

## FERMION–BOSON RELATIONS IN BCS-TYPE THEORIES

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### 1. Introduction

Supersymmetry is a mathematical idea in search of physical relevance, but so far the search has not been successful in the area where it originated: particle physics. The present report is an outcome of my attempt to understand the physical meaning of supersymmetry and to find examples of it in more familiar low energy physics.

Iachello and collaborators [1] have observed that there is a kind of supersymmetry in nuclear physics. For certain nuclei in the mass number region of platinum, the low energy spectra of even–even nuclear species and neighboring even–odd species can be described by the same empirical formula based on group theory. I have been aware, on the other hand, that in theories of the BCS-type, there always is a simple relation between the mass (energy gap) of the basic fermion and those of the bosons (collective modes) [2]. To use the language of particle physics, the dynamically induced masses of the pion, quark and  $\sigma$  meson stand in the ratio 0:1:2 (subject to higher order corrections). In terms of the effective  $\sigma$  model (or Higgs or Ginsburg–Landau) Lagrangian, this implies that the self-coupling and the Yukawa constants are related by  $\lambda = f^2$ . Generic relations of this nature emerge in any BCS–Heisenberg-type four-fermion short-range interaction theory [3,11]. It is gratifying that such relations have been experimentally established in superconductors and superfluid helium 3, as I will discuss later.

My speculation, then, is that the Iachello relation in nuclear physics may also be a manifestation of the BCS mechanism which is known to

account for the nuclear pairing phenomenon. An immediate question that arises is whether the BCS or Ginsburg–Landau theory has a supersymmetry of which these relations are a consequence. I do not know the answer yet. Before coming to nuclear physics, however, I will first discuss the other examples to show the origin of the relations.

### 2. Superconductivity

We use the two-component formalism in which the spin-up electron and spin-down hole span a  $\tau$ -spin space for the quasiparticle. The free quasiparticle Hamiltonian has the form

$$H = K\tau_3 + m\tau_1, \quad (1)$$

where  $K$  is the kinetic energy measured from the Fermi surface and  $m$  is the gap parameter. (We deal only with states in the vicinity of the Fermi surface.) The charge operator is  $e\tau_3$ , so a  $\tau_1$  or  $\tau_2$  term will break charge conservation. The gap in the  $\tau_1$  direction will then generate a Goldstone mode in the  $\tau_2$  direction. The collective modes in the  $\tau_1$  and  $\tau_2$  directions (called amplitude and phase modes) are the analogs of  $\sigma$  and  $\pi$  mesons in particle physics.

In terms of the four-fermion interaction Lagrangian

$$G\tau_3\tau_3' \sim G[(\tau_1\tau_1' + \tau_2\tau_2' + \dots)] \quad (2)$$

(under a Fierz transformation),

a collective mode is determined as the pole of the

function

$$G/(1 - GI) = G + G(GI) + G(GI)^2 + \dots, \quad (3)$$

where the formal expansion indicates that the function is derived by summing to infinite order chains of “bubble” perturbation theory diagrams.  $I$  is a two-point loop integral with appropriate vertex operators,  $\tau_1$  or  $\tau_2$ . Its energy ( $E$ ) dependence can be displayed by a dispersion integral

$$I(z) = \int \rho(x) dx / (x - z), \quad x > 4m^2, \quad z = E^2, \quad (4)$$

where the absorptive parts  $\rho$  for  $\tau_1$  and  $\tau_2$  modes satisfy the relation

$$\rho_1(x) = \rho_2(x)(1 - 4m^2/x). \quad (5)$$

From this, it follows that

$$I_1(z = 4m^2) = I_2(z = 0). \quad (6)$$

But  $GI_2(z = 0) = 1$ , as a result of the Ward identity corresponding to the breaking of  $\tau_3$  invariance. Thus, one sees that the  $\tau_2$  and  $\tau_1$  modes have a pole at  $E = 0$  and  $2m$ , respectively. It looks as if these states correspond to composites of two fermions with maximum and zero binding energy. I do not have a simple physical explanation for this, but technically it depends on whether the vertex operator anticommutes or commutes with the mass operator. Another way of putting it is that, in the particle theory analog,  $\pi$  and  $\sigma$  are s- and p-wave bound states of quarks, and the extra factor in the absorptive part for the latter is just the p-wave phase space factor.

