DOES FADDEEV'S ANOMALY EXIST IN GAUSS LAW CONSTRAINTS?

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The anomalous commutation relation for the Gauss law constraint recently claimed by Faddeev is examined in the perturbative calculation at one-loop level. It is found that in the presence of chiral fermions the commutation relation for the operator $\tilde{G}^a = \partial_i E^{ai} + f^{abc} A_i^b E^{ci}$ has the anomalous term found by Faddeev, but for the true constraint operator $G^a = \tilde{G}^a - \bar{\psi}\gamma^0 t^a \psi$ there exist no anomalies.

Recently, Faddeev [1] claimed that there exists an additional term in the commutation relation of the Gauss law constraint for the Yang–Mills theory with chiral fermions. Following Dirac's method of quantizing gauge theories [2], we start with $A_0^a = 0$, set up commutation relations between A_i^a and $E_j^b = \dot{A}_j^b$ ($i, j = 1, \ldots, 3$), and restrict state vectors $\Psi[A]$ satisfying the Gauss law constraint

$$G^a(x)\Psi[A] = 0, \tag{1}$$

with

$$G^a(x) = \partial_i E^{ai} + f^{abc} A_i^b E^{ci} - \bar{\Psi}\gamma^0 t^a \Psi. \tag{2}$$

Faddeev's claim is that in the presence of chiral fermions ψ_L or ψ_R, the usual consistency condition for the Gauss law constraint breaks down by the term proportional to d^{abc};

$$\left[G^a(x), G^b(y)\right]_{x^0=y^0} = \mathrm{i} f^{abc} G^c(x)\delta^{(3)}(x-y) - (\mathrm{i}/12\pi^2)\Delta n\, d^{abc}\varepsilon_{ijk}\partial_i A_j^c(y)\,\partial_k \delta^{(3)}(x-y), \tag{3}$$

where $d^{abc} = \frac{1}{2}\mathrm{Tr}(t^a\{t^b, t^c\})$ with $[t^a, t^b] = \mathrm{i} f^{abc} t^c$ and $\mathrm{Tr}\, t^a t^b = \frac{1}{2}\delta^{ab}$, and Δn = (the number of ψ_L) − (the number of ψ_R).

Faddeev has argued that the above anomalous term is an allowed form in the generalized sense of the projective representation. His derivation is an extension of the derivation of non-abelian anomalies given by Wess and Zumino [3]: Assuming the existence of anomalous terms we have consistency conditions for them, called cocycle conditions in the cohomology theory. We note here that the coefficient of the anomalous term in eq. (3) is not determined by the cocycle condition alone.

In this paper we examine Faddeev's anomalous commutation relation in the perturbative calculation. We have found that the operator $\tilde{G}^a(x) \equiv \partial_i E^{ai} + f^{abc} A_i^b E^{ci}$ actually has an anomalous term of the same form as in eq. (3). However, for the true Gauss law operator $G^a = \tilde{G}^a - \bar{\psi}\gamma^0 t^a \psi$ the anomalous term vanishes. Therefore, there is no indication at least on the one-loop level that the Gauss law constraint is of the second kind.

We choose V − A theory of fermions coupled to the non-abelian gauge fields. The standard technique to extract equal-time commutators from T-products is given by Bjorken, Johnson, and Low (BJL) [4]. We should carefully extract T-products from T*-products in theories with derivative couplings such as the non-abelian gauge theories. However, in our following discussion at one-loop level, the contribution to the

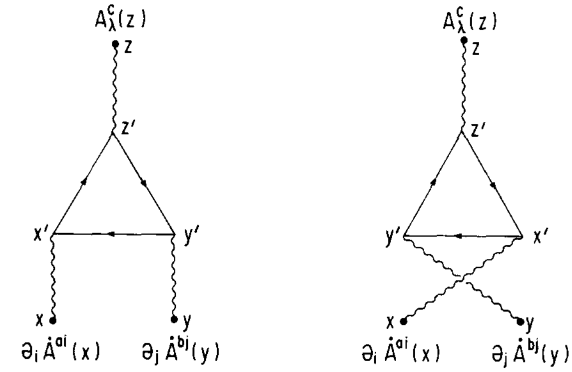

Fig. 1. Diagrams responsible for the Faddeev anomaly in the equal-time commutator $[\tilde{G}^a(x), \tilde{G}^b(y)]$.

