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BCS and IBM

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The BCS theory of fermionic pairing and condensation is used to understand the interacting boson model. Results from BCS are incorporated into an effective Hamiltonian that after symmetry-breaking and second-order corrections yields an IBM-type Hamiltonian with coefficients determined by well-known nuclear constants. The $O(6)$ and $O(5)$ chains are shown to be largely of spontaneous origin. Supersymmetry aspects of the model are also discussed.

1. INTRODUCTION

The interacting boson model [1–7] is well established as a unified description of collective levels of heavy nuclei. The BCS theory of fermionic pairing and condensation [8] is likewise a tour-de-force with wide-ranging applications, not the least of which is to the nucleus [9]. The work we describe here unifies these two paradigms of modern nuclear physics so that the former becomes the inevitable manifestation of the latter, and the latter the cause of the former. BCS theory provides the microscopic rationale for the interacting boson model [10].

The interacting boson model (or IBM for short) postulates that two types of bosons, of spin 0 and spin 2 (known as the s and d bosons) describe collective motions of heavy nuclei. These form a Hilbert space of $U(6)$ symmetry. The bosons are identified with nucleon pairs, so that half the number of nucleons outside major closed shells gives the $U(6)$ quantum number. Some nuclei seem to have the symmetry of a diagonal subgroup chain of $U(6)$. For these, the IBM provides a beautifully simple description of the energy levels and transition matrix elements, in a prescription that unifies whole regions of the table of isotopes. In the end, however, the IBM is a phenomenological theory, and its parameters (other than the crucial one of which representation applies to which nucleus) have to be obtained from experiment.

In contrast, BCS is a microscopic theory; when described as a theory of spontaneous symmetry-breaking as in this application, it has no free parameters. It allows two kinds of bosonic excitations–the π, which are oscillations along the rim of the "Mexican hat" and are massless (see Fig. 1), and the σ, which are radial oscillations and massive. In addition, there are fermionic excitations which break up

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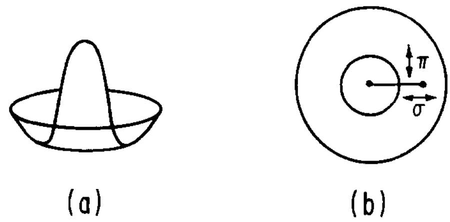

FIG. 1. The Mexican hat, from the side (a) and the top (b) showing the radial (sine) and phase (π) oscillations.

the Cooper pairs and promote the fundamental fermions above the BCS sea. BCS theory implies mass relations between these bosonic and fermionic excitations of the system. These relations can be mimicked by an appropriate choice of coupling coefficients in the equivalent Ginsburg–Landau [11] type effective theory, thus determining *at this effective level* all the parameters. After symmetry-breaking and second-order corrections the Hamiltonian of the system can be written as a sum of Casimirs (of the IBM type) with coefficients determined by well-known nuclear constants [10].

In our application of BCS to the nucleus, we have to come to terms with the finiteness in both volume and particle number of our system. We work in infinite nuclear matter with a symmetry that comes from shell structure (which is caused by finiteness!). Since we use a discrete basis, the Hamiltonian is written without a spatial integral and diagrams are calculated in the limit of zero kinetic energy. In the end we correct for finiteness by an overall fuzz-factor C—this makes our results quantitavely uncertain by about a factor of two while retaining the precision of the infinite medium in the *relative* sizes of the different terms.

The problem of the fixed number of particles is also serious, since the BCS ground state is a superposition of states with different particle number. Indeed, the BCS sea must be looked upon as existing in *a fictitious, unphysical* Hilbert space to which the Hilbert spaces of real nuclei are tangent spaces, So we perform the calculations in the BCS system and in the end project to fixed-particle-number subspaces. The problem of finiteness is central to any theory of nuclear matter that hopes to illuminate real nuclei. Our prescriptions are imperfect but (for our purposes) inescapable answers to the special problems that the nucleus presents us with.

