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COMMUTATOR ANOMALY FOR THE GAUSS LAW CONSTRAINT OPERATOR

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A formula useful for evaluating the commutator anomaly for the Gauss law constraint operator is presented. This formula makes the perturbative calculation of the anomaly much simpler than previous attempts. Furthermore, the commutator anomaly is shown to be derived directly from the non-abelian chiral anomaly and the perturbative calculation is connected to the argument based on cohomology.

1. Introduction

Progress has been made recently in the understanding of anomalies from the viewpoint of cohomology theory [1], [2]. This theory is based on an operation δ which satisfies $\delta^2 = 0$. If a certain quantity $\alpha_n(A; g_1, \ldots, g_n)$ should satisfy $\delta\alpha_n = 0$ in order for α_n to be consistently defined, then α_n is given as a sum of a general solution $\delta\beta_{n-1}$ and one of special solutions $\gamma_n^{(i)}$ $(i = 1, 2, \ldots)$ namely,

$$\alpha_n = \delta\beta_{n-1} + \gamma_n^{(i)} . \tag{1.1}$$

It is said that $\gamma_n^{(i)}$ belongs to ith cohomology class of n-cohomology group and α_n satisfying $\delta\alpha_n = 0$ is an n-cocycle while $\delta\beta_{n-1}$ is an n-coboundary.

The gauged Wess-Zumino term $\alpha_1(A; g)$ [3] is a finite integration of the non-abelian chiral anomaly [4] and is defined by

$$Z[A]U_g = Z[A]\,e^{i\alpha_1(A;g)}, \tag{1.2}$$

where U_g is a gauge transformation operator, corresponding to a group element g, and $Z[A]$ is the fermionic determinant,

$$Z[A] = \det\left(\not\partial + \not A \frac{1-\gamma_5}{2} \right). \tag{1.3}$$

Group property of the gauge transformation is ensured by the fact that α_1 is a 1-cocycle, that is, $\delta\alpha_1 = 0$.

Faddeev [2] considered that U_g satisfies the following equation, when applied on a wave functional $\Psi[A]$ of gauge theories

$$\Psi[A]U_{g_1}U_{g_2} = \Psi[A]U_{g_1 \cdot g_2} e^{i\alpha_2(A;\, g_1, g_2)}. \tag{1.4}$$

Since the gauge transformation is assumed not to violate the associative law, α_2 satisfies $\delta\alpha_2 = 0$, that is, α_2 is a 2-cocycle. For infinitesimal gauge transformations, eq. (1.4) becomes the commutator anomaly for the Gauss law constraint operator, since it generates gauge transformations on the wave functional in the Weyl gauge $A_0^a = 0$. Faddeev has given a candidate for the infinitesimal 2-cocycle α_2 which is linear in the gauge field $A(x)$.

The non-abelian chiral anomaly was found perturbatively [4], while Faddeev's anomaly was found by cohomological argument. The latter had not been confirmed perturbatively. Then two of us (M.K. and A.S.) [5] tried to derive Faddeev's anomaly in perturbation theory using the Björken-Johnson-Low (BJL) limit [6] and the Pauli-Villars regularization. We have shown that there exist no anomalies linear in $A(x)$ for the true Gauss law constraint operator $G^a = \tilde{G}^a - g\bar{\psi}\gamma^0 t^a \psi [\tilde{G}^a = D_i^{ab}(A)E^{ib}]$ whereas $\tilde{G}^a$ gives an anomaly identical to the form claimed by Faddeev. Later, Jo [7] reproduced our result, and furthermore he has estimated terms proportional to $A(x)^2$ and $A(x)^3$, giving a commutator anomaly for G^a in a different form from Faddeev's one. In fact, the Faddeev and Jo forms of anomaly belong to the same cohomology class as noted by Jo and independently by Fujiwara [8] in discussion of the current commutator anomaly in the non-linear σ model with the Wess-Zumino term. The form of anomaly that appears in a certain regularization scheme is not known from cohomology theory alone.

In the present paper we will show a systematic evaluation of the commutator anomaly in the Pauli-Villars regularization scheme, where we make full use of the Ward identity. In our method of computation, the relation between the non-abelian anomaly and the commutator anomaly becomes transparent, which has not been clearly understood so far*.

In sect. 2 we will present a basic formula which relates Green functions involving two Gauss law operators to Green functions with two insertions of the γ_5 vertex

* In abelian theory, there are earlier references (ref. [12]) discussing a relationship between the current commatators and the Adler anomaly.

into loops of regulator fields. This relation becomes simple in the BJL limit. This formula can be proved in any space-time dimension and to all orders of perturbation.

In sect. 3 we will evaluate the commutator anomaly at the one-loop level in the space-time dimension four, making use of the formula obtained in sect. 2. Not only the odd parity contribution but also the even part are computed. The even parity part of the commutator anomaly is shown to be exactly eliminated by adding to the action local counterterms which are determined so as to sweep out the even parity part of the non-abelian chiral anomaly. The odd parity part of the commutator anomaly coincides with that of Jo [7]. Our method of calculation, however, is much simpler than his and is useful for further generalizations.

Next, we intend to derive the commutator anomaly for the Gauss law constraint operator directly from knowledge of the non-abelian chiral anomaly. We have successfully identified the commutator anomaly with $\omega_3^2(A, w)$ which is fixed by the non-abelian anomaly $\omega_4^1(A, w)$. The commutator anomaly so obtained agrees with the result obtained by the perturbative calculation in sect. 3. This is the content of sect. 4.

Thus we can understand that the commutator anomaly is a different manifestation of the non-abelian anomaly. If the latter is cancelled somehow, then the former disappears.

2. A basic formula to evaluate the commutator anomaly for the Gauss law constraint operator

We consider a system of chiral fermions $\psi(x)$ coupled to non-abelian gauge fields $A_\mu^a(x)$, which is described by the following lagrangian,

$$\mathscr{L} = \bar{\psi}(x) i\gamma^\mu \left(\partial_\mu - igt^a A_\mu^a \frac{1-\gamma_5}{2} \right) \psi(x) - \tfrac{1}{4}(F_{\mu\nu}^a)^2, \tag{2.1}$$

where we follow the convention of Björken and Drell [9] for the γ matrices and t^a is hermitian, satisfying $[t^a, t^b] = if_{abc}t^c$. Sometimes we will use the following notations: $A_\mu = -igt^a A_\mu^a$, $A_{\mu\mathrm{L}} = A_\mu(1-\gamma_5)/2$, $A = A_\mu \mathrm{d}x^\mu$, $F_{\mu\nu} = -igt^a F_{\mu\nu}^a$, and $F = \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. The Gauss law constraint operator is given by

$$\begin{aligned} G^a(x) &= \partial^i E_i^a + gf_{abc}A^{ib}E_i^c - g\bar{\psi}\gamma^0 t^a \psi \\ &= \tilde{G}^a(x) - g\bar{\psi}\gamma^0 t^a \psi, \end{aligned} \tag{2.2}$$

where $E_i^a = \dot{A}_i^a$ $(i = 1,2,3)$. In the Weyl gauge, $n^\mu A_\mu^a = 0$ $[n^\mu = (1,0,0,0)]$, physical states are assumed to vanish if any G^a operate on them.