anomaly comes from the diagrams without derivative coupling, so that we have no such delicate problem. For the T-product

$$T^{abc,\rho}(p,q) \equiv \int d^4x \int d^4y \int d^4z \exp[-i(px+qy+rz)]\langle 0|TG^a(x)G^b(y)A^{c,\rho}(z)|0\rangle, \tag{4}$$

the equal-time commutator reads

$$\lim_{p^0-q^0\to\infty} (p^0-q^0)T^{abc,\rho}(p,q)$$
$$= \int d^4x \int d^4y \int d^4z \exp[-i(px+qy+rz)](-2i)\delta(x^0-y^0)\langle 0|T[G^a(x),G^b(y)]A^{c\rho}(z)|0\rangle. \tag{5}$$

If we restrict ourselves to the calculation at one-loop level, the diagrams including the three-point vertex of gauge fields contribute to the terms proportional to f^{abc}. Several triangle graphs can generate the anomalous terms proportional to d^{abc}. We classify these diagrams in figs. 1–3 according to their dependence on $\partial_i E^{ai}$ and $-\bar{\psi}\gamma^0 t^a\psi$. The BJL limits of figs. 1, 3, and the total sum of figs. 1–3 contribute to $[\tilde{G}^a(x), \tilde{G}^b(y)]_{x^0=y^0}$, $[\bar{\psi}\gamma^0 t^a\psi(x), \bar{\psi}\gamma^0 t^b\psi(y)]_{x^0=y^0}$, and $[G^a(x), G^b(y)]_{x^0=y^0}$, respectively.

Remembering that we have assumed $A^a_0 = 0$ from the beginning, we take the gauge boson propagator

$$D_{\mu\nu}(p) = \frac{-i}{p^2+i\varepsilon}\left(g_{\mu\nu} - \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n} + \frac{p_\mu p_\nu}{(p\cdot n)^2}\right), \tag{6}$$

where we have introduced a time-like vector n^μ with $n_\mu n^\mu = 1$, in order to keep apparent Lorentz-covariance.

The contribution to $T^{abc,\rho}(p,q)$ of the two diagrams in fig. 1 leads to

$$T^{abc,\rho}_{(1)}(p,q) = [p_{\mu'} - n_{\mu'}(p\cdot n)](p\cdot n)[q_{\nu'} - n_{\nu'}(q\cdot n)](q\cdot n)$$
$$\times D^{\mu'}_\mu(p) D^{\nu'}_\nu(q) D^\rho_\lambda(r)(2\pi)^4\delta^{(4)}(p+q+r)A^{abc,\mu\nu\lambda}(p,q|0), \tag{7}$$

where

$$A^{abc,\mu\nu\lambda}(p,q|0) \equiv [\mathrm{Tr}(t^a t^b t^c)A^{\mu\nu\lambda}(p,q|0) + \mathrm{Tr}(t^b t^a t^c)A^{\nu\mu\lambda}(q,p|0)], \tag{8}$$

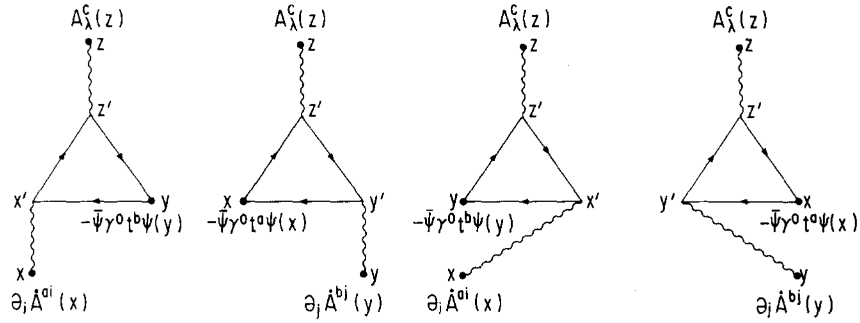

Fig. 2. Diagrams responsible for the anomaly in the equal-time commutator $[\tilde{G}^a(x), -\bar\psi\gamma^0 t^b\psi(y)] + [-\bar\psi\gamma^0 t^a\psi(x), \tilde{G}^b(y)]$.

