HAMILTON–JACOBI FORMALISM FOR STRINGS ☆

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It is shown that a Hamilton–Jacobi-type formalism can be set up to deal with the classical dynamics of relativistic strings and other one-dimensional extended systems. A special feature is that the formalism involves two evolution parameters which are treated on an equal footing. The corresponding Hamilton–Jacobi functions turn out to be vector potentials or Clebsch potentials, and in this sense we find a link between the string model and gauge field theory.

The relativistic string is an interesting example of extended objects which permit a simple geometrical description. Its relevance, of course, derives from the success of the string model of hadrons as a phenomenological theory, as well as the close physical and mathematical link that seems to exist between the string model and quantum chromodynamics (QCD). It has recently been pointed out [1] that the path ordered phase factor, which is a natural stringlike construct in QCD, can be shown to satisfy, under certain restricted conditions, the equations for a quantized string.

In the present article we will try to reverse the process. We will reanalyze the classical string mechanics by developing a Hamilton–Jacobi formalism appropriate to systems having two evolution parameters. The ensuing results can be interpreted in terms of an abelian gauge field. This program goes smoothly up to the semi-classical level, but logical uncertainties appear when one tries to develop a full quantum theory or to extend it to non-abelian cases.

Our starting point is the Schild form of string lagrangian [2] (in the pseudo-euclidean metric)

$$L = \tfrac{1}{4}\{y_\mu, y_\nu\}\{y_\mu, y_\nu\} ,\qquad \{y_\mu, y_\nu\} \equiv \partial(y_\mu, y_\nu)/\partial(\sigma, \tau) , \tag{1}$$

where $y_\mu(\sigma, \tau)$ denotes a point lying on the two-dimensional surface swept out by a string. This form is the square of the usual lagrangian representing the surface area element, and hence does not have a purely geometrical meaning. As has been shown by Schild, however, it is nevertheless equivalent to the latter as a consequence of the equation of motion

$$\{y_\mu \{y_\mu, y_\nu\}\} = 0 , \tag{2}$$

which implies L = const. The equivalence is secured by choosing this constant to have a fixed value, $-C^2 = -1/2(1/2\pi\alpha')^2$, where α' is the Regge slope parameter. Recently, Eguchi [3] has shown how the Schild lagrangian may be used for quantization of the string, and thereby provide an alternative to the usual Virasoro equations. Our aim is somewhat different, being directed at a generalization of the hamiltonian dynamics.

The Hamilton–Jacobi formalism is based on the differential one-form relation

$$\mathrm{d}S = \sum_i p_i\, \mathrm{d}q_i - H\, \mathrm{d}t ,\qquad H = H(p_i, q_i) ,\quad S = S(q_i, t) , \tag{3}$$

from which all the rest follows. We replace this with a two-form relation

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$$\mathrm{d}S_1 \wedge \mathrm{d}T_1 + \mathrm{d}S_2 \wedge \mathrm{d}T_2 = \sum_{i>j}^{1,N} p_{ij}\, \mathrm{d}q_i \wedge \mathrm{d}q_j - H\, \mathrm{d}\sigma \wedge \mathrm{d}\tau , \tag{4}$$

$$S_m = S_m(q_i, \sigma, \tau) , \quad T_m = T_m(q_i, \sigma, \tau) , \quad m = 1, 2 ,$$

$$H = H(p_{ij}, q_k) .$$

There are N coordinates q_i, $N(N-1)/2$ momenta p_{ij}, and two evolution parameters σ, τ which are treated on an equal footing. On the left-hand side, an exact two-form is written in terms of pairs of one-forms, or "Clebsch potentials" (see below) S_m, T_m. Obviously we have

$$p_{ij} = \sum_m \partial(S_m, T_m)/\partial(q_i, q_j) ,$$

$$-H = -H(p_{ij}, q_k) = \sum_m \partial(S_m, T_m)/\partial(\sigma, \tau) ,$$

$$0 = \sum_m \partial(S_m, T_m)/\partial(\sigma, q_i) , \tag{5}$$

$$0 = \sum_m \partial(S_m, T_m)/\partial(q_i, \tau) .$$

When a functional relation $H = H(p_{ij}, q_k)$ is given, these represent Hamilton–Jacobi-type equations for the S_m and T_m. The last two equations are constraints, and because of them we have taken two pairs. For example, the ansatz $S_1(\sigma, \tau)$, $T_1(\sigma, \tau)$, $S_2(q_i)$, $T_2(q_i)$ would do the job.