Let us define the commutator anomaly $\mathscr{A}^{ab}(x, y)$ for the Gauss law constraint operator by

$$\left[G^a(x), G^b(y)\right]_{x^0=y^0} = -igf_{abc}G^c(x)\delta^{(D-1)}(x-y) + \mathscr{A}^{ab}(x, y). \quad (2.3)$$

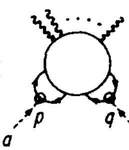
This is extracted from the Feynman amplitudes by taking the BJL limit [6] as follows:

$$\begin{aligned}
&\lim_{p^0-q^0\to\infty}(p^0-q^0)\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ldots\exp(-i(px+gy+rz+\cdots))\\
&\quad\times\langle 0|TG^a(x)G^b(y)A^c_\mu(z)\cdots|0\rangle\\
&=\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ldots\exp(-i(px+qy+rz+\cdots))(-2i)\delta(x^0-y^0)\\
&\quad\times\langle 0|T\left[G^a(x), G^b(y)\right]A^c_\mu(z)\ldots|0\rangle. \quad (2.4)
\end{aligned}$$

In the Pauli-Villars regularization scheme $\mathscr{A}^{ab}(x, y)$ can be related to the BJL limit of the amplitudes with two insertions of the γ_5 vertex into loops of the regulator fields. The basic formula to be proved in this section is as follows:

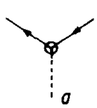
$$\begin{aligned}
&\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ldots\exp(-i(px+qy+rz+\cdots))(-2i)\delta(x^0-y^0)\\
&\quad\times\langle 0|T\mathscr{A}^{ab}(x, y)A^c_\mu(z)\ldots|0\rangle
\end{aligned}$$

$$=\lim_{p^0-q^0\to\infty}\lim_{M\to\infty}\frac{p^0-q^0}{p^0q^0}\times$$



, (2.5)

where the blob stands for the sum of Feynman graphs with an arbitrary number of external fields. Two lines with isospin indices a and b are attached through the $M\gamma_5$ vertices to the regulator fields with a mass M,



$$= gM\gamma_5 t^a. \quad (2.6)$$

If we denote the regulator field by $\Psi(x)$, then eq. (2.5) can be rewritten as

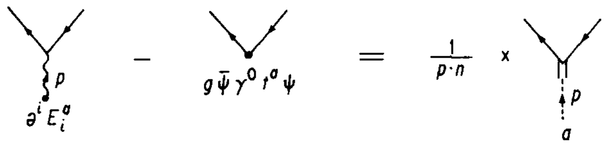
$$\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ldots\exp(-i(px+qy+rz+\cdots))$$

$$\times(-2i)\delta(x^0-y^0)\langle 0|T\mathscr{A}^{ab}(x,y)A_\mu^c(z)\ldots|0\rangle$$

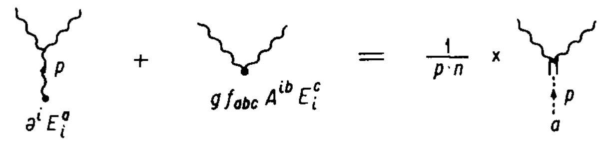
$$=\lim_{p^0-q^0\to\infty}\lim_{M\to\infty}\frac{p^0-q^0}{p^0q^0}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ldots\exp(-i(px+qy-rz+\cdots))$$

$$\times g^2\langle 0|T\overline{\Psi}M\gamma_5t^a\Psi(x)\overline{\Psi}M\gamma_5t^b\Psi(y)A_\mu^c(z)\ldots|0\rangle. \qquad (2.7)$$

Now we give a diagrammatical proof of eq. (2.5). First we notice the following identities at the tree level,

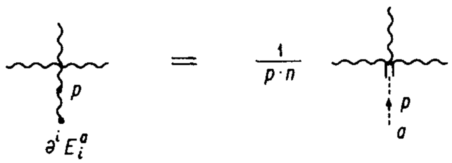


(2.8a)

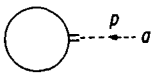


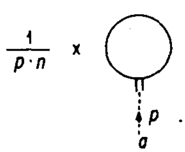
(2.8b)

and



(2.8c)

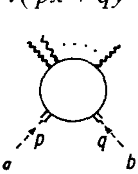
where  represents multiplication of ip_μ, after amputating the external leg of the gauge field with a momentum p and an isospin a. These identities (2.8a) through (2.8c) show that insertion of the Gauss law operator $G^a(p)$ is reduced to



Therefore, we have

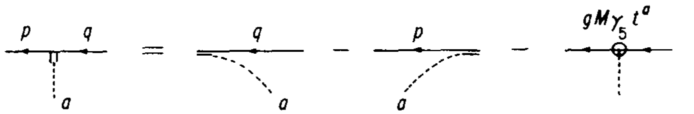
$$\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ldots\exp(-i(px+qy+rz+\cdots))\langle 0|TG^a(x)G^b(y)A_\mu^c(z)\ldots|0\rangle$$

$$=\frac{1}{p\cdot n}\frac{1}{q\cdot n}\times$$

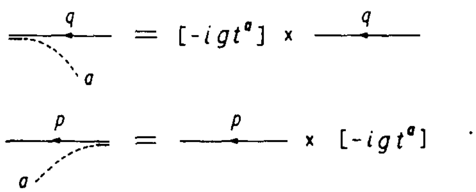


(2.9)

Next we use the Ward identities in the gauge theories in order to transform the right hand side of eq. (2.9). In our case of chiral theory, however, the Ward identities give anomalies which should be taken carefully. Since we have regularized Feynman amplitudes by introducing regulator fields, we can shift loop momenta freely. Naive Ward identities, however, are broken through the regulator field, namely,



where



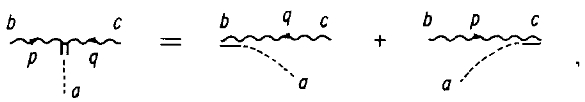
(2.10a)

As is easy to understand, eq. (2.10a) represents an algebraic identity,

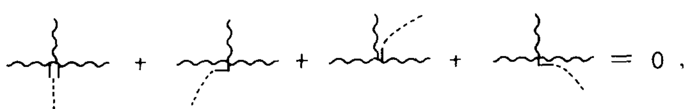
$$\frac{i}{\not p - M} igt^a\gamma^\mu(p-q)_\mu \frac{1-\gamma_5}{2}\frac{i}{\not q - M}$$

$$= [-igt^a]\frac{1-\gamma_5}{2}\frac{i}{\not q - M} - \frac{i}{\not p - M}\frac{1+\gamma_5}{2}[-igt^a] - \frac{i}{\not p - M} gt^a M\gamma_5 \frac{i}{\not q - M}.$$

Other tree identities are the same as those of the vector gauge theories in the Weyl gauge $A_0^a = 0$:

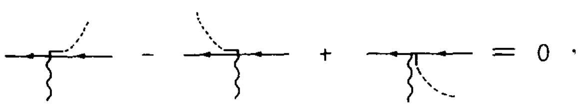


(2.10b)



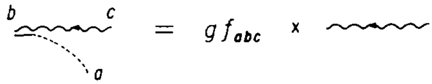
(2.10c)

(2.10d)

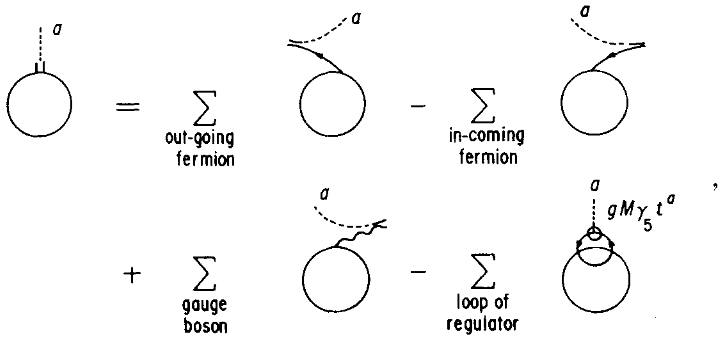


(2.10e)

where



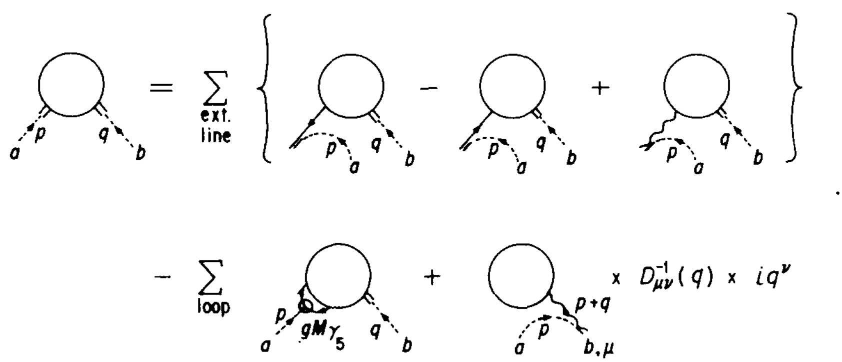
Combining tree identities (2.10a) through (2.10e), we obtain the following Ward identity valid in all orders,



(2.11)

where the last term represents the non-abelian anomaly [10].