This article is divided into seven parts. The first recapitulates the ideas of BCS theory relevant to this work; for example, the mass relations and the couplings in the equivalent Ginsburg–Landau picture. The second sets up the problem in the nucleus, using the $O(6)$ subgroup chain for specificity. The third details calculations for the bosonic interactions. The fourth gives our adaptations for finiteness for the bosonic part and comparisons to the IBM for the $O(6)$ chain. The fifth part returns to the fermions. After a short discussion of quasi-supersymmetry in the BCS system we compute the fermionic interactions and list the results. The sixth section describes the $SU(3)$ subgroup (the $U(5)$ chain does not fit naturally into the BCS scenario and we will not deal with it other than to point out that an $O(6)$ theory with explicit symmetry-breaking looks very similar to the $U(5)$ chain). Finally we end with discussions, conclusions, and last but not least, speculations.

2. MODES AND MASSES IN BCS

Let us begin with the BCS Lagrangian with the usual four-fermion interaction [8, 12–14]:

$$H(x) = \sum_{s=\downarrow,\uparrow} [\psi_s^\dagger(x)\,\varepsilon_p \psi_s(x)] - 2g\psi_\uparrow^\dagger(x)\,\psi_\downarrow^\dagger(x)\,\psi_\downarrow(x)\,\psi_\uparrow(x). \tag{1}$$

Here $\psi_\uparrow$ and $\psi_\downarrow$ are spin-$\frac{1}{2}$ fermions, ε_p is the kinetic energy, and g is a fundamental coupling which goes to zero outside a shell of thickness Λ centered at the Fermi surface.

This can be rewritten using the two-component (Nambu) notation [12–14]

$$\Psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow^\dagger(x) \end{pmatrix} \tag{2}$$

as

$$H(x) = \Psi^\dagger\,\varepsilon_p \tau_3 \Psi - g\,\Psi^\dagger \tau_3 \Psi \Psi^\dagger \tau_3 \Psi + \text{constant terms.} \tag{3}$$

To linearise, we make the BCS ansatz that there is a Cooper pair condensate and $\langle\psi\psi\rangle$, $\langle\psi^\dagger\psi^\dagger\rangle \neq 0$:

$$\phi \sim \Psi^\dagger \tau_1 \Psi. \tag{4}$$

Here we choose only one linear combination $\langle\psi^\dagger\psi^\dagger\rangle + \langle\psi\psi\rangle$ to be non-zero since the other (proportional to τ_2) can be obtained by a rotation in the Nambu space (this is called the "phase" rotation). Now we write the Hamiltonian as the sum of "free" and "interacting" parts:

$$\begin{aligned} H &= H_0 + H_I, \\ H_0 &= \Psi^\dagger\,\varepsilon_p \tau_3 \Psi + \Psi^\dagger \tau_1 \Psi \Delta, \\ H_I &= -g\,\Psi^\dagger \tau_3 \Psi \Psi^\dagger \tau_3 \Psi - \Psi^\dagger \tau_1 \Psi \Delta \end{aligned} \tag{5}$$

and calculate diagrams using the "bare" propagator

$$iG(p) = i\,\frac{(p_0 + \tau_3 \varepsilon_p + \tau_1 \Delta)}{(p_0^2 - \varepsilon_p^2 - \Delta^2 + i\varepsilon)}. \tag{6}$$

Self-consistency requires that we choose Δ so that H_I gives no further self-energy corrections proportional to τ_1. This gives us the gap equation with Δ the mass of the fermionic excitations. (The gap energy is that required to break up a Cooper pair and is therefore 2Δ). To see how the four-fermion term can give contributions proportional to τ_1 we use the Fierz identity for Pauli matrices

$$(\tau_3)_{ij}(\tau_3)_{kl} = \frac{1}{2}\sum_{A=1}^{4} \tau^A_{kj}(\tau_3\tau^A\tau_3)_{il} \tag{7}$$

where $\tau^A = (\tau_1, \tau_2, \tau_3, 1)$ to write

$$\begin{aligned} &-\tfrac{1}{2}\Psi^\dagger\tau_3\Psi\Psi^\dagger\tau_3\Psi \\ &\quad = \tfrac{1}{4}[\Psi^\dagger\Psi\Psi^\dagger\Psi + \Psi^\dagger\tau_3\Psi\Psi^\dagger\tau_3\Psi - \Psi^\dagger\tau_1\Psi\Psi^\dagger\tau_1\Psi - \Psi^\dagger\tau_2\Psi\Psi^\dagger\tau_2\Psi]. \end{aligned} \tag{8}$$

