a+\sum_{i=1}^{n-1}(pa^i)a^i \tag{2.31}$$

so that by definition of f_a,

$$\frac{\partial f_a}{\partial p_a}=\sum_{k=1}^{n}a_k\frac{\partial f_k}{\partial p_a}=\sum_{k=1}^{n}a_k\left(\sum_{j=1}^{n}\frac{\partial p_j}{\partial p_a}\frac{\partial f_k}{\partial p_j}\right)=\sum_{j,k=1}^{n}a_j a_k\frac{\partial f_k}{\partial p_j}. \tag{2.32}$$

We also have to determine the effects of orthogonal changes of coordinates on the derivatives of demand functions.

Thus, let

$$x_i=\sum_{j=1}^{n}b_{ij}y_j, \tag{2.33}$$

where $(b_{ij})_{i,j=1}^{m}$ is a real orthogonal matrix. Let p,q denote price vectors in x,y coordinates, respectively.

It follows from (2.7) and (2.33) that

$$\frac{\partial u}{\partial y_j}=\sum_{i=1}^{n}b_{ij}\frac{\partial u}{\partial x_i}=\sum_{i=1}^{n}b_{ij}\lambda p_i=\lambda q_j$$

so that the Lagrange multiplier is invariant, and denoting the demand function in the q, y coordinates by $g(q)$, we obtain the formula

$$\frac{\partial f_k}{\partial p_j}=\sum_{i=1}^{n}b_{ki}\frac{\partial y_i}{\partial p_j}=\sum_{i=1}^{n}b_{ki}\left(\sum_{m=1}^{n}b_{jm}\frac{\partial g_i}{\partial q_m}\right)$$

$$=\sum_{i,m=1}^{n}b_{ki}b_{jm}\frac{\partial g_i}{\partial q_m}. \tag{2.34}$$

Substituting (2.34) in (2.32), we see that

$$\frac{\partial f_a}{\partial p_a}=\sum_{i,m=1}^{n}\left(\sum_{k=1}^{n} b_{ki}a_k\right)\left(\sum_{j=1}^{n} b_{jm}a_j\right)\frac{\partial g_i}{\partial q_m}. \tag{2.35}$$

To use (2.35) we need to compute each $\partial g_i/\partial q_m$.

Let p be a price vector in B such that the Gaussian curvature of the indifference surface through $x(p)$ does not vanish [at $x(p)$]. Choose the orthogonal system $(b^1,\dots,b^n)$ so that b^n is perpendicular to the indifference surface at $x(p)$ (thus b^n is proportional to Du and to p) and $b^1,\dots,b^{n-1}$ are the principal directions of the indifference surface [compare Kannai (1981)]. Let $b^i=(b_{1i},\dots,b_{ni})$, and assume without loss of generality that $x(p)$ is the origin. Set $\pi_j=\sum_{i=1}^{n} b_{ij}\omega_i$, and $\mu=-\lambda$. In this coordinate system, the usual implicit differentiation formula takes the form (recall that λ is invariant and that $\|p\|=\|q\|$)

$$\begin{bmatrix}\lambda_1 & & 0 & \alpha_1 & 0\\ & \ddots & & \vdots & \vdots\\ 0 & & \lambda_{n-1} & \alpha_{n-1} & 0\\ \alpha_1 & \cdots & \alpha_{n-1} & \lambda_n & \|p\|^2\\ 0 & \cdots & 0 & \|p\|^2 & 0\end{bmatrix}\begin{bmatrix}\dfrac{\partial g_1}{\partial q_1} & \cdots & \dfrac{\partial g_1}{\partial q_n}\\ & & \\ \dfrac{\partial g_n}{\partial q_1} & \cdots & \dfrac{\partial g_n}{\partial q_n}\\ \dfrac{\partial \mu}{\partial q_1} & \cdots & \dfrac{\partial \mu}{\partial q_n}\end{bmatrix}$$

$$=\begin{bmatrix}-\mu & & 0\\ & \ddots & -\mu\\ \pi_1-g_1 & \cdots & \pi_n-g_n\end{bmatrix}, \tag{2.36}$$

where $\lambda_1,\dots,\lambda_{n-1}\neq 0$ and by (quasi) concavity $\lambda_i<0$, $1\leqq i\leqq n-1$. (Thus λ_i here is the same as λ_i^* in the proof of Proposition 2.3.) Denote the value of the determinant

$$\begin{vmatrix}\lambda_1 & & 0 & \alpha_1\\ & \ddots & & \vdots\\ 0 & & \lambda_{n-1} & \alpha_{n-1}\\ \alpha_1 & \cdots & \alpha_{n-1} & \lambda_n\end{vmatrix}$$

by Δ. By an easy induction (or otherwise) one computes that

$$\Delta=\lambda_1\dots\lambda_n-\sum_{i=1}^{n}\alpha_i^2\prod_{j\neq i,n}\lambda_j. \tag{2.37}$$

