EFFECTIVE ABELIAN GAUGE FIELDS ☆

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For a non-abelian gauge field coupled to external source trajectories, one may introduce a pair of intermediary abelian fields. By integrating over the non-abelian variables one obtains a nonlinear lagrangian for the abelian fields which are related by a dielectric coefficient. The string model corresponds to a limiting case of such an abelian system.

In earlier publications [1,2] we have explored certain mathematical links between quantum chromodynamics (QCD) and the string model. From the QCD point of view, on the one hand, the path-ordered phase factor is a natural construction whose properties may be studied under small variations of the path. It is found that some limiting forms of second-order variations yield equations bearing a correspondence to quantum string equations. From the string model point of view, on the other hand, it is possible to formulate its classical mechanics as a generalized Hamilton–Jacobi (H–J) system in which two evolution parameters appear symmetrically. In such a formulation, the Hamilton–Jacobi function (which is a scalar field in the case of point mechanics) is represented by an abelian gauge field, showing again that the link between string and gauge theory is a rather close and natural one. To be a little more precise, the H–J system describing a family of classical string histories (or minimal surfaces) is given by

$$H = \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} = \text{const.},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ belongs to a restricted class of gauge fields representing the field of tangential planes to the minimal surfaces.

Quantization of this string system via functional integration has been undertaken by Eguchi [2] and by Lüscher and Symanzik [3] with some interesting results. The purpose of the present note, however, is different from that of these authors. Rather than seek quantization of the string system as such, we will show that a quantum non-abelian gauge theory can be partially reduced to an effective nonlinear abelian theory which in turn reduces in limiting cases to the string system. Thus the effective abelian theory acts as an interpolation between full QCD and classical string dynamics. In other words, a relation between these two theories is established through a route which bypasses the question of quantum string dynamics.

We start by recapitulating the earlier results [2,4] on the H–J formulation, augmenting them with additional equations required for mathematical consistency. The Euler equations to determine a minimal surface $x_\mu(\sigma, \tau)$ are cast in the form:

$$\partial(x_\mu, x_\nu)/\partial(\sigma, \tau) = \partial H/\partial p_{\mu\nu} = p_{\mu\nu}, \tag{1a}$$

$$\partial(p_{\mu\nu}, x_\nu)/\partial(\sigma, \tau) = -\partial H/\partial x_\nu = 0, \tag{1b}$$

from which follows

$$H = \tfrac{1}{4} p_{\mu\nu} p_{\mu\nu} = \text{const.} \tag{2}$$

Consider a family $C(\xi, \eta)$ of generally nonintersecting minimal surfaces whose points $x_\mu(\sigma, \tau; \xi, \eta)$ satisfy eq. (1). It can be regarded as generating a system of curvilinear coordinates $(\sigma, \tau, \xi, \eta)$ and a field of tangential planes $p_{\mu\nu}(x)$. Conversely, define a gauge field $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ of a restricted variety such that (1) it satisfies eq. (2), and (2) is what we will call a sheet field, namely a field of tangential planes which can be

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parametrized as in eq. (1a). Then it follows, with the aid of the Bianchi identities for $F_{\mu\nu}$, that eq. (1b) is also satisfied, so $F_{\mu\nu} = p_{\mu\nu}$ describe a family of minimal surfaces.

A sheet field $p_{\mu\nu}(x)$ may be characterized by the following properties

$$p_{\mu\nu}p^*_{\mu\nu} = \epsilon_{\mu\nu\lambda\rho}p_{\mu\nu}p_{\lambda\rho} = 0, \quad \partial_\mu(\chi p_{\mu\nu}) = 0, \qquad \text{(3a, b)}$$

where χ is some scalar field. The first relation is obvious. To prove the second, consider the identities involving the mapping $x_\mu \leftrightarrow \sigma, \tau, \xi, \eta$:

$$\partial(x_\mu, x_\nu)/\partial(\sigma,\tau) = \tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}[\partial(\xi,\eta)/\partial(x_\lambda, x_\rho)]\chi^{-1}(x),$$