We now would like to set up the corresponding Hamilton-type equations of motion. For this purpose we will regard all the variables q_i, p_{ij} as independent. Physically, the momenta represent planes tangent to an evolving world sheet, so clearly we have introduced too many degrees of freedom. We will have to make sure that these degrees of freedom are properly suppressed as a result of the equations of motion (i.e., on the "mass shell"), so the equations become integrable. But integrability is guaranteed if the tangential planes are generated by a pair of global functions as in eq. (5). The Hamilton equations should then be derived by noting that the exterior derivatives of eq. (4) are zero:

$$0 = \sum_{i>j} \mathrm{d}p_{ij} \wedge \mathrm{d}q_i \wedge \mathrm{d}q_j - \left(\sum_{i>j} \frac{\partial H}{\partial p_{ij}}\, \mathrm{d}p_{ij} + \sum_i \frac{\partial H}{\partial q_i} \mathrm{d}q_i\right) \wedge \mathrm{d}\sigma \wedge \mathrm{d}\tau . \tag{6}$$

Equating the coefficients of $\mathrm{d}p_{ij}$ and $\mathrm{d}q_i$, respectively, we find

$$\{q_i, q_j\} = \partial H/\partial p_{ij} , \quad \sum_j \{p_{ij}, q_j\} = -\partial H/\partial q_i . \tag{7}$$

Substituting them back into eq. (4), we also obtain the "on-shell" relation

$$\sum \mathrm{d}S_m \wedge \mathrm{d}T_m = \left(\sum_{i>j} p_{ij} \frac{\partial H}{\partial p_{ij}} - H\right) \mathrm{d}\sigma \wedge \mathrm{d}\tau \equiv L\, \mathrm{d}\sigma \wedge \mathrm{d}\tau , \tag{8}$$

which defines the lagrangian L from the hamiltonian H. Furthermore, it follows from eq. (7) that

$$\frac{\partial H}{\partial \tau} = \sum_{i>j} \frac{\partial p_{ij}}{\partial \tau} \{q_i, q_j\} - \sum_{i,j} \frac{\partial q_i}{\partial \tau} \{p_{ij}, q_j\} = 0 ,$$

$$\frac{\partial H}{\partial \sigma} = \sum_{i>j} \frac{\partial p_{ij}}{\partial \sigma} \{q_i, q_j\} - \sum_{i,j} \frac{\partial q_i}{\partial \sigma} \{p_{ij}, q_j\} = 0 , \tag{9}$$

so the hamiltonian is a constant of motion, being independent of the evolution parameters. Eq. (7) is the set of Hamilton-type equations that we were looking for. To specialize to the Schild equations, we choose H to be

$$H = \frac{1}{2} \sum_{\mu>\nu} p_{\mu\nu}^2 . \tag{10}$$

Eq. (7) then reduces to eq. (2) (with the identification $q = y$), and we have $H = L = -C^2$.

Let us now go back to the Hamilton–Jacobi system (5). It is a straightforward matter to check that a set of functions S_m, T_m satisfying eq. (5) do actually generate a family of surfaces that evolve according to eq. (7). Suppose, then, that a solution $y_\mu(\sigma, \tau)$ spans in the six-dimensional space $(y_\mu, \sigma, \tau) \equiv y_\alpha$, $\alpha = 1, \ldots, 6$, a two-surface domain D bounded by a closed curve Γ. According to eq. (8), the action is

$$I_D = \int_D L\, \mathrm{d}\sigma\, \mathrm{d}\tau = \sum \int_D \mathrm{d}S_m \wedge \mathrm{d}T_m = \int_\Gamma A_\alpha\, \mathrm{d}y_\alpha , \tag{11}$$

$$A_\alpha \equiv \sum S_m \partial_\alpha T_m \quad \left(\text{or alternatively } - \sum T_m \partial_\alpha S_m\right) .$$

If A is regarded as a vector potential, the corresponding field $F_{\alpha\beta}$ is given by

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha = \sum_m \frac{\partial(S_m, T_m)}{\partial(y_\alpha, y_\beta)}, \tag{12}$$

and eqs. (5) and (1) may be stated as

$$\{y_\mu, y_\nu\} = p_{\mu\nu} = F_{\mu\nu}, \quad F_{5\mu} = F_{6\mu} = 0,$$
$$H = -F_{56} = \frac{1}{2}\sum_{\mu>\nu} F^2_{\mu\nu} = -C^2. \tag{13}$$

We realize at this point that the ansatz for the left-hand side of eq. (4) in terms of scalar functions S_m, T_m could have been replaced with

$$\sum_\alpha \mathrm{d}A_\alpha \wedge \mathrm{d}y_\alpha, \tag{14}$$

from the beginning. However, the fields $F_{\mu\nu}$ do not satisfy the sourceless Maxwell equations; instead, they are constrained by eq. (13c). We also notice that the $F_{\mu\nu}$ can, in general (i.e., except at singular points), be reduced to one nonvanishing component as they represent a tangent plane. This property is characterized by

$$F_{\mu\nu} F_{\lambda\rho}\, \epsilon_{\mu\nu\lambda\rho} = 0, \tag{15}$$

together with eq. (13c). Under these circumstances, the "Clebsch potentials" [4] S, T may be more natural than the vector potentials.