The inverse of the first matrix (on the left) in (2.36) can be given explicitly as

$$\begin{bmatrix} \dfrac{1}{\lambda_1} & & 0 & 0 & \dfrac{-\alpha_1}{\lambda_1\|p\|^2} \\ 0 & & \dfrac{1}{\lambda_{n-1}} & 0 & \dfrac{-\alpha_{n-1}}{\lambda_{n-1}\|p\|^2} \\ 0 & \ldots & 0 & 0 & \dfrac{1}{\|p\|^2} \\ \dfrac{-\alpha_1}{\lambda_1\|p\|^2} & \cdots & \dfrac{-\alpha_{n-1}}{\lambda_{n-1}\|p\|^2} & \dfrac{1}{\|p\|^2} & \dfrac{-\Delta}{\lambda_1\ldots\lambda_{n-1}\|p\|^4} \end{bmatrix}.$$

Applying this inverse to the right-hand side of (2.36), we get the formulas

$$\begin{aligned} &\frac{\partial g_i}{\partial q_m}=-\frac{\mu\delta_{i,m}}{\lambda_i}-\frac{\alpha_i(\pi_m-g_m)}{\lambda_i\|p\|^2}, && 1\leqq i,m\leqq n-1, \\ &\frac{\partial g_i}{\partial q_n}=\frac{\alpha_i(g_n-\pi_n)}{\lambda_i\|p\|^2}, && 1\leqq i\leqq n-1, \\ &\frac{\partial g_n}{\partial q_m}=\frac{\pi_m-g_m}{\|p\|^2}, && 1\leqq m\leqq n. \end{aligned} \tag{2.38}$$

Setting

$$\sum_{k=1}^{n} b_{ki}a_k=c_i \tag{2.39}$$

and substituting in (2.35), we arrive at

$$\frac{\partial f_a}{\partial p_a}=-\sum_{i=1}^{n-1}\frac{c_i^2\mu}{\lambda_i}-\sum_{i=1}^{n-1}\sum_{m=1}^{n}\frac{c_ic_m\alpha_i(\pi_m-g_m)}{\lambda_i\|p\|^2}+\sum_{m=1}^{n}\frac{c_nc_m(\pi_m-g_m)}{\|p\|^2}. \tag{2.40}$$

Now let $\{p^r\}_{r=1}$ be any sequence of points at which the demand f is differentiable such that $p^r\to p^0$. It suffices to show that from every such sequence one can choose a subsequence satisfying (2.5), so we can assume that the corresponding unit vectors $b^1(p^r),\ldots,b^{n-1}(p^r)$ also converge [the sequence $b^n(p^r)$ converges by definition]. Then the coefficients c_i [equal by (2.39) to the inner product of a with $b^i(p^r)$] also converge to c_i^0, $1\leqq i\leqq n$. Let us now assume that $\lambda_i(x(p^0))=0$ for $1\leqq i\leqq k$, $\lambda_i(x(p^0))<0$ for $i>k$. By assumption $c_i^0\neq 0$ for $1\leqq i\leqq k$. Moreover, by the Fenchel conditions for concavity

[Fenchel (1956), Kannai (1981) and see the proof of (2.2)] $\alpha_i(x(p^0))=0$ for $1 \leqq i \leqq k$. Also, $\mu=-\lambda<0$, and $\lambda_i(p^r)<0$. Hence it follows from (2.40) that $(\partial f_a/\partial p_a)(p^r) \to -\infty$.

Remark. The assumption in the n-commodity case of Theorem 2.1 that p^0 is strictly positive can be dropped by basing the proof of that Theorem on eq. (2.40).