The right-hand side of eq. (2.9) contains two contractions with external momentum. Applying the above Ward identity to one of the contractions, we get



(2.12)

In the last term of eq. (2.12), we intentionally attached an external gauge boson leg which had once been amputated and contracted with the momentum q. This term is further reduced to

$$-f_{abc}(q\cdot n)\frac{1}{(p+q)\cdot n}\times$$ 

(2.13)

[See eq. (3.11) of ref. [10].] In this procedure, we have to use the axial gauge propagator with a parameter α.

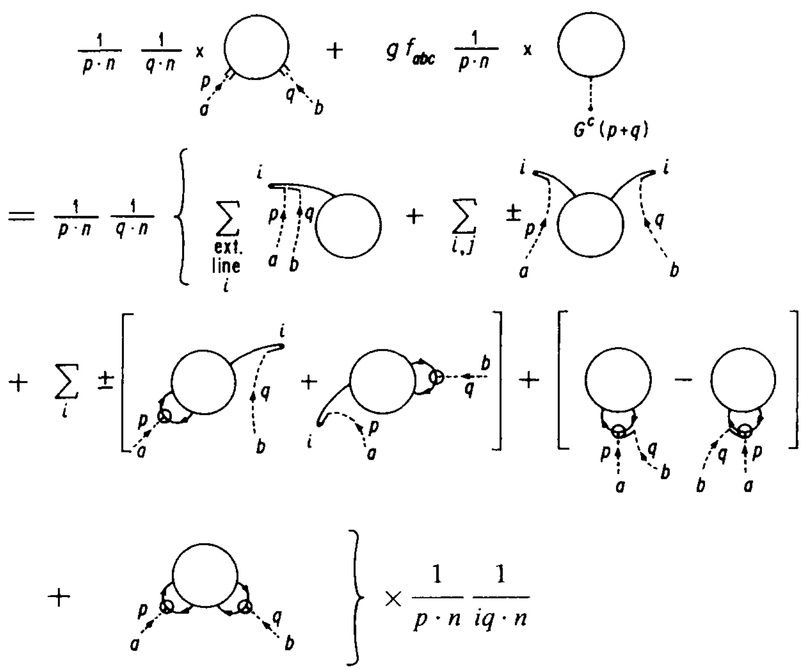
$$D_{\mu\nu}(q) = \frac{-i}{q^2 + i\varepsilon}\left[g_{\mu\nu} - \frac{q_\mu n_\nu + q_\nu n_\mu}{q\cdot n} + \frac{q_\mu q_\nu}{(q\cdot n)^2}\right] - i\alpha\frac{q_\mu q_\nu}{(q\cdot n)^2},$$

since we need the inverse of the propagator. For non-vanishing α we have

$$D^{-1}_{\mu\nu}(q) = i\left[q^2 g_{\mu\nu} - q_\mu q_\nu + \frac{1}{\alpha} n_\mu n_\nu\right].$$

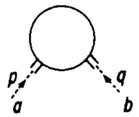
The final result (2.13), however, does not depend on α and therefore it is valid also in the limit of $\alpha \to 0$. If the reader is not content with the above argument, he is able to arrive at the same result treating the two cases that the external gauge boson leg with a momentum q is attached to a fermion line and to a gauge boson line, separately. In the BJL limit, (2.13) gives the canonical term of the commutator, the 1st term of the right-hand side of eq. (2.3).

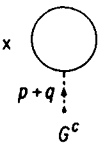
Now we apply the Ward identities to the other contraction in the right-hand side of eq. (2.12) and we obtain



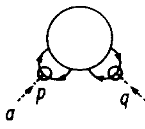
(2.14)

where summation should be taken over all external lines, with a + sign for gauge bosons and outgoing fermions, and with a − sign for incoming fermions. In the right-hand side of eq. (2.14), 1st and 4th terms depend on $(p+q)$, and 2nd and 3rd terms have extra propagators with a momentum linear in p or q, so that these four terms vanish after taking the BJL limit of $(p^0-q^0)\to\infty$. Therefore, only the last term survives after the BJL limit, namely,

$$\lim_{p^0-q^0\to\infty}\Bigg[\frac{p^0-q^0}{p^0q^0}$$


$$+\ g f_{abc}\frac{p^0-q^0}{p^0}\times$$


$$\Bigg]$$

$$=\lim_{p^0-q^0\to\infty}\ \lim_{M\to\infty}\frac{p^0-q^0}{p^0q^0}\times$$


$$,\qquad (2.15)$$

which completes our proof of eq. (2.5) or eq. (2.7).

At the end of this section two comments are in order:

(1) The formula (2.5) is valid for any space-time dimension. Therefore, it will be useful for calculation of the commutator anomalies in higher dimensional space-time.

(2) If we increase the number of gauge transformation operations $G^a(x)$, the number of $M\gamma_5$ insertions increases, so that generalization of this kind of formula (2.5) to higher order cocycles may also be possible.

3. Perturbative calculation of the commutator anomaly for the Gauss law constraint operator

In this section, we explicitly evaluate the right-hand side of eq. (2.5) at the one-loop level in the 4-dimensional space-time. Relevant diagrams are listed in fig. 1. Let us start with the triangle graphs, (A_1) and (A_2). The amplitude for the graph (A_1) reads

$$\Gamma_\mu^{(A_1)}(r,q,p)^{cba}=g^3\mathrm{tr}(t^ct^bt^a)\Gamma_\mu^{(A)}(r,q,p), \qquad (3.1a)$$

with

$$\Gamma_\mu^{(A)}(r,q,p)=-\sum_i c_iM_i^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma_\mu\frac{1-\gamma_5}{2}\frac{1}{\not k-\not r-M_i}\gamma_5\frac{1}{\not k+\not p-M_i}\gamma_5\frac{1}{\not k-M_i}\right], \qquad (3.1b)$$

where we have introduced two regulator fields $i=1$ and 2 to tame quadratic divergences. The coefficients c_i's are fixed by the conditions, $1+\Sigma_ic_i=0$ and $\Sigma_ic_iM_i^2=0$.