and

$$A^{\mu\nu\lambda}(p,q|m) = (-1)\int \frac{d^4 l}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not{l}+\not{p}-m+i\varepsilon}\, i\gamma^\mu \frac{1-\gamma^5}{2}\, \frac{i}{\not{l}-m+i\varepsilon}\, i\gamma^\nu\frac{1-\gamma^5}{2}\, \frac{i}{\not{l}-\not{q}-m+i\varepsilon}\, i\gamma^\lambda\frac{1-\gamma^5}{2}\right). \tag{9}$$

The same contributions from figs. 2 and 3 are given by

$$T^{abc,\rho}_{(2)}(p,q) = \left\{-\left[p_{\mu'}-n_{\mu'}(p\cdot n)\right](p\cdot n)D^{\mu'}_{\mu}(p)(in_\nu) - (in_\mu)\left[q_{\nu'}-n_{\nu'}(q\cdot n)\right](q\cdot n)D^{\nu'}_{\nu}(q)\right\}$$
$$\times D^\rho_\lambda(r)(2\pi)^4\delta^{(4)}(p+q+r)A^{abc,\mu\nu\lambda}(p,q|0), \tag{10}$$

$$T^{abc,\rho}_{(3)}(p,q) = (in_\mu)(in_\nu)D^\rho_\lambda(r)(2\pi)^4\delta^{(4)}(p+q+r)A^{abc,\mu\nu\lambda}(p,q|0), \tag{11}$$

respectively.

The Pauli–Villars method is used to regularize the amplitude $A^{\mu\nu\lambda}$, since it treats three vertices of fermions with gauge fields equally.

The regularized $A^{\mu\nu\lambda}$ is now

$$A^{\mu\nu\lambda}(p,q)_R = \lim_{M^2\to\infty}\int_{M^2}^0 dm^2 \frac{\partial}{\partial m^2}A^{\mu\nu\lambda}(p,q|m) \tag{12}$$

$$= \lim_{M^2\to\infty} \tfrac{1}{2}\cdot 4\left(g^{\alpha\mu}g^\beta_\rho - g^{\alpha\beta}g^\mu_\rho + g^{\mu\beta}g^\alpha_\rho - i\varepsilon^{\alpha\mu\beta}_{\rho}\right)\left(g^{\rho\nu}g^{\gamma\lambda} - g^{\rho\gamma}g^{\nu\lambda} + g^{\rho\lambda}g^{\nu\gamma} - i\varepsilon^{\rho\nu\gamma\lambda}\right)$$
$$\times \frac{i}{(4\pi)^2}\int_{M^2}^0 dm^2 \int_0^1 \prod_{i=1}^3 du_i\,\delta\left(1-\sum_{i=1}^3 u_i\right)\left(-\frac{1}{2}\,\frac{g_{\alpha\beta}Q^2_\gamma + g_{\beta\gamma}Q^1_\alpha + g_{\gamma\alpha}Q^3_\beta}{V+m^2-i\varepsilon} + \frac{Q^1_\alpha Q^3_\beta Q^2_\gamma}{[V+m^2-i\varepsilon]^2}\right), \tag{13}$$

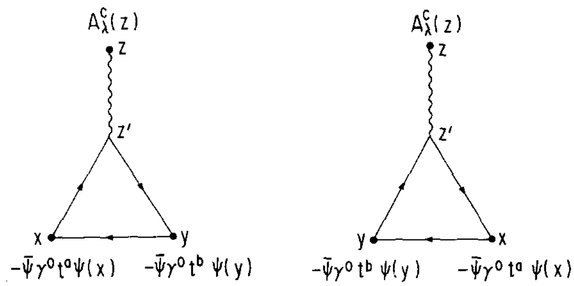

Fig. 3. Diagrams responsible for the anomaly in the equal-time commutator in $[-\bar\psi\gamma^0 t^a\psi(x), -\bar\psi\gamma^0 t^b\psi(y)]$.

where

$$V = -u_1u_3p^2 - u_2u_3q^2 - u_1u_2(p+q)^2,$$
$$Q^1 = u_2(p+q) + u_3p, \quad Q^2 = -u_3q - u_1(p+q), \quad Q^3 = -u_1p + u_2q. \tag{14}$$

