BCS MECHANISM AND THE INTERACTING BOSON MODEL

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Some consequences of the BCS mechanism in nuclei are studied in the effective sigma-model formalism. With adaptations to finite systems, the O(6)–O(5) symmetry breaking chain in the interacting boson model is reproduced reasonably well both in magnitude and form. The same holds to a lesser degree for the U(3)–O(3) chain. Supersymmetry aspects of the model are also discussed.

The interacting boson model (IBM) [1] and its supersymmetric extension [2] are well established as systematic phenomenological descriptions, based on group theory, of low-lying nuclear excitations. The theoretical origins of these intriguing results have not been clear so far. We assert here that the BCS mechanism, which is known to be useful in describing the pairing of nucleons in nuclei, also reproduces the main features of the IBM. Inherent in BCS are relations between bosonic and fermionic excitations which can be cast into a quasi-supersymmetry [3], suggesting that the apparent supersymmetry might also be understood in this light.

First we outline the basic ideas which have been described elsewhere [4]. One postulates that nucleons, in particular those above a major closed shell, participate in the BCS mechanism to form condensed Cooper pairs. Taking the results for bulk media as a guide, there will be single fermion excitations with mass m_f equal to the energy gap, and collective bosonic excitations which are evinced as poles in the fermion pair-bubble propagator [5,6]. The latter serve to restore the symmetries broken by condensation [7] and come in two types, a phase (or Goldstone or pi) mode with zero mass m_0 (in the absence of explicit symmetry breaking) and an amplitude (or Higgs or sigma) mode with mass m_1. Characteristically, these masses have the simple ratios [5,6]

$$m_0 : m_f : m_1 = 0:1:2. \qquad (1)$$

(This is a special case of the more general relation $m_0^2 + m_1^2 = 4m_f^2$.) The low-energy properties of the system due to these excitations can be well represented by an effective hamiltonian of the sigma-model (Ginsburg–Landau–Gell-Mann–Lévy) type, where the coupling constants are related so as to yield the above mass relations.

In BCS one works with a superposition of states with different particle numbers, and applying it to finite systems like nuclei poses some conceptual problems. The particle number degree of freedom is conjugate to the phase mode which has a coherence length comparable to the nuclear size. Hence its physical meaning becomes questionable. As discussed below, we handle the problem by projecting the effective hamiltonian onto states of fixed particle number.

We choose H_0 as an effective hamiltonian that conforms to BCS-type symmetry breaking and also to one of the symmetry schemes of the IBM. All nucleons above (or holes below) a closed magic shell are assumed to participate in generating a Cooper-pair condensate and its spin 0 and 2 bosonic excitations. As we add nucleons to the system, however, the symmetry of the states in the valence shell changes, and so does the symmetry of the bosonic excitations. So the starting hamiltonian H_0 is to be interpreted as spanning a Hilbert space which forms a tangent bundle, at a particular valence shell, to the Hilbert space that represents nuclei of all mass numbers. The fermions in the valence shell couple to the Cooper pairs by Yukawa-type terms. The coupling constants of the interactions are so arranged that symmetry breaking generates masses for the bosons and fermions that satisfy eq. (1). We compute boson–boson, boson–fermion, and fermion–fermion interactions to second order, keeping only energy-conserving processes (as in S-matrix theory). The sum of these terms now defines a new effective hamiltonian H, which replaces H_0, and is to be compared to the IBM hamiltonian. In fact, H can be rewritten as a sum of Casimirs of the relevant groups with coefficients determined by well-known nuclear constants.

In this article we mainly discuss the case with the starting symmetry $\mathrm{U}(1)\times\mathrm{O}(6)\sim\mathrm{U}(1)\times\mathrm{SU}(4)$, which breaks to $\mathrm{O}(5)\sim\mathrm{Sp}(4)$. (According to the IBM, the chain O(6)–O(5)–O(3) is relevant to nuclei in the platinum region where $j=3/2$ proton and neutron shells are active.) Since proton and neutron pairs cannot mix in the usual BCS theory based on four-fermion interactions we consider them separately for now. We write H_0 as a sum of U(6) (of which the U(1) represents the proton or neutron number) and O(6) invariant bosonic interactions and O(6) invariant fermion–boson interactions:

$$H_0=\sum\pi_n^\dagger\pi_n+g^2\left(\sum\phi_n^\dagger\phi_n-c^2\right)^2$$
$$-\tfrac{1}{2}g^2\sum(\phi_n^\dagger\phi_m-\phi_m^\dagger\phi_n)^2+\Psi^\dagger\Gamma_7K\Psi$$
$$\tfrac{1}{2}g\sum\Psi^\dagger(1+\Gamma_7)\Gamma_n\Psi\phi_n+\tfrac{1}{2}g\sum\Psi^\dagger(1-\Gamma_7)\Gamma_n\Psi\phi_n^\dagger. \tag{2}$$

Here we have introduced twelve composite boson fields ϕ_n, $\phi_n^\dagger$, and their canonical conjugates π_n, $\pi_n^\dagger$, $n=1, 2, ..., 6$ representing spin 0 and 2 states formed out of nucleon pairs. Ψ represents nucleon states in the active $j=3/2$ shell and is defined as $\Psi=(\psi,\psi^\dagger)/\sqrt{2}$.