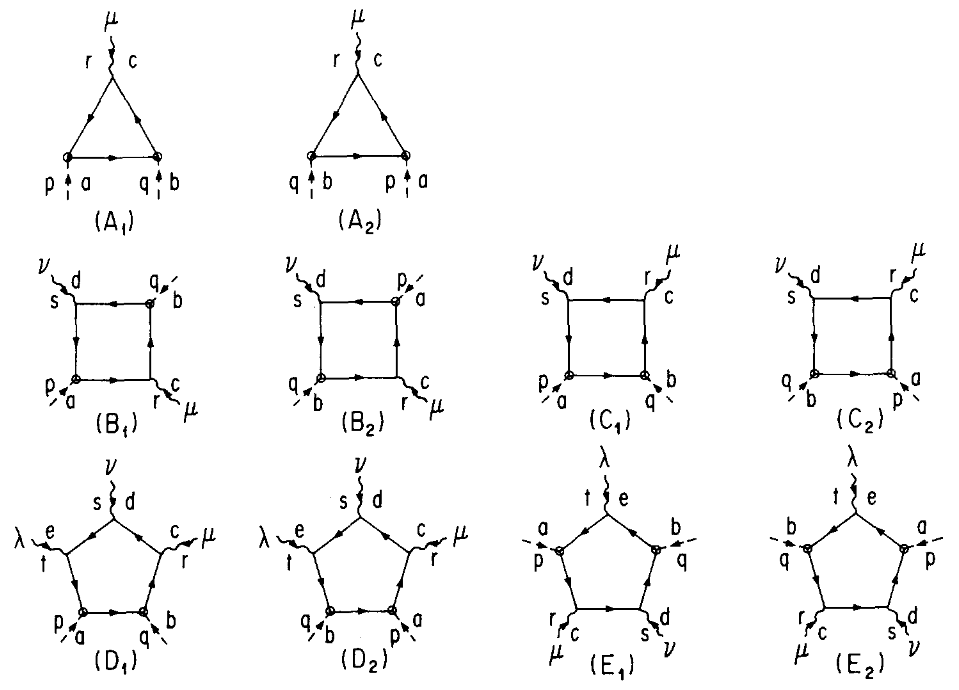


Fig. 1. (A): The triangle graphs contributing to the commutator anomaly. (B): The box diagrams with diagonal insertions of $M\gamma_5$ vertices. (C): The box diagrams with neighboring insertions of $M\gamma_5$ vertices (Two graphs are abbreviated, which are obtained by interchange of two external gauge boson lines.) (D): The pentagon diagrams with neighboring insertions of $M\gamma_5$ vertices (Ten graphs are abbreviated, which are obtained by permutation of three external gauge boson lines.) (E): The pentagon diagrams with alternate insertions of $M\gamma_5$ vertices (Ten graphs are abbreviated as in (D).)

The divergent part as $M \to \infty$ of the integral (3.1b) is $M^2 \times$ [const or $\ln M^2$] $\times$ [p_μ or q_μ], but it does not contribute to the anomaly. The reason is as follows: the Lorentz index μ is contracted with the gauge boson propagator $D_{\mu\mu'}(r)$ which is non-vanishing only for $\mu \neq 0$ in the Weyl gauge. Then $\Gamma_\mu(p, q)$ does not provide a factor p^0 or q^0 to kill off the factor $(p^0 - q^0)/p^0 q^0$ and to survive in the BJL limit. We frequently use this kind of reasoning in the following.

Third-order terms in the external momenta are finite as $M \to \infty$ and the relevant terms involving p^0 or q^0 are obtained as,

$$\Gamma_\mu^{(\mathrm{A})}(r, q, p) \underset{\mathrm{BJL}}{=} -\frac{i}{3(4\pi)^2}\left[q_\mu q^2 - p_\mu p^2 + 2r_\mu(r \cdot p)\right], \tag{3.2}$$

where $\underset{\mathrm{BJL}}{=}$ indicates that we have thrown away terms independent of $(p^0 - q^0)$.

Next we evaluate the box diagrams, (B) and (C). The amplitude for (B_1) is given by

$$\Gamma_{\nu\mu}^{(\mathrm{B}_1)}(s, q, r, p)^{dbca} = -g^4 \operatorname{tr}(t^d t^b t^c t^a)\Gamma_{\nu\mu}^{(\mathrm{B})}(s, q, r, p), \tag{3.3a}$$

with

$$\Gamma_{\nu\mu}^{(\mathrm{B})}(s,q,r,p) = -\sum_i c_i M_i^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma_\nu \frac{1-\gamma_5}{2} \frac{1}{\not k - \not s - M_i} \gamma_5 \frac{1}{\not k + \not p + \not r - M_i} \right.$$

$$\left. \times \gamma_\mu \frac{1-\gamma_5}{2} \frac{1}{\not k + \not p - M_i} \gamma_5 \frac{1}{\not k - M_i}\right]. \quad (3.3\mathrm{b})$$

After taking the trace of the γ matrices, finiteness of the integral is obvious and we obtain

$$\Gamma_{\nu\mu}^{(\mathrm{B})}(s,q,r,p) \underset{\mathrm{BJL}}{\sim} -\frac{i}{3(4\pi)^2}\left[g_{\mu\nu}(p\cdot q) - i\varepsilon_{\mu\nu\alpha\beta}(r+s)^\alpha q^\beta\right]. \quad (3.4)$$

The amplitude for (C_1) reads

$$\Gamma_{\nu\mu}^{(\mathrm{C}_1)}(s,r,q,p)^{dcba} = -g^4 \mathrm{tr}(t^d t^c t^b t^a) \Gamma_{\nu\mu}^{(\mathrm{C})}(s,r,q,p), \quad (3.5\mathrm{a})$$

with

$$\Gamma_{\nu\mu}^{(\mathrm{C})}(s,r,q,p) = -\sum_i c_i M_i^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma_\nu \frac{1-\gamma_5}{2} \frac{1}{\not k - \not s - M_i} \gamma_\mu \frac{1-\gamma_5}{2} \right.$$

$$\left. \times \frac{1}{\not k + \not p + \not q - M_i} \gamma_5 \frac{1}{\not k + \not p - M_i} \gamma_5 \frac{1}{\not k - M_i}\right]. \quad (3.5\mathrm{b})$$

The quadratic divergent terms proportional to $M^2 \times [\text{const or } \ln M^2] \times g_{\mu\nu}$ are independent of p^0 nor q^0 and they vanish in the BJL limit. Second-order terms in the external momenta are finite as $M \to \infty$ and the relevant terms in the BJL limit are obtained as

$$\Gamma_{\nu\mu}^{(\mathrm{C})}(s,r,q,p) \underset{\mathrm{BJL}}{\sim} -\frac{i}{3(4\pi)^2}\left[g_{\mu\nu} q\cdot(r-s) - i\varepsilon_{\mu\nu\alpha\beta}(r-s)^\alpha q^\beta\right]. \quad (3.6)$$

Next we come to the pentagon diagrams, (D) and (E). We notice that the amplitudes for the diagrams (E) vanish after taking the limit of $M \to \infty$. The finite part must be linear in the external momenta, but the amplitudes are at least quadratic after taking the trace of the γ matrices.

The amplitude for (D_1) is given by

$$\Gamma^{(D_1)}_{\lambda\nu\mu}(t,s,r,q,p)^{edcba} = g^5 \mathrm{tr}(t^e t^d t^c t^b t^a)\Gamma^{(D)}_{\lambda\nu\mu}(t,s,r,q,p), \tag{3.7a}$$

where

$$\Gamma^{(D)}_{\lambda\nu\mu}(t,s,r,q,p) = -\sum_i c_i M_i^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma_\lambda \frac{1-\gamma_5}{2} \frac{1}{\not k - \not t - M_i}\gamma_\nu \frac{1-\gamma_5}{2}\right.$$
$$\left.\times \frac{1}{\not k - \not t - \not s - M_i}\gamma_\mu \frac{1-\gamma_5}{2}\frac{1}{\not k + \not p + \not q - M_i}\gamma_5 \frac{1}{\not k + \not p - M_i}\gamma_5 \frac{1}{\not k - M_i}\right]. \tag{3.7b}$$

This yields the following term relevant in the BJL limit,

$$\Gamma^{(D)}_{\lambda\nu\mu}(t,s,r,q,p) \underset{\text{BJL}}{\sim} -\frac{1}{3(4\pi)^2}\varepsilon_{\mu\nu\lambda\alpha}q^\alpha. \tag{3.8}$$