The amplitude $A_{\mu\nu\lambda}$ contains both parity-conserving and -violating parts. The parity-conserving part does not contribute to the anomalous term which we are considering now. Therefore we will concentrate on the parity-violating part, which can be expressed in terms of the invariant amplitudes T_1–T_8 as follows:

$$A_{\mathrm{PV}}^{\mu\nu\lambda}(p,q) = \varepsilon^{\mu\nu\lambda\alpha}p_\alpha T_1 + \varepsilon^{\mu\nu\lambda\alpha}q_\alpha T_2 + p^\mu\varepsilon^{\nu\lambda\alpha\beta}p_\alpha q_\beta T_3 + q^\mu\varepsilon^{\nu\lambda\alpha\beta}p_\alpha q_\beta T_4$$
$$+ p^\nu\varepsilon^{\lambda\mu\alpha\beta}p_\alpha q_\beta T_5 + q^\nu\varepsilon^{\lambda\mu\alpha\beta}p_\alpha q_\beta T_6 + p^\lambda\varepsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta T_7 + q^\lambda\varepsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta T_8, \tag{15}$$

with $T_i = T_i(p^2, q^2, p\cdot q)$.

A comparison of (13) and (15) gives the following results:

$$T_1 = \frac{1}{8\pi^2}\int_{M^2}^{0} \mathrm{d}m^2 \int[\mathrm{d}u]\left(-\frac{1}{2}\frac{2(1-3u_1)}{V+m^2-\mathrm{i}\varepsilon} + \frac{[(1-u_1)p+u_2q](-u_1p+u_2q)(-u_1)}{[V+m^2-\mathrm{i}\varepsilon]^2}\right),$$
$$T_2 = \frac{1}{8\pi^2}\int_{M^2}^{0} \mathrm{d}m^2 \int[\mathrm{d}u]\left(-\frac{1}{2}\frac{-2(1-3u_2)}{V+m^2-\mathrm{i}\varepsilon} + \frac{[(1-u_1)p+u_2q](-u_1p+u_2q)(-1)(1-u_2)}{[V+m^2-\mathrm{i}\varepsilon]^2}\right),$$
$$T_3 = \frac{1}{8\pi^2}\int_{M^2}^{0} \mathrm{d}m^2 \int[\mathrm{d}u]\frac{-u_1(2-2u_1-u_2)}{[V+m^2-\mathrm{i}\varepsilon]^2}, \quad T_4 = \frac{1}{8\pi^2}\int_{M^2}^{0} \mathrm{d}m^2 \int[\mathrm{d}u]\frac{u_2(1-2u_1-u_2)}{[V+m^2-\mathrm{i}\varepsilon]^2}, \tag{16}$$
$$T_5 = -T_7 = \frac{1}{8\pi^2}\int_{M^2}^{0} \mathrm{d}m^2 \int[\mathrm{d}u]\frac{-u_1u_2}{[V+m^2-\mathrm{i}\varepsilon]^2}, \quad T_6 = -T_8 = \frac{1}{8\pi^2}\int_{M^2}^{0} \mathrm{d}m^2 \int[\mathrm{d}u]\frac{-(1-u_2)u_2}{[V+m^2-\mathrm{i}\varepsilon]^2}.$$

It should be noted that due to the following identities:

$$(p^\mu\varepsilon^{\nu\lambda\alpha\beta} + p^\nu\varepsilon^{\lambda\mu\alpha\beta} + p^\lambda\varepsilon^{\mu\nu\alpha\beta})p_\alpha q_\beta = -(p\cdot q)\varepsilon^{\mu\nu\lambda\alpha}p_\alpha + p^2\varepsilon^{\mu\nu\lambda\alpha}q_\alpha,$$
$$(q^\mu\varepsilon^{\nu\lambda\alpha\beta} + q^\nu\varepsilon^{\lambda\mu\alpha\beta} + q^\lambda\varepsilon^{\mu\nu\alpha\beta})p_\alpha q_\beta = (p\cdot q)\varepsilon^{\mu\nu\lambda\alpha}q_\alpha - q^2\varepsilon^{\mu\nu\lambda\alpha}p_\alpha, \tag{17}$$

the number of invariant tensors is six rather than eight so that the above decomposition is not unique, but this does not affect the following arguments.