The gap equation now reads (Fig. 2)

$$g\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}[\tau_1 G(p)] + i\Delta = 0, \tag{9}$$

or, doing the p_0 integral,

$$1 = g\int_{p_F - p_{A/2}}^{p_F + p_{A/2}} \frac{d^3p}{(2\pi)^3}\frac{1}{(\varepsilon_p^2 + \Delta^2)^{1/2}}, \tag{10}$$

where the integration is over a shell around the Fermi surface of thickness $2p_{A/2}$.

Hence we can make the approximation

$$\frac{d^3p}{(2\pi)^3} = \frac{4\pi p^2\,dp}{(2\pi)^3} = \frac{p_F}{2\pi^2}\,p\,dp = \frac{N_F}{2}\,d\varepsilon_p, \tag{11}$$

where N_F is the Fermi density (including the spin degeneracy $\nu = 2$) to write for the gap equation

$$1 = \frac{gN_F}{2}\int_{\varepsilon_F - \varepsilon_{A/2}}^{\varepsilon_F + \varepsilon_{A/2}} \frac{d\varepsilon_p}{(\varepsilon_p^2 + \Delta^2)^{1/2}}. \tag{12}$$

To find the resonances in the fermion–fermion interaction we now calculate the bubble chains in Fig. 3 with τ_1 or τ_2 at the vertices and look for the poles. These describe the amplitude and phase (Goldstone and Higgs) modulations, respectively,

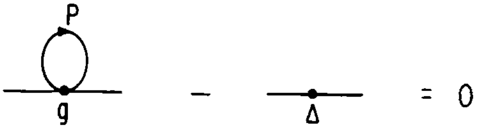

FIG. 2. Mass contributions from H_I.

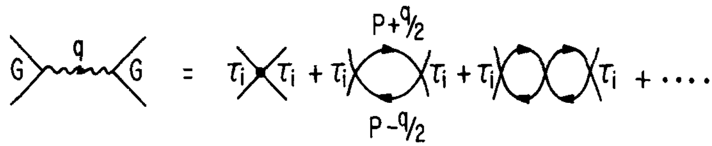

FIG. 3. Repeated fermion-fermion interactions and the equivalent meson-exchange diagram. For $\tau_i = \tau_1$ we get the σ meson and for $\tau_i = \tau_2$ the π meson.

since the condensate has been chosen to lie in the direction τ_1. An alternate approach has been described by Nambu [12] and Littlewood and Varma [15]: the poles in the gauge-invariant vertex functions are used to find the collective modes. The connection with the Ward identities is then more explicit. The π and σ modes are seen to be manifestations of symmetries of the Hamiltonian (3) under $SU(1, 1)$ rotations generated by τ_3, $i\tau_1$, and $i\tau_2$.

The matrix element for the bubbles is given by

$$M = g/(1 - J_{\pi,\sigma}(q)), \tag{13}$$

where

$$J_\pi(q) = ig \cdot \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}[\tau_2 G(p + q/2)\, \tau_2 G(p - q/2)] \tag{14}$$

and

$$J_\sigma(q) = ig \cdot \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}[\tau_1 G(p + q/2)\, \tau_1 G(p - q/2)] \tag{15}$$

are the integrals for a single bubble. Note that the poles of M are at

$$J_{\pi,\sigma}(q) = 1; \tag{16}$$

this is the condition for the existence of a mode.