Example 2.4. In order to show that (2.2) does not hold even in the two dimensional case if the partial derivatives of u are not assumed to be positive [and that (2.5) does not hold if a is perpendicular to a principal flat direction] consider the strictly quasi-concave function $u=x_2-x_1^4$ near the origin. Choose $\omega=0$. Then the demand is given by

$$f_1(p_1,p_2)=-4^{-1/3}(p_1/p_2)^{1/3}, \qquad f_2(p_1,p_2)=I/p_2+4^{-1/3}(p_1/p_2)^{4/3}.$$

As p converges to $p^0=(0,1)$, $\partial f_1/\partial p_1$ tends to $-\infty$, but $\partial f_2/\partial p_2$ is continuous at p^0.

If $n>2$ then $\partial f_i/\partial p_i$ can have oscillatory discontinuities (even if $u_j>0$ for all j), as we see in

Example 2.5. Consider the quasi-linear utility

$$u=-x_1^4/12-x_1x_2^2/2-x_1^2x_2^2/2-x_2^2+x_1+x_2+x_3, \tag{2.41}$$

and set $\omega=0$, $I=0$. Differentiating (2.41), we see that near the origin, $u_1>0$ for $1 \leqq i \leqq 3$, $u_{ii} \leqq 0$ for $1 \leqq i \leqq 2$ ($u_{33} \equiv 0$), and

$$u_{11}u_{22}-u_{12}^2=2x_1^2+x_2^2+0(x_1^3)+0(x_1x_2^2) \tag{2.42}$$

near the origin. Hence the ordering represented by u is strictly convex near the origin, and the demand is differentiable except at prices p^0 for which $f_1(p^0)=f_2(p^0)=0$ [e.g., $p^0=(1,1,1)$]. Solving for $\partial x_2/\partial p_2(=\partial f_2/\partial p_2)$ we see that

$$\frac{\partial x_2}{\partial p_2}=\mu\frac{x_1^2+x_2^2}{2x_1^2+x_2^2+0(x_1^3)+0(x_1x_2^2)}. \tag{2.43}$$

Hence $(\partial x_2/\partial p_2)(p^r) \to \mu/2$ if $p^r \to p^0$ is such that $x_2(p^r) \equiv 0$ [e.g., $p^r=(1-1/3r^3,1,1)$], whereas $(\partial x_2/\partial p_2)(p^r) \to \mu$ if $x_1(p^r) \equiv 0$ [e.g., $p^r=(1-1/2r^2, 1-3/r, 1)$].

Without concavifiability one cannot hope even for (2.1), as shown by our next example:

Example 2.6. Let

$$u=\frac{x_2-\alpha-x_1^4}{1-x_1}. \tag{2.44}$$

[The ordering induced by (2.44) is a very slight modification of the strictly convex, non-concavifiable ordering exhibited by Fenchel (1956).] Differentiating (2.44), we get

$$\begin{aligned} &u_1=\frac{u}{1-x_1}-\frac{4x_1^3}{1-x_1}, \qquad u_2=\frac{1}{1-x_1}, \\ &u_{11}=\frac{2u}{(1-x_1)^2}-\frac{12x_1^2}{1-x_1}-\frac{8x_1^3}{(1-x_1)^2}, \qquad u_{12}=\frac{1}{(1-x_1)^2}, \qquad u_{22}=0. \end{aligned} \tag{2.45}$$

Choosing $x_1^0=0$, $\alpha>0$, $x_2^0>\alpha$, we see from (2.45) that $u_1>0$, $u_2>0$, near x^0, and that

$$u_2^2u_{11}-2u_1u_2u_{12}+u_1^2u_{22}=-\frac{12x_1^2}{(1-x_1)^3} \tag{2.46}$$

[so that the ordering induced by u is indeed strictly convex, and the demand is not differentiable at prices p^0 for which $f(p^0)=x^0$]. Using (2.7), (2.45) and (2.46) in (2.16), we compute that

$$\frac{\partial f_2}{\partial p_2}=\lambda\left[\frac{-x_2u}{(1-x_1)^3}+\frac{4x_1^3x_2}{(1-x_1)^3}+\frac{12x_1^2x_2}{(1-x_1)^2}+\frac{u^2}{(1-x_1)^2}-\frac{8x_1^3u}{(1-x_1)^2}+\frac{16x_1^6}{(1-x_1)^2}\right]$$

$$\Bigg/\left[\frac{-12x_1^2}{(1-x_1)^3}\right]=\left[\frac{\lambda u}{(1-x_1)^2}\left(u-\frac{x_2}{1-x_1}\right)+\underset{x_1\to 0}{0(x_1^2)}\right]\Bigg/\left[\frac{-12x_1^2}{(1-x_1)^3}\right].$$