The  $\tau_1$  (amplitude) mode was experimentally detected a few years ago in certain superconductors whose gap parameters are susceptible to external forces like laser beams [4]. Actually, the mode comes out as a genuine bound state somewhat below  $2m$ . The theoretical explanation has been worked out along the lines sketched above [5]. The massless  $\tau_2$  (phase or Goldstone) mode, on the

other hand, mixes with the  $\tau_3$  (density) mode, and turns into the plasmon by the familiar Anderson–Higgs–Englert–Brout mechanism. Both  $\tau_1$  and  $\tau_2$  modes have the same canonical velocity

$$v = v(F)/\sqrt{3}, \quad (7)$$

where  $v(F)$  is the Fermi velocity. So it is a simple matter to write down an effective  $\sigma$ -model Lagrangian for the collective modes, which can be explicitly derived from a BCS theory. [As was mentioned in section 1, the Yukawa coupling ( $f$ ) and the boson self-coupling ( $\lambda$ ) are related as

$$\lambda = f^2 = 1/(dI_2(z)/dz)_{z=0}, \quad (8)$$

where  $\lambda/2$  is the coefficient of the quartic term in the Lagrangian.]

### 3. Helium 3

Superfluid helium 3, occurring at temperatures in the millikelvin range, has two phases, A and B<sup>6</sup>. It is believed that the Cooper pairs of two helium atoms are formed in a triplet p state in either phase. In principle, the total angular momentum  $j$  can be 0, 1, or 2, and the pairs can condense in any state in the space spanned by these 9 states; also, recall that the degeneracy of a state of angular momentum  $j$  is  $2j + 1$ . Actually, however, the A and B phases correspond to  $j = 1$  and 0, respectively. The nine states may be expressed by a scalar  $S$ , vector  $V_i$ , or tensor  $T_{ik}$  order parameter transforming as

$$\tau_\alpha \sigma \cdot p, \quad \tau_\alpha (\sigma \times p)_i, \quad \text{or} \quad \tau_\alpha \sigma_i p_k \quad (\alpha = 1, 2), \quad (9)$$

respectively, where  $\sigma$  is the spin, and  $p$  is the relative momentum of the Cooper pair. As before, the  $\tau_\alpha$ 's operate in the space of particle and hole components, so there are 18 collective modes altogether.

What symmetries are broken by the Cooper pairings? This question leads to another: what is the precise nature of the forces responsible for the

pairing? The answer is not entirely clear, but it appears that they are not greatly different in different  $j$  channels, meaning that intrinsic spin-orbit interaction is small. The pairing in a particular  $j$  state will therefore break the conservation of total angular momentum  $j$  (if  $\neq 0$ ) and spin (or equivalently, orbital) angular momentum, in addition to the usual violation of particle number. Each violation should generate a Goldstone mode (exact or approximate), whereas the remaining collective states should be massive. Again, these have a simple mass spectrum as shown in table I [7]. Note in particular that for a given  $j$ , the phase and amplitude modes satisfy a sum rule

$$m_1^2 + m_2^2 = 4m^2 \tag{10}$$

in terms of the fermion mass  $m$ . This is a consequence of a theorem which generalizes eq. (6):

$$aI_1(a) + (1 - a)I_2(a) = \text{const} = I_2(0). \tag{11}$$

Vertices appearing in this problem generally give rise to the linear combination on the left-hand side, and the phase and amplitude modes are orthogonal and complementary in the sense  $a \leftrightarrow 1 - a$ .

The above theoretical predictions are roughly in agreement with observation. The phase and amplitude modes behave differently under particle–hole reflection relative to the Fermi surface. Since this symmetry is not exact, both modes can be excited by external density waves. Perhaps the above sum rule is to be more trusted than the individual mass values.