To calculate the integrals we resort to dispersion theory techniques. Using the expression for the propagator we find, for ε_q set to zero (i.e., for zero momentum transfer),

$$J_{\pi,\sigma}(q) = 2ig \int \frac{d^4p}{(2\pi)^4} \frac{[(p_0^2 - q_0^2/4) - \varepsilon_p^2 \mp \Delta^2]}{[(p_0 - q_0/2)^2 - \varepsilon_p^2 - \Delta^2 + i\varepsilon)][(p_0 + q_0/2)^2 - \varepsilon_p^2 - \Delta^2 + i\varepsilon]} \tag{17}$$

where the $-$ and $+$ signs are for the π and σ modes, respectively. We do the p_0 integral and pick up the residues at the two poles

$$p_0 = \pm q_0/2 - (\varepsilon_p^2 + \Delta^2)^{1/2} + i\varepsilon \tag{18}$$

to get (writing $q_0 = \omega$)

$$J_\pi (q_0^2 = \omega^2) = g \int_{p_F - p_{A/2}}^{p_F + p_{A/2}} \frac{d^3p}{(2\pi)^3} \frac{1}{(\varepsilon_p^2 + \Delta^2)^{1/2}} \left[1 - \frac{\omega^2}{(\omega^2 - 4(\varepsilon_p^2 + \Delta^2))} \right]. \tag{19}$$

The expression for $J_\sigma(\omega^2)$ is similar, with the ω^2 in the numerator replaced by $\omega^2 - 4\Delta^2$. Note that

$$1 = J_\pi(\omega^2 = 0) = J_\sigma(\omega^2 = 4\Delta^2) \tag{20}$$

by the gap equation. Hence the poles of M are at $\omega^2 = 0$ and $4\Delta^2$ for the π and σ modes, respectively. The masses of the modes are therefore [16]

$$m_\pi = 0, \qquad m_\sigma = 2\Delta. \tag{21}$$

(For $\varepsilon_q \neq 0$, we find instead of the above the dispersion relation [15], $\omega^2 = m_{\pi,\sigma}^2 + v_F^2 q^2/3$). Using Eq. (11) and then the substitution $\kappa = 2(\varepsilon_p^2 + \Delta^2)^{1/2}$, we get

$$J_{\pi,\sigma}(\omega^2) = \frac{gN_F}{2} \int_{4\Delta^2}^{\Lambda^2} \frac{d\kappa^2}{(\kappa^2 - \omega^2)(1 - 4\Delta^2/\kappa^2)^{\pm 1/2}}, \tag{22}$$

where the $\pm$ signs refer to the π and σ modes respectively and $\Lambda^2 \gg 4\Delta^2$ is the shell-thickness cutoff.

To evaluate the effective coupling of the σ and π modes we write (from Fig. 3)

$$-\frac{G_{\pi,\sigma}^2}{\omega^2 - m_{\pi,\sigma}^2} = \frac{g}{1 - J_{\pi,\sigma}(\omega^2)} \tag{23}$$

and expand $J_{\pi,\sigma}$ about the mass-shell value of ω^2, namely $\omega^2 = 0, 4m^2$ for π, σ. Using the gap equation again, we get

$$G_{\pi,\sigma}^2 = g(dJ_{\pi,\sigma}/d\omega^2)^{-1} \tag{24}$$

with the derivative evaluated on the mass shell. We find

$$G^2 = G_{\pi,\sigma}^2 \simeq 8\Delta^2/N_F, \tag{25}$$

since the cutoff $\Lambda^2 \gg 4\Delta^2$. Here we have approximated the spectral density $(1 - 4m^2/\kappa^2)^{\pm 1/2} \simeq 1$. We need an approximation of this kind to use the sigma model in which the π and σ modes are identical prior to symmetry breaking.

If we evaluate $G_{\pi,\sigma}$ exactly at $\omega^2 = 0$ we get

$$\begin{aligned} G_\pi^2 &= 8\Delta^2/2N_F, \\ G_\sigma^2 &= 3(8\Delta^2)/2N_F \end{aligned} \tag{26}$$

so that the average of the two couplings at this momentum transfer is still G^2. However, at arbitrary ω^2, this might not be true. This difference of the π and σ couplings is the primary problem with modelling the BCS model by the σ model.