The diagrams with more than 6 external lines vanish in the limit of $M \to \infty$. Here, we summarize the results obtained so far:

$$\Gamma^{(A)}_{\mu} \underset{\text{BJL}}{\sim} -\frac{i}{3(4\pi)^2}\left[q_\mu q^2 - p_\mu p^2 + 2r_\mu(r\cdot p)\right], \tag{3.9a}$$

$$\Gamma^{(B)}_{\nu\mu} \underset{\text{BJL}}{\sim} -\frac{i}{3(4\pi)^2}\left[g_{\mu\nu}(p\cdot q) - i\varepsilon_{\mu\nu\alpha\beta}(r+s)^\alpha q^\beta\right], \tag{3.9b}$$

$$\Gamma^{(C)}_{\nu\mu} \underset{\text{BJL}}{\sim} -\frac{i}{3(4\pi)^2}\left[g_{\mu\nu}q\cdot(r-s) - i\varepsilon_{\mu\nu\alpha\beta}(r-s)^\alpha q^\beta\right], \tag{3.9c}$$

$$\Gamma^{(D)}_{\lambda\nu\mu} \underset{\text{BJL}}{\sim} -\frac{1}{3(4\pi)^2}\varepsilon_{\mu\nu\lambda\alpha}q^\alpha, \tag{3.9d}$$

$$\Gamma^{(E)}_{\lambda\nu\mu} = 0. \tag{3.9e}$$

Now we perform the Laurent expansion of these amplitudes divided by p^0q^0, with respect to (p^0-q^0), and retain only terms proportional to $(p^0-q^0)^{-1}$. We discard constant or positive power terms if they exist, since they correspond to local terms in $x^0 - y^0$ of the Green function defined as a T^* product [6]. They do not exist in the Green function defined as a T product from which we get the equal-time commutator by taking the BJL limit. Taking into account the combinatorics, we obtain the anomaly $\mathscr{A}$ defined in eq. (2.5) as follows:

$$\mathscr{A}^{(\mathrm{A})} = \lim_{p^0 - q^0 \to \infty} \frac{i(p^0 - q^0)}{2p^0 q^0}\left[g^3 \mathrm{tr}(t^c t^b t^a)\Gamma_\mu^{(\mathrm{A})}(r,q,p) + (a \leftrightarrow b,\ p \leftrightarrow q)\right]$$

$$= -\frac{g^3}{48\pi^2} r^0 r_\mu \mathrm{tr}(t^c[t^b, t^a]) \tag{3.10a}$$

for the triangle graphs (A);

$$\mathscr{A}^{(\mathrm{B})} = \lim_{p^0 - q^0 \to \infty} \frac{i(p^0 - q^0)}{2p^0 q^0}\left[-g^4 \mathrm{tr}(t^d t^b t^c t^a)\Gamma_{\nu\mu}^{(\mathrm{B})}(s,q,r,p) + (a \leftrightarrow b,\ p \leftrightarrow q)\right]$$

$$= \frac{ig^4}{48\pi^2}\varepsilon_{0\nu\mu\alpha}(r+s)^\alpha \mathrm{tr}(t^d t^b t^c t^a - t^d t^a t^c t^a), \tag{3.10b}$$

$$\mathscr{A}^{(\mathrm{C})} = \lim_{p^0 - q^0 \to \infty} \frac{i(p^0 - q^0)}{2p^0 q^0}\Big[\big\{-g^4 \mathrm{tr}(t^d t^c t^b t^a)\Gamma_{\nu\mu}^{(\mathrm{C})}(s,r,q,p)$$

$$+ (p \leftrightarrow q,\ a \leftrightarrow b)\big\} + \{r \leftrightarrow s,\ \mu \leftrightarrow \nu,\ c \leftrightarrow d\}\Big]$$

$$= \frac{ig^4}{48\pi^2}\varepsilon_{0\nu\mu\alpha}(r-s)^\alpha \mathrm{tr}(\{t^d, t^c\}[t^b, t^a]) - \frac{g^4}{48\pi^2} g_{\mu\nu}(r-s)_0 \mathrm{tr}([t^d, t^c][t^b, t^a]) \tag{3.10c}$$

for the box diagrams (B) and (C);

$$\mathscr{A}^{(\mathrm{D})} = \lim_{p^0 - q^0 \to \infty} \frac{i(p^0 - q^0)}{2p^0 q^0}\Big[\big\{g^5 \mathrm{tr}(t^e t^d t^c t^b t^a)\Gamma_{\lambda\nu\mu}^{(\mathrm{D})}(t,s,r,q,p) + (p \leftrightarrow q,\ a \leftrightarrow b)\big\}$$

$$+ \text{permutation of } \{(rst), (\mu\nu\lambda), (cde)\}\Big]$$

$$= -\frac{ig^5}{48\pi^2}\varepsilon_{0\lambda\nu\mu}\mathrm{tr}\{(t^e t^d t^c + t^d t^c t^e + t^c t^e t^d - t^e t^c t^d - t^c t^d t^e - t^d t^e t^c)[t^b, t^a]\} \tag{3.10d}$$

for the pentagon diagrams (D).

By inverse Fourier transformation, we obtain the commutator anomaly as follows:

$$
\begin{aligned}
&\left[G^a(x), G^b(y)\right]_{x^0=y^0} + igf_{abc}G^c(x)\delta^{(3)}(x-y) \\
&= \frac{g^2}{48\pi^2}\varepsilon_{0\mu\nu\lambda}\Big[\mathrm{tr}[t^a, t^b](\partial_\mu A_\nu A_\lambda + A_\mu\partial_\nu A_\lambda + A_\mu A_\nu A_\lambda) \\
&\qquad + \mathrm{tr}\left\{t^a\partial_\mu(A_\nu t^b A_\lambda)\right\}\Big]\delta^{(3)}(x-y) \\
&\quad - \frac{g^2}{96\pi^2} igf_{abc}(\partial^i E_i^c + gf_{cde}A^{id}E_i^e)\delta^{(3)}(x-y). \qquad (3.11)
\end{aligned}
$$

The even parity part in the commutator anomaly (3.11) has not been evaluated in previous literature. On the other hand the odd parity part, proportional to $\varepsilon_{0\mu\nu\lambda}$, has been computed by Jo [7]. Our result agrees with that of Jo*, but our method of estimating the anomaly is much simpler than his. The basic formula given in the previous section simplified the computation. Our method is a generalization of the discussion given in the previous paper by two of us [5], where the identity (2.8a) was used to show that no linear term in $A(x)$ appears in the commutator anomaly if we take the Pauli-Villars regularization.

The form of the anomaly suggested by Faddeev and us is different, but they belong to the same cohomology class. This is understood as follows: multiplying the commutator by $u^a(x)$ and $v^b(y)$ and integrating it over x and y, we obtain the infinitesimal 2-cocycle defined by eq. (1.4). Faddeev's cocycle and our's differ only by an infinitesimal coboundary:

$$2i\alpha_2^{(\mathrm{Faddeev})} = -\frac{1}{24\pi^2}\int \mathrm{tr}(u\,\mathrm{d}v - v\,\mathrm{d}u)\,\mathrm{d}A, \qquad (3.12a)$$

$$2i\alpha_2^{(\mathrm{our's})} = -\frac{1}{48\pi^2}\int \mathrm{tr}\{[u,v](\mathrm{d}AA + A\,\mathrm{d}A + A^3) + u\,\mathrm{d}A\,vA - uAv\,\mathrm{d}A\}, \qquad (3.12b)$$

$$\alpha_2^{(\mathrm{Faddeev})} - \alpha_2^{(\mathrm{our's})} = \delta_u\beta(A,v) - \delta_v\beta(A,u) - \beta(A,[u,v]), \qquad (3.12c)$$

$$2i\beta(A,u) = \frac{1}{48\pi^2}\int \mathrm{tr}\,u(\mathrm{d}AA + A\,\mathrm{d}A + A^3), \qquad (3.12d)$$

* By comparison of our result with that of Jo, one must take into account the difference in the definition of $\varepsilon_{\mu\nu\lambda\rho}$ and t^a: $(\varepsilon)_{\mathrm{our's}} = -(\varepsilon)_{\mathrm{Jo's}}$, $(t^a)_{\mathrm{our's}} = +i(t^a)_{\mathrm{Jo's}}$. In addition, another (-1) should be multiplied, since Jo took the right-handed fermion instead of the left-handed fermion.

where $u = -iu^a t^a$. In eq. (3.12c), δ_u stands for the usual variation under an infinitesimal gauge transformation, and the totality of the right-hand side of eq. (3.12c) is rather an infinitesimal 2-coboundary.