The eight-component Ψ can form a Majorana representation of Spin(7) with seven matrices $\Gamma_1, ..., \Gamma_7$, where Γ_7 multiplying the kinetic energy K (relative to the Fermi surface) is diagonal (1 in the ψ and -1 in the $\psi^\dagger$ subspace) while the other six Γ's are off-diagonal. The twelve modes $(1\pm\Gamma_7)\,\Gamma_n$ represent the $\psi\psi$ and $\psi^\dagger\psi^\dagger$ pairings carrying nucleon number ±2. In the strict O(6) limit (which we assume for simplicity) there is only one coupling constant g in order to satisfy the mass constraint eq. (1).

The hamiltonian (2) is written without a spatial integral as we are dealing here with a finite system on a discrete basis. To justify spontaneous symmetry breaking, however, we will pretend that the system is part of an infinite homogeneous medium, but consider only bosons with zero or near zero momentum. To this end, a boson kinetic energy term $v_\mathrm{F}^2/3\nabla\phi_n^\dagger\cdot\nabla\phi_n$, derived from the BCS theory, is added to eq. (2), where v_F is the Fermi velocity of nucleons. The calculation is thereby simplified except for some delicate problems of the handling of non-uniform limits and the adaptation of the results to the finite system.

We assume ϕ_1 and $\phi_1^\dagger$ (the spin 0 mode) to develop an expectation value c (up to an arbitrary phase) and shift the fields accordingly. There emerge two kinds of Goldstone modes π_1 and π_i, $i=2, ..., 6$. They are associated with the breaking of U(1) and O(6) symmetry respectively. The massive (Higgs) modes, denoted by σ_1 and σ_i, have a common mass

$$m_\sigma^2=4g^2c^2. \tag{3}$$

The fermion field acquires a mass term and Yukawa couplings

$$m_\mathrm{f}\Gamma_1+\frac{g}{\sqrt{2}}(\Gamma_1\sigma_1+\mathrm{i}\Gamma_7\Gamma_1\pi_1+\Gamma_i\pi_i+\mathrm{i}\Gamma_7\Gamma_i\sigma_i),$$
$$m_\mathrm{f}=\tfrac{1}{2}m_\sigma. \tag{4}$$

The remaining terms are cubic and quartic self-couplings among the bosons. We write the σ and π fields in terms of particle fields a and b and hole fields $a^\dagger$ and $b^\dagger$ respectively (in an energy diagonal basis). We are interested in all two-particle interactions to lowest order (tree approximation) and so collect all the first- and second-order diagrams where energy is conserved because of the mass relations (1). Although the one-particle diagrams are superficially of order g^4, the relation (3) (used in the vertices and propagators) conspires to make them of the same order g^2 as the original quartic terms. Similar relations also hold in the fermion diagrams.

We first discuss the boson–boson interactions. Computation is straightforward when the external bosons are σ's. The π bosons must be treated with care to show that their contributions are not only free of infrared divergence but also finite in the zero mass limit. We have computed them assuming that all π modes have the same small mass, which does not change the coupling, the latter being based on the underlying BCS theory. The resultant bosonic hamiltonian can be brought to the form

$$H_b = m_\sigma N_\sigma + (1/8c^2V) \times [-3N_6(\sigma)^2 + 9N_6(\sigma) - 3O_6(\sigma) + 2O_5(\sigma) - N_6(\pi)^2 - O_6(\pi) + 2O_5(\pi) + 2O_5(\sigma, \pi)]. \quad (5)$$

Here V is the normalization volume of the bosons, which may be equated to the nuclear volume. $N_6(\sigma)$ is the U(6) number operator for the σ's, $O_l(\sigma$ or $\pi)$ the quadratic $O(l)$ Casimir, and $O_l(\sigma, \pi)$ stands for a similar form with mixed products of σ and π generators. For example,

$$N_6(\sigma) = \sum a_n^\dagger a_n,$$

$$O_6(\sigma) = \tfrac{1}{2}\sum L_{mn}^2 = -\tfrac{1}{2}\sum (a_m^\dagger a_n - a_n^\dagger a_m)^2, \quad (6)$$

etc. Self-energies are partially included in that the Casimir operators, when converted to normal products, contain linear terms in N.