$$\chi = \partial(\sigma,\tau,\xi,\eta)/\partial(x_1,x_2,x_3,x_4). \qquad (4)$$

The right-hand side is to be identified with $p_{\mu\nu}$. Clearly $\chi p^*_{\mu\nu}$ satisfies the Bianchi identities, or eq. (3b), except at the singularities of the mapping. Thus a sheet field is proportional to the dual of a restricted gauge field satisfying eq. (3a). If $p_{\mu\nu} = F_{\mu\nu}$ is a minimal surface field, both $F_{\mu\nu}$ and $F^*_{\mu\nu}$ are sheet fields, and there is a reciprocity between the restricted gauge fields $F_{\mu\nu}$ and $S^*_{\mu\nu} \equiv \chi F^*_{\mu\nu}$. (A convenient way to represent $F_{\mu\nu}$ is to set $B_\mu = S\partial_\mu T$, $F_{\mu\nu} = \partial_\mu S \partial_\nu T - \partial_\nu S \partial_\mu T$ in terms of two scalar functions S and T[2].)

In this way we are naturally led to the concept of a dielectric medium characterized by two fields $F_{\mu\nu}$ and $S_{\mu\nu}$, with χ playing the role of a dielectric coefficient. Furthermore, an explicit example constructed for a spinning string (see below) shows that the failure of Bianchi identities for $S^*_{\mu\nu}$ may be attributed to the quark sources at the ends of the string. Let us therefore consider a model nonlinear lagrangian

$$L = f(X) - \tfrac{1}{2}S_{\mu\nu}F_{\mu\nu} - j_\mu B_\mu,$$
$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \quad X = \tfrac{1}{4}S_{\mu\nu}S_{\mu\nu}, \qquad (5)$$

with a given external source j. A_μ and $S_{\mu\nu}$ are independent variables. One may call $F_{\mu\nu}$ and $S_{\mu\nu}$ the force field and source field, respectively. The Euler equations are:

$$\partial_\mu S_{\mu\nu} = j_\nu, \quad (\mathrm{d}f/\mathrm{d}X)S_{\mu\nu} = F_{\mu\nu}, \quad \text{or } \mathrm{d}f/\mathrm{d}X = \chi^{-1}. \quad (6)$$

The string system satisfying eq. (2) corresponds to $f(X) \sim \sqrt{X}$. In this case $S_{\mu\nu}$ cannot be eliminated in terms of $F_{\mu\nu}$, as would be possible in general.

To illustrate the situation, take the results of ref. [2]. A family of strings of length $2L$ bodily spinning around the z axis with angular velocity $\omega = 1/L$ gives rise to the force field

$$F_{\mu\nu} = (\mathrm{i}\boldsymbol{E}, \boldsymbol{B}),$$

$$\boldsymbol{E} = C(x/\rho, y/\rho, 0)/(1-\rho^2\omega^2)^{1/2},$$

$$\boldsymbol{B} = C(0,0,\rho\omega)/(1-\rho^2\omega^2)^{1/2},$$

$$-L < \rho = (x^2+y^2)^{1/2} < L. \qquad (7)$$

This is related to the source field

$$S_{\mu\nu} = (\mathrm{i}\boldsymbol{D}, \boldsymbol{H}) = \chi F_{\mu\nu},$$

$$\boldsymbol{D} = C(x/\rho^2, y/\rho^2, 0),$$

$$\boldsymbol{H} = C(0,0,\omega),$$

$$-\tfrac{1}{2}S_{\mu\nu}S_{\mu\nu} = C^2(1-\rho^2\omega^2)/\rho^2 = C^2\chi^2, \qquad (8)$$

and the $S_{\mu\nu}$ do satisfy Maxwell's equations $\partial_\mu S_{\mu\nu} = j_\nu$, where j corresponds to the current of the "quarks" running around the cylindrical surface of radius L with density C per unit area. Note that the dielectric coefficient varies from ∞ to 0 between the middle and the ends of the string.