Going to quantum theory, we may associate with the action integral (11) an amplitude

$$W = \exp(\mathrm{i}I_D)$$
$$= \exp\left(\mathrm{i}\int_\Gamma A_\mu\, \mathrm{d}x_\mu\right) \exp\left(-\mathrm{i}C^2 \int \mathrm{d}\sigma\, \mathrm{d}\tau\right). \tag{16}$$

The first factor is the usual Wilson loop factor, whereas the second measures the area in the (σ, τ) space [3], which does not have a direct physical significance. Eq. (16) may be interpreted as a semiclassical approximation to the full quantum theory. However, it is not entirely obvious what the full theory should be. One possibility would be to generate the $F_{\mu\nu}$ by keyboard variations $\delta\sigma_{\mu t}$ of the boundary Γ, and express the constraint as

$$\left[\sum_\mu (\delta/\delta\sigma_{\mu t})^2_z - C^2\right] W = 0, \quad z \in \Gamma, \tag{17}$$

where W is now freed from the semiclassical form (16). This is precisely the equation derived before [1]. However, it does not quite correspond to the eq. (13c) as only half of the $F_{\mu\nu}$ are generated this way. Generating the other half by means of twisting variations of Γ is conceivable, but the question is whether such operations can be adequately defined or not.

For now, we will content ourselves with the semiclassical approximation, and treat the spinning string (rod) as an example. The solution to eq. (2) for this case is

$$y_0 = \tau, \quad y_1 + \mathrm{i}y_2 = \mathrm{e}^{\mathrm{i}\omega\tau\rho(\sigma)},$$
$$\sigma = \frac{1}{C\omega}\int_0^{\rho\omega} (1 - x^2)^{1/2}\, \mathrm{d}x, \quad y_3 = 0. \tag{18}$$

The fields $F_{\mu\nu}$ are then

$$F_{\rho 0} = C/(1 - \omega^2\rho^2)^{1/2}, \quad F_{\rho\theta} = C\omega\rho/(1 - \omega^2\rho^2)^{1/2},$$
$$\text{others} = 0. \tag{19}$$

The corresponding potentials (naturally not unique) are found to be

$$S_2 = \rho, \quad T_2 = C(t + \omega\theta\rho^2)/(1 - \omega^2\rho^2)^{1/2}$$
$$(-1 < \omega\rho < 1, \theta = \tan^{-1} y_2/y_1), \tag{20}$$
$$A_\rho = -T_2 \partial_\rho S = -T_2, \quad \text{others} = 0,$$

We note here that the potentials are not single valued; in particular, ρ has to be extended to negative values in order to cover the entire rod. This latter feature seems inevitable because of the oriented nature of the string.

Under this choice of gauge (20), a rectangular loop integral visiting the points (t_1, ρ), $(t_1, -\rho)$, $(t_2, -\rho)$, (t_2, ρ) at fixed θ reduces to two spacelike segments at times t_1 and t_2, so the factor

$$\exp\left(\mathrm{i}\int_{-1/\omega}^{1/\omega} A_\rho\, \mathrm{d}\rho\right)\Bigg|_t = \exp\left[-\mathrm{i}\int_{-1/\omega}^{1/\omega} \frac{C\, \mathrm{d}\rho}{(1 - \omega^2\rho^2)^{1/2}} - \mathrm{i}\theta \int_{-1/\omega}^{1/\omega} \frac{C\omega\rho^2}{(1 - \omega^2\rho^2)^{1/2}}\, \mathrm{d}\rho\right]$$
$$= \exp[-\mathrm{i}\pi C(2\omega t + \theta)/2\omega^2] \tag{21}$$

may be interpreted as the wave function of the rod. Indeed, the two integrals are nothing but the total energy E and angular momentum $l = \pi C/2\omega^2$ of the spinning rod, related by the Regge formula $l = \alpha' E^2$.

We also find that the total "magnetic flux" in the z-direction is

$$2\pi \int_{-1/\omega}^{1/\omega} F_{\rho\theta}\, \rho \mathrm{d}\rho = 2\pi l\,. \tag{22}$$

The condition of single valuedness of the wave function then leads in the usual manner to the quantization of l as well as of the flux, whatever the significance of the latter might be. Although such an interpretation seems natural, it leaves some open questions. For example, what is the meaning of the amplitude when the integral does not extend from end to end?

If one is curious about how special the above example is, it is instructive to try a more general ansatz: $S_2 = \rho$, $T_2 = tf + \theta\rho(1-f^2)^{1/2}$ which satisfies eq. (13c) for arbitrary $f(\rho)$. Minimizing the energy E for fixed angular momentum l, however, leads one back to the previous solution.

Finally, we will briefly comment on two generalizations. First, eq. (4) is a special case of the more general form

$$\sum_{i>j} p_{ij}\, \mathrm{d}q_i \wedge \mathrm{d}q_j + \sum_i P_i\, \mathrm{d}q_i \wedge \mathrm{d}\sigma + \sum_i Q_i\, \mathrm{d}q_i \wedge \mathrm{d}\tau - H\, \mathrm{d}\sigma \wedge \mathrm{d}\tau\,, \tag{23}$$

$$H = H(p_{ij}, P_i, Q_i, q_i)\,.$$

Such a form will be relevant to strings weighted with mass points.

The second comment concerns non-abelian extensions. There is no problem in formally generalizing the quantum equations (16) and (17) as has been done before [1], but from the classical string point of view it is not so natural because only one set of $F_{\mu\nu}$ may be associated with the tangential plane at each point. Thus the non-abelian degrees of freedom must be reduced to a locally abelian level.

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