Hence

$$\frac{\partial f_2}{\partial p_2}=\lambda u\alpha/12x_1^2+(\lambda ux_1^4-0(x_1^2))/12x_1^2$$

so that

$$\lim_{p\to p^0}\frac{\partial f_2}{\partial p_2}=+\infty.$$

3. The weakly convex, C^1-Lipschitzian case

In this section we drop the requirement of strict convexity, so that demand may be set-valued. If $f(p)$ is a set and x is a bundle in the relative interior of $f(p)$, we cannot approximate the behavior of demand near x by the derivatives of demand at prices near p, even when such derivatives exist. Nonetheless, under a smoothness assumption slightly weaker than the one used above, we can obtain an upper bound on the Giffen effect even in the set-valued case. It was shown in Jordan (1982) that if preferences can be represented by a concave utility function, then the demand for a commodity cannot 'jump' in the same direction as a change in its own price. We will show here that if the utility function is also C^1, and its derivative is uniformly Lipschitzian on compact subsets of X, then the geometrical argument used in that paper can be strengthened to show that demand cannot have an infinite slope in the wrong direction.

To motivate the result, consider the two-commodity case and two price vectors $p^0=(p_1^0,p_2^0)$ and $p=(p_1^0+\delta,p_2^0)$, which differ only by an increase of δ in the price of commodity one. Let $x^0\in f(p^0)$ and $x\in f(p)$, with $x_1>x_1^0$, so that commodity one is, for these two price vectors, a Giffen good. This situation is depicted in fig. 1 which is drawn under the additional assumption that $x_1^0-\omega_1>0$. At the beginning of the proof of Theorem 3.1 below, we will derive as a simple consequence of concavity that $(x_1-x_1^0)/\delta\leqq[(\lambda-\lambda^0)/\delta](x_1^0-\omega_1)/\lambda^0$, where λ^0 and λ are the Lagrange multipliers: $Du(x^0)=\lambda^0p^0$ and $Du(x)=\lambda p$. The difference quotient $(\lambda-\lambda^0)/\delta$ can be bounded above as follows. Map x onto the budget line $p^0(y-\omega)=I$ and map x^0 onto the budget line $p(y-\omega)=I$, as depicted in fig. 1. Denote the resulting commodity bundles x^a and x^b respectively. By revealed preference $u(x^0)\geqq u(x^a)$ and $u(x)\geqq u(x^b)$, so $u(x^0)-u(x^b)\geqq u(x^a)-u(x)$. Hence by concavity, $Du(x^b)(x^0-x^b)\geqq Du(x^a)(x^a-x)$. The vector (x^0-x^b) is simply p^0 multiplied by a scalar which is $0(\delta)$, so by the Lipschitz condition, the distance $\|Du(x^b)-\lambda^0p^0\|$ is also $0(\delta)$. Similarly (x^a-x) is p multiplied by a scalar which is $0(\delta)$, and $\|Du(x^a)-\lambda p\|$ is $0(\delta)$. This approximation can be applied to the above inequality to obtain an upper bound on the difference quotient $(\lambda-\lambda^0)/\delta$, and thus on the Giffen effect $(x_1-x_1^0)/\delta$. In fact, since the Lipschitz condition is uniform on compact subsets of X, this upper bound is also.

Theorem 3.1. Suppose that $\gtrsim$ is non-satiated on X and is representable by a utility function $u: X\to R$ which is concave and has a derivative which is uniformly Lipschitzian on compact subsets of X. That is, for each compact set $K\subset X$ there is a constant c such that

$$\|Du(x')-Du(x)\|\leqq c\|x'-x\| \tag{*}$$

for all x', $x\in K$. Then for each compact set $K\subset X$ there exists a constant k such that for each p^0, $p\in R^n$ with $p_1^0\neq p_1$ and $p_j^0=p_j$ for all $j>1$, and each $x^0\in f(p^0)\cap K$, $x\in f(p)\cap K$,