What does this abelian theory have to do with QCD? This is the next question which we will address. Qualitatively speaking first, the answer turns out to be as follows. In a non-abelian gauge theory without symmetry breaking, one can meaningfully introduce external sources to the extent of specifying their classical trajectories, as is usually done in terms of the Wilson loop. The energy of the system will then be a functional of these trajectories. An effective abelian gauge theory is so constructed as to give the same energy as in the original theory.

As was shown elsewhere [6], it is convenient to let the non-abelian gauge field $G_{\mu\nu}$ couple to an external trajectory indirectly through a Dirac sheet attached to the latter. The internal index of the source is carried by an auxiliary field which is defined over the sheet and to be integrated over, together with the gauge field variables. We will slightly generalize this procedure by using the fields $F_{\mu\nu}$ and $S_{\mu\nu}$ instead of singular Dirac sheets.

Thus consider the lagrangian

$$L = (1/4g^2)\,\mathrm{tr}\,G_{\mu\nu}G_{\mu\nu} - \tfrac{1}{2}S_{\mu\nu}(\mathcal{F}_{\mu\nu} - F_{\mu\nu}) + B_\mu j_\mu,$$

$$\mathcal{F}_{\mu\nu} = \partial_\mu(\tfrac{1}{2}\mathrm{i}\, u^+\overleftrightarrow{D}_\nu u) - \partial_\nu(\tfrac{1}{2}\mathrm{i}\, u^+\overleftrightarrow{D}_\mu u)$$

$$= u^+G_{\mu\nu}u + \mathrm{i}D_\mu u^+D_\nu u - \mathrm{i}D_\nu u^+D_\mu u$$

$$+ \tfrac{1}{2}\mathrm{i}\, u^+[\partial_\mu, \partial_\nu]\, u - \tfrac{1}{2}\mathrm{i}[\partial_\mu, \partial_\nu]\, u^+u,$$

$$D_\mu = \partial_\mu - \mathrm{i}A_\mu, \tag{9}$$

where $G_{\mu\nu}$ and A_μ are SU(n) matrices, and u and u^+ are scalar fields in the fundamental representation. The last two terms of $\mathcal{F}_{\mu\nu}$ are zero if u and u^+ are topologically nonsingular, which we will assume for the time being. As $S_{\mu\nu}$ and $F_{\mu\nu}$ can be trivially integrated out, eq. (9) is actually equivalent to the direct coupling $\frac{1}{2}u^+\overleftrightarrow{D}_\mu u j_\mu$ of A_μ to j_μ, so that [‡1]

$$D_\mu G_{\mu\nu} = j_\nu\phi, \quad \phi_{\alpha\beta} = u_\alpha u^+_\beta - n^{-1}\delta_{\alpha\beta}u_\gamma u^+_\gamma. \tag{10}$$

Suppose now one reverses the order of integration of eq. (9), and first integrates over the non-abelian variables A_μ, u and u^+ for fixed $S_{\mu\nu}$ and B_μ. Since u and u^+ are dummy fields except on the source, one may conveniently choose their nature and domain to make the problem tractable and well defined. One will obtain as a result an effective action which depends on $S_{\mu\nu}$. If this action turns out to be representable by a local function, then the latter will play the role of f postulated in eq. (5).

It is useful to take up first a two-dimensional problem, i.e., a flat sheet with all other dimensions dropped. $S_{\mu\nu}$ and $G_{\mu\nu}$ have only one component $S_{12} = S$, $G_{12} = G$, respectively. The field equations (in the source-free region) read

$$\bar{D}_\mu\bar{G} = 0 \tag{11}$$

where the bar is used to indicate on-shell (stationary) quantities. Now consider the following constraints for ϕ:

$$\bar{D}_\mu\phi = 0, \quad [\bar{D}_1, \bar{D}_2]\,\phi = [\bar{G}, \phi] = 0, \tag{12a, b}$$