Table I  
Collective modes in He 3, B phase.  $H = K\tau_3 + m\tau_1\sigma \cdot p$

Mode	$j$	Mass	Loop integral	Comment
$\tau_1\sigma \cdot p$	0	$2m$	$I_1$	particle number Goldstone
$\tau_2\sigma \cdot p$	0	0	$I_2$	
$\tau_1(\sigma \times p)_i$	1	0	$I_2$	spin Goldstone
$\tau_2(\sigma \times p)_i$	1	$2m$	$I_1$	
$\tau_1\sigma_1 p_k$	2	$\sqrt{2/5}(2m)$	$\frac{2}{3}I_1 + \frac{3}{5}I_2$	
$\tau_2\sigma_i p_k$	2	$\sqrt{3/5}(2m)$	$\frac{3}{5}I_1 + \frac{2}{3}I_2$	

#### 4. Nuclear physics

The BCS theory has been applied with success to account for the pairing phenomenon in nuclei [8]. Typically, the ground state of a heavy even–even nucleus is filled with singlet proton pairs and neutron pairs, whereas an even–odd nucleus has one quasi-particle added or missing. The excited states of a nucleus can be due to single-particle as well as pair excitations, the latter giving rise to collective models with  $j = 0, 2, \dots$

In the interacting boson model of Arima and Iachello [9,1], one treats the pairs as bosons, and introduces an effective Hamiltonian with quartic self-couplings, which is determined on the basis of group theory. Thus, take  $j = 0$  and  $j = 2$  bosons only, considered to be degenerate in the first approximation. The relevant invariance group is then  $U(6) = SU(6) \times (U(1))$ . Next, break it down to a chain of subgroups. There are three distinct chains:

$$\begin{aligned} U(6) &\supset U(5) \supset SO(5) \dots, \\ U(6) &\supset SU(4) \supset SO(6) \supset SO(5) \dots, \\ U(6) &\supset SU(3) \dots \end{aligned} \tag{12}$$

The question of which chain is relevant is where physics comes in, and the answer depends on the mass number region one is considering. In the platinum region, for example, it is the second chain. At any rate, one then writes down an effective Hamiltonian as a linear combination of Casimir operators corresponding to the chain of groups in terms of bose operators. This seems to work well in general. It has been pointed out [1], moreover, that in the platinum region, the even–odd nuclei have single protons in a  $D_{3/2}$  shell, for which the natural invariance group is  $SU(4)$ , leading to the same chain of subgroups as for the bosons. As it turns out, the same Hamiltonian can indeed describe both even–even and even–odd cases, hence suggesting a supersymmetry [1]. Real supersymmetry, however, should involve a supergroup like  $U(6/4)$  which includes fermion–boson transitions. Such a scheme still works, although not as well as before.

Now, I come back to the scenario already developed, i.e., the BCS–Ginsburg–Landau theory, as applied to the present problem. Let us say that Cooper pairs, each formed out of nucleons in the same shell, collectively condense into a ground state, which can be excited by breaking pairs up or placing them into new states. In the case of  $j = 3/2$  shell, the pair can have  $j = 0$  or 2.

According to the standard methods, a nucleon should be described by combining four particle and four hole (complex conjugate) states into a 8-component wave function  $\psi$ . Then one writes an analog of the Hamiltonian, eq. (1), in this space. The  $\psi$  can form a representation of SO(8) and Spin(7) (Majorana), so there are seven anticommuting  $\Gamma$  matrices  $\Gamma_1, \Gamma_2, \dots, \Gamma_7$ . One can choose them in such a way that  $\Gamma_0 = +1(-1)$  for particle (hole) states, and  $\Gamma_s \equiv \Gamma_1$  behaves as a scalar (or possibly a mixture of scalar and tensor). Then the Hamiltonian will take the form

$$H = K\Gamma_0 + m\Gamma_s. \quad (13)$$

A remark is in order before proceeding further. The emergence of the parameter  $m$  is a result of interactions among many nucleons in various shells, not just in the states described by the wave function under consideration. The collective modes are generated by all of them, which contribute to the loop integral  $I$ , eq. (4). For example,  $j = 1/2$  shells will contribute to  $j = 0$ , and  $j = 5/2$  states to  $j = 0, 2, 4$ , collective modes. The relation between  $m$  in eq. (13) and the masses of the collective modes, therefore, is no longer so clear. On the other hand, to the extent that  $j = 0$  and 2 modes are approximately degenerate while others can be ignored, it would be reasonable to suppose that the symmetry group SU(4) of the fermions under consideration and the symmetry group SO(6) of the collective modes are physically identical. So the loop integrals will still maintain the same properties as before.