The underlying BCS theory gives

$$g_i^2 = [(dI_i(z)/dz)_{z=0}]^{-1} \approx 8m_f^2/N_F,$$

where $I_i(z)$ is the pair-bubble propagator for a mode i of energy $E=\sqrt{z}$. So that from eqs. (1) and (3) we get

$$c^2 = N_F/8, \quad N_F = (\nu/2\pi^2)\, p_F^2/v_F, \quad (7)$$

where N_F is the density of states at the Fermi surface (here the valence 3/2 shell) with momentum p_F and velocity v_F, and ν the degeneracy for the valence shell. Thus for $p_F \sim 240$ MeV and $\nu = 4$ (for $j=3/2$), $c \sim 75$ MeV (if the O(3) breaking due to core deformation is ignored). Being dependent on ν, c is not a universal constant.

Eq. (5) is not yet applicable to actual nuclei for two reasons: the finiteness (in both number and size) of the system and the coupling to density modes. Regarding the first point, the bosons represent normal modes of fluctuations of Cooper pairs. This would be valid if the system were large compared to the coherence length $\lambda = v_F/\sqrt{3}m_\sigma$. From the empirical value ~ 1.5 MeV of the energy gap $(=m_\sigma)$, we find $\lambda \sim 20$ fm, which is larger than the nuclear radius ~ 7 fm for mass number $A \sim 200$.

Regarding the second point, the π_1 mode for both protons and neutrons is expected to couple to the isospin density $(\sim \Gamma_7)$ and get absorbed into a plasmon mode. With a plasmon energy of ~ 10 MeV, the π_1 contributions would be suppressed. The π_i modes, on the other hand, would not disappear even though they could couple to quadrupole deformations $(\sim \Gamma_1\Gamma_i)$.

We propose to deal with the finite size effect by a correction factor $C<1$. C represents a reduction of interaction between bosons due to the finite system size and should actually be diffent for different diagrams. The momentum uncertainty $\sim 1/R$ (where R is the nuclear radius) reduces the propagator relative to its zero-momentum limit (determined only by the mass of the exchanged particle). A typical value might be $1/C \sim 1 + (\lambda/R)^2 \sim 3$. The finite N problem is handled by projecting the effective hamiltonian (5) from the energy diagonal basis to subspaces diagonal in particle number. Specifically, we write $a = (u+v)/\sqrt{2}$, $b = (u-v)/\mathrm{i}\sqrt{2}$, where u (v) annihilates a particle (hole) pair, and keep only u (or v) terms (nucleons above and holes below a magic shell can both be considered "particles" in this picture). As for the coupling to densities, we ignore it for the moment. The result is

$$H_b' = \tfrac{1}{2}m_\sigma N_6 + (C/8c^2V)\,(-N_6^2 + \tfrac{9}{2}N_6 - O_6 + \tfrac{3}{2}O_5), \quad (8)$$

where N and O now refer to single bosonic modes u or v.

The boson mass $m_\sigma = 2m_f$ has been redefined by a factor 1/2 due to the projection. Identifying $m_\sigma/2$ with the pairing energy, we get $m_\sigma \sim 3$ MeV. Eq. (3) then gives $g \sim 0.02$, which justifies the perturbation theory we have used.

From eq. (8) we find for the coefficient of the Casimir terms

$$C/8c^2V \sim 40C/A \text{ MeV}, \quad (9)$$

which gives $\sim 200C$ keV for the present case of $A \sim 200$.

In the IBM the two phenomenological coefficients for O(6) and O(5) in the platinum region vary from -30 to -50 keV and from 40 to 60 keV, respectively. Thus eq. (8) reproduces the average behavior of the data with $C \sim 1/4$. We also note that the A dependence of eq. (9) is consistent with the data in the platinum and barium regions.

It is to be noted that the IBM (in the original version discussed here) regards both proton and neutron pairs indiscriminately, whereas our formula should apply to each of them separately. The BCS mechanism does not directly provide for a mixing of proton and neutrons, but this could occur through coupling to density modes. In that case the degeneracy ν is effectively doubled and the coefficient in eq. (9) reduced by 1/2.