The above results carry over to the analysis of BCS systems in larger Hilbert spaces. In general the Fermi density is given by

$$N_F = \frac{\nu p_F^2}{2\pi^2 v_F}, \tag{27}$$

where ν is the spin degeneracy. With this allowance for the enlarged space, the main results Eqs. (21) and (25) remain unchanged.

3. BCS in the Nucleus

We concentrate on the case of $O(6)$ symmetry since it is the most amenable to a BCS interpretation. The symmetry is dictated by that of the valence shell; while this assumption is not essential to the description of the even–even nuclei it is important to the even–odd nuclei (as we shall see, however, the supersymmetry aspects of this model do not coincide with those in the extensions of the IBM). We have a symmetry dictated by the shell structure and a discrete basis, and so we write the Hamiltonian without an integral; however, we also work in an infinite medium in the limit of zero momentum. In any case, consider the valence fermions to be in a spin $\frac{3}{2}$ shell. These pair up to form spin 0 and spin 2 bosons. So the starting symmetry is that of the valence shell, that is, $U(4) \sim O(6)$. We also impose a $U(1)$ symmetry for particle number conservation:

$$\begin{aligned} H_0 = {} & \sum_n \pi_n^\dagger \pi_n + 2G^2 \left[\sum_n \phi_n^\dagger \phi_n - c^2 \right]^2 \\ & - G^2 \sum_{nm} [\phi_n^\dagger \phi_m - \phi_m^\dagger \phi_n]^2 + \Psi^\dagger \Gamma_7 K \Psi \\ & + G/\sqrt{2} \sum_n [\Psi^\dagger (1+\Gamma_7) \Gamma_n \Psi \phi_n + \Psi^\dagger (1-\Gamma_7) \Gamma_n \Psi \phi_n^\dagger]. \end{aligned} \tag{28}$$

In the above ϕ_n, $\phi_n^\dagger$ with $n = 0, 1, ..., 5$ are composite bosonic fields ($\phi \sim \psi\psi$) and π_n, $\pi_n^\dagger$ their canonical conjugates. The first term is the usual bosonic kinetic energy, the second an $U(6)$-invariant symmetry-breaking piece, and the third has $O(6) \times U(1)$ invariance ($L_{mn} = -i(\phi_m^\dagger \phi_n - \phi_n^\dagger \phi_m)$ is like an $O(6)$ angular momentum, and $\frac{1}{2} \sum L_{mn}^2$ is therefore the Casimir invariant). Ψ represents nucleon states in the valence shell (considered in this infinite medium approach as the Fermi surface) and is written as an eight-component Majorana fermion $\Psi = (\psi_{3/2}, \psi_{1/2}, ..., \psi_{-3/2}, \psi_{3/2}^\dagger, ..., \psi_{-3/2}^\dagger)/\sqrt{2}$. The Γ_n's are 8×8 Clifford algebra matrices. That is, they transform as vectors under $O(6)$, anticommute among themselves, and serve to couple the Ψ's together to form spin 0 and spin 2 composites. $\Gamma_7 = \prod_n \Gamma_n = (1, 1, 1, 1, -1, -1, -1, -1)$ is used to construct the kinetic energy piece for the fermions since it gives positive and negative energies to the particles and holes, respectively. The other Γ's are off-diagonal. The last two terms in Eq. (28) are arranged to conserve particle number; that is, they only allow terms like $\psi\psi\phi^\dagger$ (and not, for instance, $\psi\psi\phi$). The coefficients are chosen, as we shall see, to reproduce the results of the last section.