where ϕ is defined by eq. (10) and with $\bar{u}u = 1$. Eq. (12a) is a generalization of the current conservation along the boundary of S. Eq. (12b) is a secondary constraint following from eq. (12a), and together they form a closed set because of eq. (11). Actually the converse (b → a) is also true if the matrix G is nondegenerate. This is because, by the definition of ϕ, any variation $D_\mu\phi$ of ϕ amounts to an SU(n) transformation. One may therefore regard eqs. (13) and (14) as essentially equivalent (except for cases of measure zero in functional integration). A consequence of eq. (12a) is to eliminate the terms $D_\mu u^+D_\nu u - D_\nu u^+D_\mu u$ in the lagrangian (9), as may be verified easily. The quantum action is thus given by

$$Z = \int \exp\Big(-\sum L\Delta\Big)\mathcal{D}G\,\mathcal{D}\phi\,\mathcal{D}S\,\mathcal{D}B,$$

$$L = (1/2g^2)\,\mathrm{tr}\, G^2 - \mathrm{i}S\,\mathrm{tr}(G\phi) + \mathrm{i}SF + \mathrm{i}j_\mu B_\mu, \tag{13a}$$

$$[G, \phi] = 0, \tag{13b}$$

choosing G as the integration variable instead of A_μ. The factor i in this euclidean form is of the usual origin, all other quantities being defined as real. The G and ϕ integrations are to be done in each volume element Δ under the constraint (13b), an off-shell generalization of (12b). [One may also carry out the integration in two steps, first with respect to fluctuations of G around $\bar{G}$, and then with respect to $\bar{G}$ under eq. (12b).] Since first-order terms will vanish for symmetry reasons, one may resort to perturbation expansion and reduce the G-dependent part of Z to

$$\exp\Big(-\sum\frac{1}{2g^2}\,\mathrm{tr}\, G^2\Delta - \tfrac{1}{2}S^2(\mathrm{tr}\, G\phi)^2\Delta^2\Big)\mathcal{D}G\,\mathcal{D}\phi, \tag{14}$$

for sufficiently small Δ. This leads to an effective S-dependent action

$$\exp\Big(-\sum CS^2\Delta\Big) = \exp\Big(-\int CS^2\, \mathrm{d}^2x\Big),$$

$$C = \tfrac{1}{2}\langle \mathrm{tr}(G\phi)^2\Delta\rangle_{G,\phi} = kg^2, \tag{15}$$

with some constant k, as is obvious from eq. (14).

In the case of SU(2), for example, G and ϕ are isovectors which are parallel or antiparallel according to eq. (12). Writing $G = \boldsymbol{G}\cdot\boldsymbol{\tau}/\sqrt{2}$, $\boldsymbol{\phi} = \boldsymbol{\phi}\cdot\boldsymbol{\tau}/2$, $\boldsymbol{\phi}^2 = 1$, we have

$$C = \tfrac{1}{4}\langle G^2\Delta\rangle = \tfrac{3}{4}g^2. \tag{16}$$

Applied to a single current loop, this gives the correct

‡1 The relation between this coupling and the Wilson loop has been discussed by Wadia and Hosotani [7,8] and in ref. [9]. The u and u^+ may be either classical or Grassmann numbers.

area law [10] since $S^2 = 1$ in the interior of the loop, and zero outside [‡2].

The above results, eq. (13) etc., have been derived on the basis of eq. (12), which works only in two dimensions. In higher dimensions, the field equations do not guarantee that these conditions are consistent and closed. Nevertheless, it is possible to argue that the basic procedure is still valid, if only approximately.

Assume again that $S_{\mu\nu}$ is a sheet field. As remarked before, any variation $D_\mu\phi$ of ϕ is an SU(n) transformation, so one may write

$$D_\mu\phi = -\mathrm{i}[v_\mu, \phi],$$

or

$$\bar{D}_\mu\phi = 0, \quad \bar{D}_\mu = \partial_\mu - \mathrm{i}\bar{A}_\mu, \quad \bar{A}_\mu \equiv A_\mu - v_\mu, \qquad (17)$$

with some v_μ. Restricted to the subspace $S_{\mu\nu}$, this leads to an analog of eq. (12).