At the end of this section, we show that the even parity part, proportional to $\tilde{G}^c$, can be eliminated by addition of local counterterms $I_C[A]$ to the action. We determine the form of local counterterms so as to eliminate the even parity part of the non-abelian chiral anomaly. In the Pauli-Villars regularization scheme, the vacuum expectation value of the covariant divergence of the current $j^a_\mu = \bar{\psi}\gamma_\mu((1-\gamma_5)/2)t^a\psi$ is given by

$$\begin{aligned}\langle D^{ab}_\mu(A) j^{b\mu}(x)\rangle_A &= -i\Lambda^2 \mathrm{tr}(t^a\partial^\mu A_\mu)\\ &\quad+\frac{i}{48\pi^2}\mathrm{tr}\left[t^a\left\{D_\mu\partial^\mu\partial^\nu A_\nu+\partial_\mu[F^{\mu\nu},A_\nu]+\partial_\mu(A^\nu A^\mu A_\nu)\right\}\right]\\ &\quad+\frac{1}{24\pi^2}\varepsilon^{\mu\nu\lambda\rho}\mathrm{tr}\left[t^a\partial_\mu\left(A_\nu\partial_\lambda A_\rho+\tfrac{1}{2}A_\nu A_\lambda A_\rho\right)\right],\end{aligned} \tag{3.13}$$

where $\langle\quad\rangle_A$ denotes the vacuum expectation value in the presence of the background gauge field A. The first term on the right-hand side of eq. (3.13) represents a quadratically divergent part. If we add the following local counterterms to the action,

$$\begin{aligned}I_C(A) \equiv \int \mathrm{d}^4x\Big[&-\tfrac{1}{2}\Lambda^2\mathrm{tr}(A_\mu A^\mu)\\ &-\frac{1}{96\pi^2}\mathrm{tr}\left\{(\partial_\mu A^\mu)^2+F^{\mu\nu}[A_\mu,A_\nu]-\tfrac{1}{2}A_\mu A_\nu A^\mu A^\nu\right\}\Big],\end{aligned} \tag{3.14}$$

then the definition of the current is modified so that the corresponding chiral anomaly contains only the odd parity part, that is

$$\begin{aligned}\delta_u(W[A]+I_C[A]) &= -\int \mathrm{d}^4x\, u^a D^{ab}_\mu(A)\frac{\delta}{\delta A^b_\mu}(W[A]+I_C[A])\\ &= -\frac{i}{24\pi^2}\int\omega^1_4(A,u),\end{aligned} \tag{3.15}$$

where $W[A]$ is the logarithm of the fermionic determinant multiplied by $-i$. We used the notation of Zumino [1] which will be explained in the next section.

The Gauss law operator is modified by addition of the action $I_C[A]$ as follows:

$$G^a \to \hat{G}^a = G^a + \Delta G^a, \tag{3.16a}$$

where

$$\Delta G^{a} = \frac{g^{2}}{96\pi^{2}}\left(\partial^{i}E_{i}^{a} - g f_{abc} A^{ib} E_{i}^{c} \right). \tag{3.16b}$$

The commutation relation of E_i^a with A_i^a is not changed. We can confirm that the even parity part is dropped from the commutator anomaly for $\hat{G}^a$.

4. Derivation of the commutator anomaly from the non-abelian chiral anomaly

In the previous section, we have explicitly evaluated the Feynman amplitudes with two $M\gamma_5$ insertions to obtain the commutator anomaly. It is a straightforward calculation, but still lacks an understanding of the reason why the final result so obtained is connected to a geometrical quantity, a 2-cocycle $\int\omega_3^2$. The purpose of this section is to clarify this point. We will show that the commutator anomaly can be obtained without explicit evaluation of the amplitudes, if the non-abelian chiral anomaly is computed explicitly in the perturbation theory and is known to be ω_4^1 + [even parity part]. Therefore, we can say that the commutator anomaly for the Gauss law constraint operator is a different manifestation of the non-abelian chiral anomaly and that if the latter is cancelled in a certain model, the former is also cancelled in that model.

Let us consider the regularized fermionic determinant in the background gauge field A,

$$\mathrm{e}^{iW[A,V]} = \int \mathrm{D}\psi\, \mathrm{D}\bar{\psi}\, \mathrm{D}\Psi\, \mathrm{D}\bar{\Psi}\, \mathrm{e}^{i\int \mathrm{d}x \mathcal{L}_{\mathrm{F}}}, \tag{4.1a}$$

$$\mathcal{L}_{\mathrm{F}} = \bar{\psi} i(\partial\!\!\!/ + A\!\!\!/_{\mathrm{L}})\psi + \bar{\Psi} i(\partial\!\!\!/ + A\!\!\!/_{\mathrm{L}})\Psi - M\bar{\Psi}U\Psi, \tag{4.1b}$$

where ψ represents a chiral fermion and Ψ stands for the Pauli-Villars regulator field. A unitary matrix U is defined by $U = \mathrm{e}^{\gamma_5 u}$ which serves as a source field to generate $M\gamma_5$ vertices. Meanwhile V is defined by $V = \mathrm{e}^{-u}$. For convenience, we introduce a real parameter τ in the following way,

$$U_\tau = \mathrm{e}^{\tau\gamma_5 u}, \qquad V_\tau = \mathrm{e}^{-\tau u}.$$

By noting that the functional integral (4.1a) is invariant under the change of integration variables, $\psi_{\mathrm{L}}' = \mathrm{e}^{-u}\psi_{\mathrm{L}}$, $\Psi_{\mathrm{L}}' = \mathrm{e}^{-u}\Psi_{\mathrm{L}}$, we obtain the following identity,

$$W[A,V] = W\left[A^{V^\dagger}, 1\right], \tag{4.2}$$

where $A^{V^\dagger} = VAV^\dagger + V\mathrm{d}V^\dagger$ is a gauge transform of A. This identity implies that amplitudes with two $M\gamma_5$ insertions in the loop of regulator fields are related to those with two gauge transformations applied on the external gauge fields instead of $M\gamma_5$ insertions. Actually, on the one hand we have

$$\partial_\tau W[A, V_\tau]|_{\tau=0} = -\int \mathrm{d}x \langle \overline{\Psi} M \gamma_5 u \Psi(x) \rangle_A , \tag{4.3a}$$

$$\begin{aligned}\partial_\tau^2 W[A, V_\tau]|_{\tau=0} = &-\int \mathrm{d}x \langle \overline{\Psi} M u^2 \Psi(x) \rangle_A \\ &+ i \int \mathrm{d}x\, \mathrm{d}y \langle \overline{\Psi} M \gamma_5 u \Psi(x) \overline{\Psi} M \gamma_5 u \Psi(y) \rangle_A \\ &- i \left\{ \int \mathrm{d}x \langle \overline{\Psi} M \gamma_5 u \Psi(x) \rangle_A \right\}^2 ,\end{aligned} \tag{4.3b}$$

where the 3rd term subtracts the disconnected part from the 2nd term; on the other hand, starting from $W[A^{V_\tau^\dagger}, 1]$, we obtain another expression for (4.3). Let us denote A_τ for $A^{V_\tau^\dagger}$ generated by a one-parameter family of gauge transformations. First we notice that the τ-derivative of the gauge field A_τ induces an "infinitesimal gauge transformation" as if $u(x)$ were an "infinitesimal" gauge parameter, that is,

$$\begin{aligned}\partial_\tau A_\tau &= \mathrm{d}u + [A_\tau, u] \\ &= \delta_u A_\tau .\end{aligned} \tag{4.4}$$