There should be  $6 \times 2 = 12$  collective modes, which come in six pairs  $\Gamma_k, i\Gamma_0\Gamma_k$  ( $k = 1, \dots, 6$ ) of amplitude and phase modes. Thus,  $\Gamma_s$  is an analog

Table II  
Collective modes in nuclei ( $j = 3/2$  shell).  $H = K\Gamma_0 + m\Gamma_s$

Mode	$j$	Mass	Baryon # ( $B$ )	Loop integral	Comment
$\Gamma_s$	0	$2m$	$\pm 2$	$I_1$	
$\Gamma_0\Gamma_s$	0	0	$\pm 2$	$I_2$	B Goldstone
$\Gamma_t$	2	0	$\pm 2$	$I_2$	SO(6) Goldstone
$\Gamma_0\Gamma_t$	2	$\sim 2m$	$\pm 2$	$I_1$	
$\Gamma_0$	0	0		$I_1 - I_2$	B generator
$\Gamma_s\Gamma_t$	2	0		$I_1 - I_2$	SO(6) generator
$\Gamma_t\Gamma_t$	2	0		0	SO(6) generator
$\Gamma_0 - \Gamma_0\Gamma_s$	0	0		$(iE/2m)I_2$	} off diagonal elements
$\Gamma_t - \Gamma_s\Gamma_t$	2	0		$(iE/2m)I_2$	

$\Gamma_s$ : scalar,  $\Gamma_t$ : tensor

of the  $\sigma$  with  $j = 0$  at mass  $2m$ , whereas the zero-mass  $i\Gamma_0\Gamma_1$  mode is spurious because there is actually no particle number violation in a finite system. The SO(6) symmetry, if it exists in an approximate sense, is generated by  $\Gamma_k\Gamma_1$  ( $K > 1$ ). The  $\Gamma_s$  term in  $H$  breaks this down to SO(5), so there should be spurious near-zero modes  $\Gamma_t$  ( $t \neq 1$ ), and complementary modes  $i\Gamma_0\Gamma_t$  at mass  $\approx 2m$ , both with  $j = 2$ . Actually, the zero modes will in general couple to ordinary acoustic modes and get absorbed into them. These results are summarized in table II.

Finally, we will write down an effective Hamiltonian for the system. Introduce six complex boson operators  $B_i$  ( $B_i^\dagger$ ), transforming under the symmetry group SU(6)  $\times$  U(1). It turns out that the proper Hamiltonian (ignoring the mixing with acoustic modes for the moment) is given by

$$H = \psi^\dagger K \Gamma_0 \psi + f \left[ \psi^\dagger (1 + \Gamma_0) \Gamma_t \psi B_t + \text{h.c.} \right] \\ + (f^2/2) \left[ (B_i^\dagger B_i - c^2)^2 + |B_i^\dagger B_k - B_k^\dagger B_i|^2 \right].$$

Note that the nonlinear terms for the  $B_i$ 's are composed of Casimir operators of U(1) and SU(6). The terms involving the fermions, on the other hand, have the symmetry of SU(4)  $\times$  U(1)  $\approx$  SO(6)  $\times$  U(1). The latter gets broken spontaneously to SO(5), giving rise to the above mentioned zero modes. It thus seems likely that an effective

Hamiltonian constructed along this line has properties similar to the one proposed by Iachello.

I have tried to uncover in this kind of system a genuine supersymmetry involving fermionic transformations, but so far I do not have a clear answer to that.

## 5. Particle physics

Finally, I will briefly come back to particle physics. Historically, the  $\sigma$  model in hadron physics is the oldest relevant example and is still of considerable interest in many respects, but our present concern is with mass relations. In accordance with our general results, it has been claimed [10] that the  $\sigma$  meson should have a mass roughly twice the constituent quark mass, i.e.,  $\approx 700$  MeV, although the actual  $\sigma$  resonance seems to occur at  $\approx 900$  MeV. Since the QCD interaction is rather different from the ones we are considering, it is not obvious that the present arguments apply here equally well.

A more interesting case should be the electro-weak interaction [11]. Here the Higgs bosons in the Weinberg–Salam theory are analogs of the  $\pi$  and  $\sigma$ , but regarded as elementary objects. Their masses therefore are not predictable. But in composite models like Terazawa's [11], there emerge relations like ours between boson and constituent fermion masses. Similar situations may also exist in technicolor theories, subject to the uncertainties encountered in the hadronic case. Short of a specific model however, I will not indulge in further speculations at this moment.

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