We allow the spin 0 boson to develop a vacuum expectation value and call the excitations around vacuum π and σ modes in anticipation of their respective masses:

$$\begin{aligned} \phi_0 &= c + (\sigma_0 + i\pi_0)/\sqrt{2} \\ \phi_i &= (\pi_i + i\sigma_i)/\sqrt{2}, \qquad i = 1, ..., 5. \end{aligned} \tag{29}$$

Then our Hamiltonian becomes (omitting the kinetic terms for simplicity)

$$\begin{aligned} H_0 = {} & 4G^2c^2 \sum_n \sigma_n^2 + \sqrt{2}Gc\,\Psi^\dagger\Gamma_0\Psi \\ & + 2\sqrt{2}G^2c\,\sigma_0 \sum_n (\sigma_n^2 + \pi_n^2) + 4\sqrt{2}G^2c \sum_i \sigma_i(\sigma_0\sigma_i - \pi_0\pi_i) \\ & + G^2/2 \sum_n (\sigma_n^2 + \pi_n^2)^2 + G^2 \sum_{ij} (\pi_i\sigma_j - \sigma_i\pi_j)^2 \\ & + 2G^2 \sum_i (\sigma_0\sigma_i - \pi_0\pi_i)^2 \\ & + G\Psi^\dagger(\Gamma_0\sigma_0 + i\Gamma_7\Gamma_0\pi_0 + \Gamma_i\pi_i + i\Gamma_7\Gamma_i\sigma_i)\Psi. \end{aligned} \tag{30}$$

In the above $n = 0, 1, ...5$ and $i = 1, 2, .., 5$. The first two are the mass terms, the next two vertex (interaction) terms, the following three quartic and the last Yukawa terms. The σ mass is

$$m_\sigma = 2\sqrt{2}Gc \tag{31}$$

and the fermion mass

$$\Delta = \sqrt{2}Gc. \tag{32}$$

As promised, $m_\sigma = 2\Delta$. We further find the value of c using Eqs. (25) and (32) to be

$$c = \sqrt{N_F}/4. \tag{33}$$

The overall scale is fixed by the Yukawa coupling G of Eq. (25), with N_F given by Eq. (27).

The task ahead of us is to evaluate perturbation corrections to the above Hamiltonian. Since H_0 is an effective Hamiltonian that reproduces only the low-energy characteristics of BCS, it is not meaningful to carry out perturbation theory to all orders. Diagrams with internal loops depend on high-energy (and hence untrustworthy) components of the fields. We accordingly restrict ourselves to two-particle interactions and consider only tree graphs to second order; further, we include only energy-conserving (on-shell) processes. The mass relations between various fields select a subset of tree diagrams that meet the latter (on-shell) requirement. On adding these corrections to H_0 we obtain a *new* effective Hamiltonian that will be compared to the IBM. Conceptually, the corrected Hamiltonian is analogous to the S-matrix or the optical potential; its matrix elements give physical quantities directly. Unlike the S-matrix, however, scattering causes energy shifts for a finite system. Thus our results will carry a volume factor $1/V$. In the following section we detail the bosonic corrections.

4. Bosonic Interactions

We write the π and σ fields in terms of creation and annihilation operators, remembering that we will work in the limit of zero momentum:

$$\begin{aligned} \sigma_n &= (a_n + a_n^\dagger)/\sqrt{2m_\sigma V} \\ \pi_n &= (b_n + b_n^\dagger)/\sqrt{2m_\pi V}. \end{aligned} \tag{34}$$

In the above V is the volume of the nucleus and the pion has been assigned a small mass m_π which will later be taken to be zero. (We anticipate problems with convergence–more on this shortly). A typical diagram is shown in Fig. 4; a σ is exchanged between two other σ's. The contribution from this diagram is

$$\frac{(2\sqrt{2}G^2c)^2}{(-m_\sigma^2)}\, 9(4a_i^\dagger a_0^\dagger a_i a_0)\frac{1}{(2Vm_\sigma)^2} \tag{35}$$

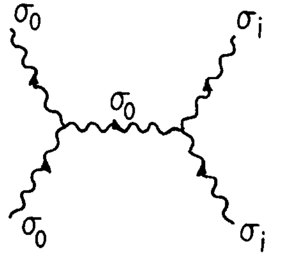

FIG. 4. A typical diagram for scattering of the massive σ mesons.