Let us consider the contribution of fig. 1. The relevant part of eq. (7) is

$$X_{(1)}^\lambda \equiv (-1)[p_{\mu'} - n_{\mu'}(p\cdot n)](p\cdot n)\times(-1)[q_{\nu'} - n_{\nu'}(q\cdot n)](q\cdot n)D_\mu^{\mu'}(p)D_\nu^{\nu'}(q)A_{\mathrm{PV}}^{\mu\nu\lambda}(p,q)_{\mathrm{R}}$$
$$= (+\mathrm{i})[p_\mu/(p\cdot n) - n_\mu](+\mathrm{i})[q_\nu/(q\cdot n) - n_\nu]A_{\mathrm{PV}}^{\mu\nu\lambda}(p,q)_{\mathrm{R}}$$
$$= n_\nu\varepsilon^{\nu\lambda\alpha\beta}p_\alpha q_\beta\{(p\cdot n)^{-1}[T_2 + p^2T_3 + (p\cdot q)T_4] + (q\cdot n)^{-1}[T_1 - (p\cdot q)T_5 - q^2T_6]$$
$$-(p\cdot n)(T_3 - T_5) - (q\cdot n)(T_4 - T_6)\}, \tag{18}$$

whose BJL limit is given by

$$\lim_{p^0-q^0\to\infty}(p^0-q^0)X_{(1)}^\lambda(p,q) = -2n_\nu\varepsilon^{\nu\lambda\alpha\beta}p_\alpha q_\beta \lim_{p^0-q^0\to\infty}(T_1 - T_2). \tag{19}$$

Therefore, what we need is the BJL limit of T_1 and T_2[‡1].

[‡1] It is useful to notice that the right-hand side of the anomalous Ward identity $(p+q)_\lambda A^{\mu\nu\lambda}(p,q)_{\mathrm{R}} = -\varepsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta(T_1 - T_2)$ is also proportional to $(T_1 - T_2)$ which is equal to $-(1/8\pi^2)\frac{1}{3}$ without taking the BJL limit.

The logarithmic divergences $\ln M^2$, superficially existing in T_1 and T_2, disappear due to the parametric integration of $\int[\mathrm{d}u](1-3u_1) = \int[\mathrm{d}u](1-3u_2) = 0$, so that T_1 and T_2 are finite. The BJL limit of T_1 and T_2,

$$\lim_{p^0-q^0\to\infty} T_1 = \frac{1}{8\pi^2}\frac{1}{12}, \qquad \lim_{p^0-q^0\to\infty} T_2 = \frac{1}{8\pi^2}\frac{5}{12}, \tag{20}$$

gives

$$\lim_{p^0-q^0\to\infty} (p^0-q^0) X^{\lambda}_{(1)}(p,q) = \lim_{p^0-q^0\to\infty} (p^0-q^0) X^{\lambda}_{(1)}(q,p) = \frac{1}{12\pi^2} n_\nu \varepsilon^{\nu\lambda\alpha\beta} p_\alpha q_\beta. \tag{21}$$

Now we can evaluate the full contribution of the BJL limit of fig. 1 as follows:

$$\lim_{p^0-q^0\to\infty} (p^0-q^0) T^{abc,\rho}_{(1)}(p,q) = \mathrm{Tr}\left(\{t^a,t^b\}t^c\right)(1/12\pi^2) n_\nu \varepsilon^{\nu\lambda\alpha\beta} p_\alpha q_\beta D^{\rho}_{\lambda}(r)(2\pi)^4 \delta^{(4)}(p+q+r)$$

$$= -\mathrm{Tr}\left(\{t^a,t^b\}t^c\right)(1/12\pi^2) n_\nu \varepsilon^{\nu\lambda\alpha\beta}$$

$$\times \int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z \exp[-\mathrm{i}(px+qy+rz)]\, \partial_\alpha \delta^{(4)}(x-y) \langle 0|\partial_\beta A^c_\lambda(y) A^{c\rho}(z)|0\rangle. \tag{22}$$

To restrict to fig. 1 is nothing but to consider the commutator of $\tilde{G}^a$ instead of G^a in eq. (5). Comparing eqs. (22) and (5), we find the expression for the anomalous commutator:

$$\text{Anomaly of } \left[\tilde{G}^a(x), \tilde{G}^b(y)\right]_{x^0=y^0} = (-\mathrm{i}/12\pi^2)\, d^{abc} \varepsilon_{ijk} \partial_i A^c_j(y)\, \partial_k \delta^{(3)}(x-y). \tag{23}$$

This anomaly is exactly identical to the one given by Faddeev.