$$S_{\mu\nu}\bar{D}_\nu\phi = 0, \quad [\bar{G}, \phi] = 0, \quad \bar{G} \equiv \tfrac{1}{2}S_{\mu\nu}\bar{G}_{\mu\nu}/|S|. \quad (18)$$

All higher derivatives of $\bar{G}$ will also commute with ϕ. Return now to the lagrangian (9). First fix $\bar{A}_\mu$ and ϕ, where $\bar{A}_\mu$ has components only in $S_{\mu\nu}$. Integrate over the four-vector v_μ, and then integrate over $\bar{A}_\mu$ and ϕ subject to eq. (18). The result of the v-integration will be, in the first approximation, just to renormalize $\bar{A}$ and g^2. Implied here is an assumption that large fluctuations away from $\bar{A}_\mu$ are not important. The next integration over $\bar{A}_\mu$ and ϕ proceeds as before. Although $\bar{A}_\mu$ propagates in four dimensions instead of two, the formula (15) remains unchanged, where Δ should be read as a four-volume element. Thus the effective abelian lagrangian is given by

$$L_{\mathrm{eff}} = \tfrac{1}{4}kg^2(S)S_{\mu\nu}S_{\mu\nu} - \tfrac{1}{2}S_{\mu\nu}F_{\mu\nu} - j_\mu B_\mu. \qquad (19)$$

Here the running scale parameter of the coupling constant has been identified with $S = |S_{\mu\nu}|$, which has a dimension (mass)^2. This seems reasonable if S is slowly varying so that a local scale can be defined. In general, however, one expects L_{eff} to be nonlocal. According to eq. (19), one sees that near a source where S is large the usual perturbative results for $g^2(S)$ will apply,

$$g^2(S) \propto 1/\ln(S/\Lambda^2), \qquad (20)$$

corresponding to a weakly modified Maxwell equation [‡3]. On the other hand, the strong coupling regime sets in for small S, or large distances. If $g^2(S)$ behaves like

$$g^2(S) \sim 1/S, \qquad (21)$$

then $f(X) \sim \sqrt{X}$ in the notation of eq. (5), which precisely corresponds to the string system. Eq. (21) is consistent with the hamiltonian strong coupling theory [11] if S is related to the lattice size a as $S \sim 1/a^2$. It is not consistent, however, with the euclidean lattice theory relation $g^2 \sim \exp(\sigma a^2)$ [12].

A few comments are in order. It was postulated after eq. (9) that u and u^+ are nonsingular, so their derivatives commute. This is equivalent to assuming ϕ to be free of monopole-type singularities: if ϕ becomes singular along a line, the phase of the corresponding u has to become singular on a (Dirac) sheet bounded by the former. There may be no a priori reason to exclude monopole singularities of ϕ. If so, one arrives at a picture that the quarks have their own strings, or the smeared string field $S_{\mu\nu}$, whereas the ϕ-monopoles have their Dirac strings represented by the singularities of u [6]. Indeed, the singular sheet field $\sigma_{\mu\nu}$ for the latter is given by

$$\sigma^*_{\mu\nu} = \tfrac{1}{2}\mathrm{i}\,(u^+[\partial_\mu, \partial_\nu]u - [\partial_\mu, \partial_\nu]u^+u),$$

$$\partial_\mu\sigma_{\mu\nu} = k_\nu, \qquad (22)$$

where k_ν is the monopole current. Thus eq. (9) contains a sheet–sheet interaction between quarks and monopoles, which manifest itself in the form of quantized vortices on the sheet $S_{\mu\nu}$. In a complementary picture, one can imagine the Dirac sheet stretched inside a magnetic current loop being pierced by a quark string. Such an interaction will cause additional contributions to the S-dependence of L_{eff}. Although these topological effects are expressed through u and u^+, they are intrinsically tied up with those of the gauge field $G_{\mu\nu}$ because of eq. (12). It remains to be seen how the determination of L_{eff} can be done in a more precise way.

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[‡2] The situation is more subtle in the case of multiple loops, and remains to be investigated.

[‡3] This is similar, but not identical, to the equations discussed in ref. [13].

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