$$(x_1-x_1^0)/(p_1-p_1^0)<k/(\|p^0\|+|I|).$$

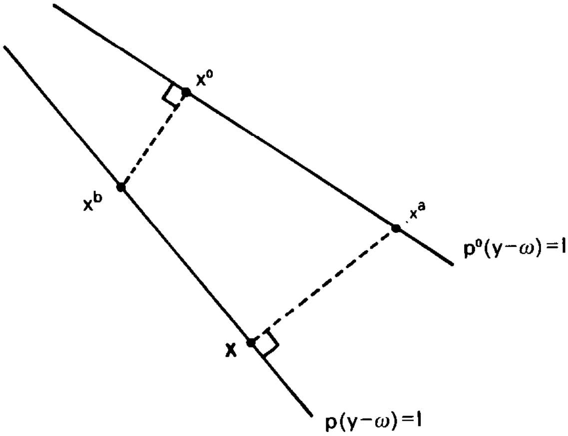

Fig. 1

Remark. The term $\|p^0\|+|I|$ arises from the zero degree homogeneity of demand. Given $(x_1-x_1^0)$, the price change $(p_1-p_1^0)$ can be made arbitrarily small by rescaling p^0 and I.

Proof. Let K be a compact subset of X, let $p^0\in R^n$, and let $x^0\in K\cap f(p^0)$. Since u is non-satiated and $f(p^0)\neq\phi$, $\|p^0\|\neq 0$. Let $\delta>0$ and define $p=(p_1^0+\delta, p_2^0,\dots,p_n^0)$. Let $x\in f(p)\cap K$, and let $\varepsilon=x_1-x_1^0$. We need to construct $k>0$, independent of p^0, p, x^0, and x, such that

$$\varepsilon/\delta<k/(\|p^0\|+|I|). \tag{*}$$

If $\varepsilon\leqq 0$, (*) will hold with $k=1$, so we will assume that $\varepsilon>0$.

Since x^0 and x are interior maxima, there exist $\lambda^0>0$ and $\lambda>0$ with $Du(x^0)=\lambda^0p^0$ and $Du(x)=\lambda p$. Since u is concave,

$$u(x)-u(x^0)\leqq\lambda^0p^0(x-x^0),\quad\text{and}$$

$$u(x^0)-u(x)\leqq\lambda p(x^0-x),\qquad\text{so}$$

$$\lambda^0p^0(x^0-x)\leqq\lambda p(x^0-x).$$

Since $\lambda^0>0$, this inequality can be written

$$(p^0-p)(x^0-x)\leqq p(x^0-x)[(\lambda-\lambda^0)/\lambda^0],\quad\text{or}$$

$$\delta\varepsilon\leqq p(x^0-x)[(\lambda-\lambda^0)/\lambda^0].$$

Since u is concave and non-satiated, the budget constraint holds with equality, so

$$p(x^0-x)=px^0-p\omega-I=(p-p^0)x^0-p^0\omega-p\omega$$

$$=(p-p^0)(x^0-\omega)=\delta(x_1^0-\omega_1). \tag{3.1}$$

Hence

$$\delta\varepsilon \leqq \delta(x_1^0-\omega_1)[(\lambda-\lambda^0)/\lambda^0], \quad \text{so}$$

$$\varepsilon/\delta \leqq (x_1^0-\omega_1)[(\lambda-\lambda^0)/\lambda^0\delta].$$

The term $(x_1^0-\omega_1)$ is clearly bounded on K, so it remains to study the term $(\lambda-\lambda^0)/\lambda^0\delta$.

We will temporarily restrict attention to values of δ satisfying a certain upper bound. Of course, the extension to large values of δ will be direct. Let $c_1=\max\{|y_1-\omega_1|:\ y\in K\}$, and choose $0<\gamma<\operatorname{dist}(K,X^c)$, where X^c denotes $R^n\backslash X$. Let $0<c_2<1$. Then if $\delta<c_2\|p^0\|$, $p^0p=\|p^0\|^2+\delta p_1^0\geqq\|p^0\|^2-\delta|p_1^0|\geqq\|p^0\|^2-\delta\|p^0\|>(1-c_2)\|p^0\|^2$. Therefore $(\delta c_1/p^0p)\|p^0\|<\delta c_1/(1-c_2)\|p^0\|<c_1c_2/(1-c_2)$. Hence, if c_2 is small enough so that $c_2/(1-c_2)<\gamma/c_1$, and $\delta<c_2\|p^0\|$, then $(\delta c_1/p^0p)\|p^0\|<\gamma$. Also, if $\delta<c_2\|p^0\|$, $\|p\|^2=\|p^0\|^2+2\delta p_1^0+\delta^2\leqq\|p^0\|^2+2\delta\|p^0\|+\delta^2<\|p^0\|^2(1+2c_2+c_2^2)=\|p^0\|^2(1+c_2)^2$. Therefore $\|p\|<\|p^0\|(1+c_2)$, so $(\delta c_1/p^0p)\|p\|<c_1c_2(1+c_2)/(1-c_2)$. Hence, if c_2 is small enough so that $c_2(1+c_2)/(1-c_2)<\gamma/c_1$, and $\delta<c_2\|p^0\|$, then $(\delta c_1/p^0p)\|p\|<\gamma$. Hence, we can fix $0<c_2<1$, independent of p^0, so that if $\delta<c_2\|p^0\|$, then $(\delta c_1/p^0p)\|p^0\|<\gamma$ and $(\delta c_1/p^0p)\|p\|<\gamma$.