On the other hand, to get the commutation relation for the true Gauss law operator $G^a(x)$, we must take into account the other contributions coming from (10) and (11). The relevant parts of them are

$$X^{\lambda}_{(2)} \equiv \left\{(\mathrm{i})\left[p_\mu/(p\cdot n) - n_\mu\right](\mathrm{i}n_\nu) + (\mathrm{i}n_\mu)(\mathrm{i})\left[q_\nu/(q\cdot n) - n_\nu\right]\right\} A^{\mu\nu\lambda}_{\mathrm{PV}}(p,q)_{\mathrm{R}}, \tag{24}$$

$$X^{\lambda}_{(3)} = (\mathrm{i}n_\mu)(\mathrm{i}n_\nu) A^{\mu\nu\lambda}_{\mathrm{PV}}(p,q)_{\mathrm{R}}. \tag{25}$$

The BJL limits of (24) and (25) are obtained as

$$\lim_{p^0-q^0\to\infty} (p^0-q^0) X^{\lambda}_{(2)}(p,q) = 2n_\nu \varepsilon^{\nu\lambda\alpha\beta} p_\alpha q_\beta \lim_{p^0-q^0\to\infty} \left[(T_1-T_2) + \tfrac{1}{4}(p^0-q^0)^2(T_3-T_4-T_5+T_6)\right], \tag{26}$$

$$\lim_{p^0-q^0\to\infty} (p^0-q^0) X^{\lambda}_{(3)}(p,q) = -2n_\nu \varepsilon^{\nu\lambda\alpha\beta} p_\alpha q_\beta \lim_{p^0-q^0\to\infty} \tfrac{1}{4}(p^0-q^0)^2(T_3-T_4-T_5+T_6), \tag{27}$$

where we have the following result using eq. (16):

$$\lim_{p^0-q^0\to\infty} \tfrac{1}{4}(p^0-q^0)^2(T_3-T_4-T_5+T_6) = -\frac{1}{8\pi^2}. \tag{28}$$

The anomaly in the commutator of G^a is the sum of contributions from $X^{\lambda}_{(1)}$, $X^{\lambda}_{(2)}$, and $X^{\lambda}_{(3)}$. From eqs. (19), (26), and (27) we see that it vanishes, contrary to Faddeev's claim:

$$\lim_{p^0-q^0\to\infty} (p^0-q^0)\left(X^{\lambda}_{(1)} + X^{\lambda}_{(2)} + X^{\lambda}_{(3)}\right) = 0. \tag{29}$$

The latter fact, the vanishing of the anomaly in the commutator of G^a, can be seen more transparently in the following way. Graphically, the contributions to G^a consist of two parts; with and without the gauge boson propagator. The sum of them can be expressed as

$$G^a = \left\{ -\left[p_{\mu'} - n_{\mu'}(p\cdot n)\right](p\cdot n) D^{\mu'}_{\mu}(p) + \mathrm{i} n_\mu \right\} J^\mu = \mathrm{i}[1/(p\cdot n)]\, p_\mu J^\mu , \tag{30}$$

where J^μ is a source current of the gauge boson. In our particular case, looking at one of the Gauss law operators, say G^a, we see that the relevant part of $T^{abc}(p,q)$ is proportional to

$$P_\mu A^{\mu\nu\lambda}(p,q|0) = -\varepsilon^{\nu\lambda\alpha\beta} p_\alpha q_\beta \left(T_2 - p^2 T_3 - p\cdot q T_4\right). \tag{31}$$

This is nothing but the anomalous divergence of the fermion triangle diagram. Adding up contributions to the other vertex G^b implies multiplying q_ν but it automatically vanishes due to the antisymmetry of eq. (31). Thus we can see that anomalous contributions vanish before taking the BJL limit.

Our conclusions derived from the above calculation are the following. Faddeev's arguments seem applicable to $\tilde{G}^a$. Our derivation of the anomalous commutator is based on the perturbative calculation of the triangle graphs. The same diagrams are responsible for giving the anomalous Ward identities of the theory. In this sense it can be said that Faddeev's anomaly in the commutator of $\tilde{G}^a$ is equivalent to the anomalous Ward identity. On the other hand, for the true Gauss law operator G^a, the corresponding anomalous terms vanish. Therefore, as our conclusion, we have no indication, at least on the one-loop level, that the Gauss law constraints are of the second kind.

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