We now turn to the fermionic sector. As an aside, we note that H_0 of eq. (2) can be factorized as [3]

$$H_0=\{Q, Q^\dagger\}, \quad Q=\Pi^\dagger\psi+\mathrm{i}g(\Phi^\dagger\Phi-c^2)\,\psi^\dagger, \tag{10}$$

where the boson field Φ and its canonical conjugate $\Pi^\dagger$ are taken to form 4×4 antisymmetric matrices multiplying the fermion field ψ. The Q's are not nilpotent, so there is no strict supersymmetry. Calculations are facilitated by the Majorana condition on Ψ which implies that $[\Psi^\dagger, O\Psi]=0$ if O is the unit matrix or the product of three or four different Γ's. As before, we keep only energy-conserving processes. One must again handle carefully the infrared problem for the Goldstone modes (as before divergent diagrams cancel) but there is an additional subtlety concerning the kinetic energy K of the fermion. The computation was made in the limit $K\to0$. We project the operators onto states of positive (negative) energy and then positive (negative) particle number and sum over the two projections. The result is

$$H'_{\mathrm{bf}}+H'_{\mathrm{f}}=\tfrac{1}{2}m_{\mathrm{f}}N_{\mathrm{f}}+(C/8c^2V)\Big(-\tfrac{1}{2}N_{\mathrm{f}}N_6 + \sum L_{1i}s_{1i}+\tfrac{1}{4}\sum s_{1i}s_{1i}\Big), \tag{11}$$

where N_{f} is the fermion number. There appear "spin–orbit" type couplings between the bosonic and fermionic generators L_{1i} and $s_{1i}=\mathrm{i}\Gamma_1\Gamma_i/2$. The spectrum of $H'_{\mathrm{b}}+H'_{\mathrm{bf}}+H'_{\mathrm{f}}$ seems more complicated than that in the supersymmetric IBM.

We comment briefly on the other symmetry chains that occur in the IBM. In the case of the U(6)–U(3)–O(3) chain, we consider the U(3)–O(3) portion to be of spontaneous origin. Adopting a pseudo-spin picture [8] (see also ref. [1]), we assume the valence fermions to consist of six ($l=1$, $s=1/2$) states out of which six ($l=0$ and 2) bosons are formed as spin singlets. (We do not mean that the $j=3/2$ and 1/2 states are nearly degenerate, but that there are six magnetic substates in the region of nuclei considered.) The bosons can be represented by a complex 3×3 symmetric matrix. A U(3) invariant hamiltonian is broken to O(3) by a condensate proportional to the unit matrix. [A similar formulation for the O(6) case, with a 4×4 antisymmetric matrix behaving as the 6 of SU(4) and breaking to $\mathrm{Sp}(4)^3$, gives results identical to eq. (8).]

The effective hamiltonian in the bosonic sector, after the particle number projection, is

$$H_{\mathrm{SU}_3}=\tfrac{1}{2}m_\sigma N_\sigma + (\tfrac{3}{32}C/c^2V)\,[-\tfrac{2}{3}N_\sigma^2-\mathrm{SU}(3)+\tfrac{3}{8}\mathrm{O}(3)]. \tag{12}$$

Here $c=92$ MeV ($\nu=6$). Eq. (12) is consistent with the few data available in the region of $A\sim160$ except that the ratio of the coefficients of O(3) and SU(3) is too small by a factor of ~2. (The SU(3) Casimir here is three times larger than the IBM.) This may indicate that there is always an intrinsic O_3 piece that is not spontaneous.

The U(5)–O(5) chain does not neatly fit into the scheme of pure dynamical symmetry breaking. Although the U(5) pseudo-spin picture with ($l=2$, $s=1/2$) fermions and a 5×5 symmetric boson matrix (for $j=0$, 2, and 4 bosons) seems natural, the $j=4$ part would have to be suppressed. Alternatively it could be regarded as a case of broken O(6).

To summarize, we have seen that the BCS mechanism can be carried further than has been attempted in interpreting low-lying nuclear excitations. An important part of symmetry chains in the IBM is reproduced fairly well. Numerical results depend only on two experimental numbers, the nuclear density and the pairing energy. The "supersymmetry" aspect of the mechanism seems, however, more complicated than that envisioned in the IBM extension, and needs further study.

We have presented specific prescriptions for adapting the infinite medium results to finite systems. Although theoretical uncertainties remain on this question, the general features are not sensitive to them, as may be seen easily.

We conclude with a couple of speculative notes. First one might ask, in view of the fact that the nuclear radius and the coherence length are equqal to within a factor of two, whether or not they mean the same thing: Cooper pairs held together by surface tension. If this is assumed for the moment, there remains only one basic parameter, the saturation nuclear density. The pairing energy would then decrease as $1/A^{1/3}$, disappearing for an infinite medium. This, however, is slower than is generally believed.

Secondly, it is interesting that the two seemingly different cases of symmetry breaking, i.e., the chirality pairing in pion physics and the nucleon pairing in nuclei, have rather similar values for the condensate. Perhaps this is not surprising since they are both rooted in quantum chromodynamics.

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