where the first term is the coupling (with the propagator in the denominator) and the last term is from the normalisation of the a's. The coefficient of the a's has the dimensions of energy per unit volume:

$$\frac{G^4 c^2}{m_\sigma^4 V^2} = \frac{1}{c^2 V^2}. \tag{36}$$

It is interesting to compare this coefficient with that of the quartic terms in H_0. A typical quartic term, when expressed in terms of the a's (we leave out particle-number-nonconserving terms) is

$$\frac{2G^2}{(2Vm_\sigma)^2} a_i^\dagger a_0^\dagger a_i a_0 \tag{37}$$

and has the coefficient

$$\frac{1}{16V^2c^2} \tag{38}$$

which we note has not only the same dimensions but also the same parametric form as the term obtained from the diagram. In fact the overall coefficient turns out to have the same $1/c^2V^2$ dependence for all the diagrams and all the quartic terms.

Let us now turn to the diagrams involving the massless π bosons, with which we expect divergence problems. The contribution from the diagram in Fig. 5, for example is

$$(2\sqrt{2}G^2c)^2\, b_0^\dagger b_0^2 \frac{1}{(2Vm_\pi))^2} \frac{1}{4m_\pi^2 - m_{\sigma^2}}. \tag{39}$$

The last term is from the propagator, where for reasons that will become clear, we have been careful to include the small effect of the π mass. Expanding it in powers of m_π^2/m_{σ^2} to first order we get for the above

$$-\frac{(2\sqrt{2}G^2c)^2}{4V^2} b_0^{\dagger 2} b_0^2 \left(\frac{1}{m_\pi^2 m_\sigma^2} + \frac{4}{m_\sigma^4}\right). \tag{40}$$

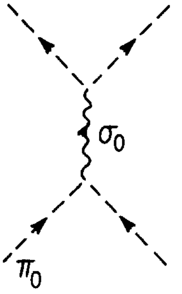

FIG. 5. A typical diagram for scattering of π mesons.

In the last bracket we see two terms: one has m_π^2 in its denominator and is divergent, while the second is finite and has the same overall coefficient as the usual σ terms. We find that when we add together all the contributions from the first- and second-order tree diagrams, *all the divergent terms cancel identically*. This propitious event is not a coincidence but attests to the self-consistency of the BCS theory as embodied in our Hamiltonian [17]. Be that as it may, it is most welcome as it takes away the specter of renormalisation with its additional cutoff parameter. Now we observe that the a's can be grouped into terms which are invariants of $U(6)$, $O(6)$, and $O(5)$:

$$\begin{aligned} N_6(\sigma) &= \sum a_m^\dagger a_n \\ O_6(\sigma) &= -1/2 \sum (a_m^\dagger a_n - a_n^\dagger a_m)^2 \\ O_5(\sigma, \pi) &= \sum (a_i^\dagger a_j - a_j^\dagger a_i)(b_i^\dagger b_j - b_j^\dagger b_i) \end{aligned} \tag{41}$$

and so on. We write the final result in terms of these invariants and multiply by a factor V to go from units of energy per unit volume to energy:

$$\begin{aligned} H_b = m_\sigma N_6(\sigma) + 1/(8c^2V)[&-3N_6(\sigma)^2 - 3O_6(\sigma) + 2O_5(\sigma) \\ &- N_6(\pi)^2 - O_6(\pi) + 2O_5(\pi) + 2O_5(\sigma, \pi)]. \end{aligned} \tag{42}$$

The self-energies have not been included because of the problem of overcounting– the loop integrals were incorporated into the underlying BCS theory. The above is our result for the bosonic case in the infinite medium. In the following section we describe the effects of finiteness in volume (V) and particle number on the Hamiltonian H_b.