As is well known, the variation of $W[A]$ under an infinitesimal gauge transformation is nothing but the chiral anomaly, which we have explicitly evaluated in the previous section [see eq. (3.15)] and we obtain

$$\begin{aligned}\lim_{M\to\infty} \partial_\tau W[A_\tau, 1] &= \lim_{M\to\infty} \delta_u W[A_\tau, 1] \\ &= -\frac{i}{24\pi^2} \int \omega_4^1(A_\tau, u) - \delta_u I_\mathrm{C}[A_\tau] ,\end{aligned} \tag{4.5}$$

where $\omega_4^1(A, u)$ is given by

$$\omega_4^1(A, u) = \mathrm{tr}\left[u\, \mathrm{d}\left(A\, \mathrm{d}A + \tfrac{1}{2} A^3\right)\right] . \tag{4.6}$$

By integrating this equation, we know the difference of $W[A]$ under a finite gauge transformation as

$$\begin{aligned}\lim_{M\to\infty} \{W[A_\tau, 1] - W[A, 1]\} = &-\frac{i}{24\pi^2} \int \{\omega_5^0(A_\tau) - \omega_5^0(A)\} \\ &- \{I_\mathrm{C}[A_\tau] - I_\mathrm{C}[A]\} ,\end{aligned} \tag{4.7}$$

where $\omega_5^0(A)$ is a Chern-Simons 5-form defined by

$$\omega_5^0(A) = \mathrm{tr}\left(AF^2 - \tfrac{1}{2}A^3F + \tfrac{1}{10}A^5\right), \tag{4.8}$$

and is related to $\omega_4^1(A,u)$ by $\delta_u\omega_5^0 = \mathrm{d}\omega_4^1$.

If we differentiate both sides of eq. (4.5) with respect to τ we obtain

$$\lim_{M\to\infty} \partial_\tau^2 W[A_\tau, 1]|_{\tau=0} = -\frac{i}{24\pi^2}\int \delta_u\omega_4^1(A,u) - \delta_u\delta_u I_C[A], \tag{4.9}$$

where u is a c-number field, and δ_u denotes a conventional gauge transformation. It turns out that $\delta_u\omega_4^1(A,u)$ is connected to ω_3^2 by a variant of the descent formula of Zumino and Stora [1]. Noting the following identity,

$$\omega_4^1(A_\tau, F_\tau, u) = \omega_4^1\left(A + \mathrm{d}V_\tau V_\tau^\dagger, F, u\right), \tag{4.10}$$

we have

$$\begin{aligned}\partial_\tau\omega_4^1(A_\tau, F_\tau, u)|_{\tau=0} &= \partial_\tau\omega_4^1(A + \tau\mathrm{d}u, F, u)|_{\tau=0} \\ &= \text{terms linear in } \mathrm{d}u \text{ of } \omega_4^1(A + \mathrm{d}u, F, u).\end{aligned} \tag{4.11}$$

The descent formula is derived from the following relation among the 5-forms,

$$\omega_5^0(A + w, F) = \omega_5^0(A,F) + \omega_4^1(A,F,w) + \omega_3^2(A,F,w) + \cdots \tag{4.12}$$

where $\omega_{5-n}^n(A,F,w)$ is the nth power term of the Taylor expansion of $\omega_5^0(A+w,F)$ with respect to w. Here w is assumed to be an arbitrary 1-form instead of the Faddeev-Popov ghost field. If we set $w = \mathrm{d}u + u\,\mathrm{d}\xi$ with $\xi(x_5)$, an arbitrary function of the extra coordinate x_5, then we have the following relations,

$$\begin{aligned}&\text{terms linear in } \mathrm{d}u \text{ of } \mathrm{d}\xi\,\omega_4^1(A + \mathrm{d}u, F, u) \\ &\quad= \text{terms proportional to } \mathrm{d}\xi\,\mathrm{d}u \text{ of } \omega_5^0(A + \mathrm{d}u + u\,\mathrm{d}\xi, F) \\ &\quad= \text{terms proportional to } \mathrm{d}\xi\,\mathrm{d}u \text{ of } \omega_3^2(A, F, \mathrm{d}u + u\,\mathrm{d}\xi) \\ &\quad= -\tfrac{1}{2}\mathrm{d}\xi\,\mathrm{tr}\left[(u\,\mathrm{d}u - \mathrm{d}u\,u)(A\,\mathrm{d}A + \mathrm{d}A\,A + A^3) + uA\,\mathrm{d}u\,\mathrm{d}A + \mathrm{d}uAu\,\mathrm{d}A\right].\end{aligned} \tag{4.13}$$

Here, ω_3^2 is obtained from eqs. (4.12) and (4.8) as

$$\omega_3^2(A,w) = -\tfrac{1}{2}\mathrm{tr}\left[w^2(A\,\mathrm{d}A + \mathrm{d}A\,A + A^3) + wAw\,\mathrm{d}A\right]. \tag{4.14}$$

Combining eqs. (4.11) and (4.13), we have

$$\partial_\tau \omega_4^1(A_\tau, F_\tau, u)|_{\tau=0} = -\tfrac{1}{2}\mathrm{tr}\big[(u\,\mathrm{d}u - \mathrm{d}u\,u)(A\,\mathrm{d}A + \mathrm{d}A\,A + A^3) + uA\,\mathrm{d}u\,\mathrm{d}A + \mathrm{d}u\,Au\,\mathrm{d}A\big]. \quad (4.15)$$

Inserting this expression into the 1st term of the right-hand side of eq. (4.9), we obtain

$$-\frac{i}{24\pi^2}\int \partial_\tau \omega_4^1(A_\tau, u)|_{\tau=0} = \int \mathrm{d}x\, \varepsilon^{\mu\nu\lambda\rho} u^a \partial_\mu u^b T^{ab}_{\nu\lambda\rho}(x), \quad (4.16)$$

where

$$T^{ab}_{\nu\lambda\rho}(x) = -\frac{i}{48\pi^2}\Big\{\mathrm{tr}[t^a, t^b](A_\nu\partial_\lambda A_\rho + \partial_\nu A_\lambda A_\rho + A_\nu A_\lambda A_\rho) - \mathrm{tr}\big(t^a A_\nu t^b \partial_\lambda A_\rho - t^b A_\nu t^a \partial_\lambda A_\rho\big)\Big\}. \quad (4.17)$$