Assume, temporarily, that $\delta<c_2\|p^0\|$. Define $x^a=x+[\delta(x_1-\omega_1)/p^0p]p$ and $x^b=x^0-[\delta(x_1^0-\omega_1)/p^0p]p^0$ (see fig. 1). Since $\delta<c_2\|p^0\|$, x^a, $x^b\in X$. Also, $p^0x^a=p^0x+\delta(x_1-\omega_1)=p^0x+(p-p^0)(x-\omega)=p^0\omega+I$, so by revealed preference, $u(x^0)\geqq u(x^a)$. Similarly, $px^b=p\omega+I$ and $u(x)\geqq u(x^b)$. Therefore

$$Du(x^a)(x^a-x)\leqq u(x^a)-u(x)\leqq u(x^0)-u(x^b)\leqq Du(x^b)(x^0-x^b).$$

Thus $[\delta(x_1-\omega_1)/p^0p]Du(x^a)p\leqq[\delta(x_1^0-\omega_1)/p^0p]Du(x^b)p^0$, so

$$(x_1-\omega_1)Du(x^a)p\leqq(x_1^0-\omega_1)Du(x^b)p^0, \quad \text{or}$$

$$[\varepsilon+(x_1^0-\omega_1)]Du(x^a)p\leqq(x_1^0-\omega_1)Du(x^b)p^0. \tag{3.2}$$

If $x_1^0-\omega_1=0$, then (3.1) implies that (*) holds with $k=1$, so we will assume that $x_1^0-\omega_1>0$. The proof in the opposite case is symmetric. By the uniform Lipschitz condition on Du over the compact set $\{y\in R^n:\|y-x\|\leqq\gamma$ for some $x\in K\}$, there is some $c_3>0$ such that

$$\|Du(x^a)-\lambda p\|<c_3\|x-x^a\| \quad \text{and} \quad \|Du(x^b)-\lambda^0p^0\|<c_3\|x^0-x^b\|.$$

That is,

$$\|Du(x^a)-\lambda p\|<c_3\delta[\varepsilon+(x_1^0-\omega_1)]\|p\|/p^0p,$$

$$\|Du(x^b)-\lambda^0p^0\|<c_3\delta(x_1^0-\omega_1)\|p^0\|/p^0p.$$

Together with (3.2), these inequalities imply that

$$[\varepsilon+(x_1^0-\omega_1)][\lambda-c_3\delta[\varepsilon+(x_1^0-\omega_1)]/p^0p]\|p\|^2$$
$$<(x_1^0-\omega_1)[\lambda^0+c_3\delta(x_1^0-\omega_1)/p^0p]\|p^0\|^2.$$

Since $\varepsilon>0$, this implies

$$[\lambda-c_3\delta[\varepsilon+(x_1^0-\omega_1)]/p^0p]\|p\|^2<[\lambda^0+c_3\delta(x_1^0-\omega_1)/p^0p]\|p^0\|^2,$$

so

$$\lambda\|p\|^2\leqq\lambda^0\|p^0\|^2+(c_3\delta/p^0p)\{(x_1^0-\omega_1)\|p^0\|^2+[\varepsilon+(x_1^0-\omega_1)]\|p\|^2\}.$$

Since $\delta<c_2\|p^0\|$, we have $\|p\|^2<(1+c_2)^2\|p^0\|^2$ and $p^0p>(1-c_2)\|p^0\|^2$, so

$$\lambda\|p\|^2<\lambda^0\|p^0\|^2+c_3\delta[\varepsilon+2(x_1^0-\omega_1)](1+c_2)^2/(1-c_2).$$

Let $c_5>\max\{|y_1-y_1^0|+2|y_1^0-\omega_1|:y,y^0\in K\}$.