As for the 2nd term on the right-hand side of eq. (4.9), we explicitly evaluated the variation of $I_C[A]$ under a gauge transformation and we get

$$-\delta_u\delta_u I_C[A] = \int \mathrm{d}x\Big\{\partial_\mu u^a \partial^\mu u^b S^{ab}_{\mathrm{I}} + u^a\partial_\mu u^b T^{ab,\mu}_{\mathrm{I}} + \partial^\mu u^a \partial^\nu u^b S^{ab}_{\mu\nu} + \Box u^a \Box u^b S^{ab}_{\mathrm{II}} + \Box u^a \partial_\mu u^b T^{ab,\mu}_{\mathrm{II}}\Big\}, \quad (4.18)$$

where

$$S^{ab}_{\mathrm{I}} = -\Lambda^2 \mathrm{tr}(t^a t^b) + \frac{1}{48\pi^2}\mathrm{tr}\big(t^a A_\mu t^b A^\mu\big),$$

$$T^{ab}_{\mathrm{I}\mu} = -\Lambda^2 \mathrm{tr}\big([t^a, t^b]A_\mu\big) + \frac{1}{48\pi^2}\mathrm{tr}[t^a, t^b] \times\Big\{\partial_\mu\partial_\nu A^\nu + [F_{\mu\nu}, A^\nu] + A_\nu A_\mu A^\nu - [\partial_\nu A^\nu, A^\mu] + \partial^\nu\big[F_{\nu\mu} + \tfrac{1}{2}(\partial_\nu A_\mu - \partial_\mu A_\nu)\big]\Big\},$$

$$S^{ab}_{\mu\nu} = \frac{1}{48\pi^2}\mathrm{tr}\big(2\{t^a, t^b\}A_\mu A_\nu - t^a A_\mu t^b A_\nu - t^b A_\mu t^a A_\nu\big),$$

$$S^{ab}_{\mathrm{II}} = -\frac{1}{48\pi^2}\mathrm{tr}(t^a t^b),$$

$$T^{ab}_{\mathrm{II}\mu} = \frac{1}{48\pi^2}\mathrm{tr}\big([t^a, t^b]A_\mu\big). \quad (4.19)$$

The resulting terms are divided into symmetric tensors, S, and antisymmetric tensors, T, under the exchange of a and b.

From eq. (4.3b), we obtain

$$\mathscr{A}=\lim_{p^0-q^0\to\infty}\lim_{M\to\infty}\frac{(-i)(p^0-q^0)}{2p^0q^0}g^2\int\mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-i(px+qy)}$$
$$\times\langle\overline{\Psi}M\gamma_5t^a\Psi(x)\overline{\Psi}M\gamma_5t^b\Psi(y)\rangle_A^{\text{connected}}$$
$$=\lim_{p^0-q^0\to\infty}\lim_{M\to\infty}\frac{(-i)(p^0-q^0)}{2p^0q^0}\frac{g^2}{2i}\int\mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-i(px+qy)}$$
$$\times\left\{\frac{\delta^2}{\delta u^a(x)\delta u^b(y)}\left(\partial_\tau^2W[A,V_\tau]|_{\tau=0}\right)+M\langle\overline{\Psi}\{t^a,t^b\}\Psi(x)\rangle_A\delta^{(4)}(x-y)\right\},\tag{4.20}$$

where the 2nd term of the integrand is local in x^0-y^0 and gives no contribution in the BJL limit. Combining eqs. (4.2), (4.9), (4.16), and (4.18), we have

$$\mathscr{A}=\lim_{p^0-q^0\to\infty}\lim_{M\to\infty}\frac{(-i)(p^0-q^0)}{2p^0q^0}\frac{g^2}{2i}\int\mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-i(px+qy)}$$
$$\times\frac{\delta^2}{\delta u^a(x)\delta u^b(y)}\left(\partial_\tau^2W[A_\tau,1]|_{\tau=0}\right)$$
$$=\lim_{p^0-q^0\to\infty}\frac{-(p^0-q^0)}{2p^0q^0}\frac{g^2}{2}\int\mathrm{d}x\,\mathrm{e}^{-i(p+q)x}$$
$$\times\left\{i(p_\mu-q_\mu)\left(\varepsilon^{\mu\nu\lambda\rho}T^{ab}_{\nu\lambda\rho}+T^{ab,\mu}_{\mathrm{I}}\right)(x)-2p\cdot qS^{ab}_{\mathrm{I}}(x)\right.$$
$$\left.-(p^\mu q^\nu+p^\nu q^\mu)S^{ab}_{\mu\nu}(x)+2p^2q^2S^{ab}_{\mathrm{II}}(x)+i\left(q^\mu p_2-p_\mu q^2\right)T^{ab,\mu}_{\mathrm{II}}(x)\right\}.\tag{4.21}$$

As explained in the previous section (see the paragraph below eq. (3.9).), we throw away positive power terms of the Laurent expansion of (4.21) with respect to (p^0-q^0), before we take the limit of $(p^0-q^0)\to\infty$. Taking the BJL limit, we get

$$\mathscr{A}=g^2\int\mathrm{d}x\,\mathrm{e}^{-i(p+q)x}\left\{i\left(\varepsilon_{0\nu\lambda\rho}T^{ab,\nu\lambda\rho}+T^{ab}_{\mathrm{I}0}\right)(x)+\tfrac{1}{2}(p^i-q^i)S^{ab}_{0i}(x)\right.$$
$$-(p^0+q^0)(p_ip^i-q_iq^i)S^{ab}_{\mathrm{II}}-\tfrac{1}{2}i(p_ip^i+q_iq^i)T^{ab}_{\mathrm{II}0}(x)$$
$$\left.+\tfrac{1}{2}i(p^0+q^0)(p^i+q^i)T^{ab}_{\mathrm{II}i}(x)\right\}.\tag{4.22}$$

Since S_{0i}^{ab} and $T_{\mathrm{II}0}^{ab}$ are linear in A_0, they vanish in the Weyl gauge. Integrating by parts and noting that S_{II}^{ab} is constant, we obtain the commutator anomaly.

$$\mathscr{A}= ig^2\int \mathrm{d}x\, \mathrm{e}^{-i(p+q)x}\left\{ \varepsilon_{0\nu\lambda\rho}T^{ab,\nu\lambda\rho} + T_{\mathrm{I}0}^{ab} - \tfrac{1}{2}\partial_0\partial_i T_{\mathrm{II}}^{ab,i}\right\}(x), \qquad (4.23)$$

which reproduces the anomaly evaluated in the previous section (3.11) after substituting the explicit forms of T's given by (4.17) and (4.19).

Now we have succeeded to derive the commutator anomaly from the non-abelian chiral anomaly. Once we know the latter by perturbative calculation or by the family index theorem [11], we obtain the former without any explicit calculation of the Feynman graphs.

5. Summary

We have evaluated the commutator anomaly for the Gauss law constraint operator in the perturbation theory. In the BJL limit, Green functions involving two Gauss law operators are equated to Green functions with two insertions of the γ_5 vertex into loops of regulator fields. The latter was explicitly evaluated and the commutator anomaly, both the odd parity part and the even part was obtained as follows:

$$\begin{aligned}&\left[G^a(x), G^b(y)\right]_{x^0=y^0} + igf_{abc}G^c(x)\delta^{(3)}(x-y)\\ &= \frac{g^2}{48\pi^2}\varepsilon^{0\mu\nu\lambda}\Big[\mathrm{tr}[t^a, t^b]\left(\partial_\mu A_\nu A_\lambda + A_\mu\partial_\nu A_\lambda + A_\mu A_\nu A_\lambda\right)\\ &\qquad\qquad + \mathrm{tr}\left\{t^a\partial_\mu\left(A_\nu t^b A_\lambda\right)\right\}\Big]\delta^{(3)}(x-y)\\ &\quad - \frac{g^2}{96\pi^2}igf_{abc}\left(\partial^i E_i^c + gf_{cde}A^{id}E_i^e\right)\delta^{(3)}(x-y). \end{aligned} \qquad (5.1)$$

The even parity part is proportional to $\tilde{G}^a$ and can be eliminated by addition of local counterterms given by eq. (3.14) to the action. Such local counterterms eliminate the even parity part of the non-abelian chiral anomaly at the same time. The odd parity part of the commutator anomaly in the Pauli-Villars regularization scheme, which we took, is different from Faddeev's form by a coboundary, but both belong to the same cohomology class.

The relation between the commutator anomaly and the non-abelian chiral anomaly has not been clear so far. The basic formula obtained in sect. 2 enabled us to clarify this point. The commutator anomaly was directly related to the non-abelian anomaly and therefore if the latter is known by perturbative calculation or by the family index theorem, the former is obtained without any further computation of the Feynman graphs. If the non-abelian chiral anomaly is cancelled in a certain model, then the commutator anomaly is also cancelled.

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