Since $\|p^0\|^2\leqq\|p\|^2+2\delta\|p\|+\delta^2$,

$$\lambda\|p\|^2<\lambda^0[\|p\|^2+2\delta\|p\|+\delta^2]+\delta c_3c_5(1+c_2)^2/(1-c_2),\quad\text{so}$$

$$(\lambda-\lambda^0)<\lambda^0[2\delta/\|p\|+\delta^2/\|p\|^2]+\delta c_3c_5(1+c_2)^2/(1-c_2)\|p\|^2,$$

$$(\lambda-\lambda^0)/\lambda^0\delta<2/\|p\|+\delta/\|p\|^2+c_3c_5(1+c_2)^2/(1-c_2)\lambda^0\|p^2\|.$$

Since $\delta<c_2\|p^0\|$, $\|p\|>(1-c_2)\|p^0\|$, so

$$(\lambda-\lambda^0)/\lambda^0\delta<\{2+c_2/(1-c_2)+c_3c_5(1+c_2)^2/(1-c_2)^2\lambda^0\|p^0\|\}/(1-c_2)\|p^0\|.$$

To bound the term $1/\lambda^0\|p^0\|$, let $0<c_6<\min\{\|Du(y)\|:y\in K\}$. Then $\|Du(x^0)\|=\lambda^0\|p^0\|>c_6$, so

$$(\lambda-\lambda^0)/\lambda^0\delta<\{2/(1-c_2)+c_2/(1-c_2)^2$$
$$+c_3c_5(1+c_2)^2/(1-c_2)^3c_6\}/\|p^0\|. \tag{3.3}$$

Let $c_7 = c_1\{\cdot\}$. Then (3.3) and (3.1) imply

$$(\varepsilon/\delta) < c_7/\|p^0\|, \tag{3.4}$$

under the restriction that $\delta < c_2\|p^0\|$.

If $\delta \geqq c_2\|p^0\|$, then since $\varepsilon < c_5$, $\varepsilon/\delta < c_5/c_2\|p^0\|$. Combining this inequality with (3.4) yields

$$\varepsilon/\delta < \max\{c_7, c_5/c_2\}/\|p^0\|. \tag{3.5}$$

To incorporate $|I|$ in the denominator, note that there is some $c_8 > 0$ such that if $q \in R^n$ and $r > c_8\|q\|$, then $\{y \in K: q(y-\omega) = r\} = \phi$. Therefore

$$(\|p^0\| + |I|)^{-1} \geqq ((1+c_8)\|p^0\|)^{-1}, \quad \text{so by (3.5)}$$

$$\varepsilon/\delta < k/(\|p^0\| + |I|),$$

where $k = (1+c_8)\max\{c_7, c_5/c_2\}$. Since the constants $c_1, \ldots, c_8$, which compose k, depend only on K, this completes the proof. □

The Lipschitz condition on Du is essential to the result. It is possible to construct a two-commodity concave C^1 utility function which violates the Lipschitz condition and the conclusion of the Theorem. Fig. 2 illustrates the preference relation and demand correspondence of such an example. The endowment of both commodities is zero, and the price of the second commodity, y, is normalized to unity, so the budget constraint is: $px + y \leqq 0$. For $p < 0$, the demand is $f(p) = (1 - \sqrt{-p}, p(\sqrt{-p} - 1))$, and at $p = 0$, the demand is the set $\{(x, y): x \leqq 1, y = 0\}$. Thus the Giffen effect is unbounded at the price vector $(0, 1)$. (The example can be rotated and translated so that prices, demands, and marginal utilities are positive.) The detailed construction of this example, which is rather lengthy, is contained in an earlier version of this paper [Hurwicz et al. (1986)].

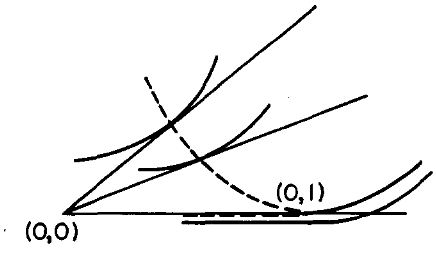

Fig. 2

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