5. Finiteness

Let us first look at the problem central to the application of BCS to most systems, that of fixed and finite particle number. The bosonic modes have well-defined interpretations in the infinite medium as fluctuations in the Cooper pairs. For a nucleus in the mass region $A \sim 200$, the coherence length can be found from the dispersion relation following Eq. (21): $\lambda = v_F/\sqrt{3}m_\sigma$. Using the empirical value of the energy gap ($= 2\Delta = m_\sigma \sim 1.5$ Mev) and the Fermi momentum $p_F \sim 240$ Mev, we find λ to be ~ 20 fm. This is larger than the radius of ~ 7 fm in this region. So the interpretation of the modes leaves something to be desired.

We deal with the finite particle number problem by projection of the Hamiltonian from an energy diagonal basis to substates diagonal in particle number.

Since the σ mode describes amplitude oscillations ($\sigma \sim \Psi^\dagger \tau_1 \Psi \sim \psi\psi + \psi^\dagger\psi^\dagger$) and the π mode phase oscillations ($\pi \sim \Psi^\dagger \tau_2 \Psi \sim i(\psi^\dagger\psi^\dagger - \psi\psi)$) we can write

$$\begin{aligned} a &= (u+v)/\sqrt{2} \\ b &= (u-v)/i\sqrt{2}, \end{aligned} \tag{43}$$

where u annihilates a pair of particles and v a pair of holes, and keep only u (or v) terms. (Nucleons above and holes below a major shell can both be considered "particles" in this picture.) Thus for the finite nucleus the π and σ modes become identified and acquire a common mass of $m_\sigma/2$. The Hamiltonian becomes

$$H'_b = 1/2m_\sigma N_6 + (1/8c^2V)[-N_6^2 - O_6 + (3/2)O_5] \tag{44}$$

where the Casimirs are now functions of the one variable u and have the same form listed in Eq. (41); i.e., $O_6 = O_6(u)$ and so on.

Now we set $m_{\sigma/2}$ equal to the pairing energy and get $m_\sigma \sim 3$ Mev. This reduces the coherence length by half from that in the infinite medium. Using Eq. (27) for $N_F, p_F \sim 240$ MeV and degeneracy $\nu = 4$, we find from Eq. (33)

$$c \sim 53 \text{ MeV} \tag{45}$$

for the value of the condensate. Then Eq. (32) gives

$$G = 0.02 \tag{46}$$

which justifies the perturbation theory we have used.

To correct for the finite volume effect we note that the momentum uncertainty Δp reduces the propagator in our tree diagrams relative to the infinite volume case. For example, the propagator

$$\frac{1}{m_\sigma^2} \to \frac{1}{m_\sigma^2 + \Delta p^2} = \frac{1}{m_\sigma^2}\left(\frac{1}{1 + \Lambda^2/R^2}\right) \sim \frac{1}{3m_\sigma^2}, \tag{47}$$

where R is the nuclear radius. Hence in this example the reduction factor C is $\sim 1/3$. C should actually be different for different diagrams but we choose to preserve the integrity of our infinite volume results by applying it as an overall correction factor:

$$H = 1/2m_\sigma N_6 + (C/8c^2V)[-N_6^2 - O_6 + (3/2)O_5]. \tag{48}$$

This is the Hamiltonian that is to be compared to the IBM. The coefficient

$$\frac{C}{8c^2V} \sim \frac{40C}{A} \text{ MeV} \sim 200C \text{ keV} \tag{49}$$

for $A \sim 200$. In the IBM the phenomenological coefficients in the platinum region vary from -30 to -50 keV for the O_6 term and from 40 to 60 keV for the O_5 term. Equation (48) reproduces these average results for $C \sim 1/4$. The dependence on A is consistent with the larger coefficients for ^{134}Ba (-74 keV and 87 keV, respectively).

A caveat remains. We have ignored the coupling of the π and σ modes to density modes. This occurs via fermion bubbles which we evaluate near the Fermi surface. The π_0 mode in both protons and neutrons is expected to couple to the isospin density ($\sim \Gamma_7$) and get absorbed into a plasmon mode. With a plasmon energy of ~ 10 MeV, the π_0 contributions would be suppressed. The π_i modes, on the other hand, would not disappear even though they can couple to quadrupole deformations ($\sim \Gamma_0 \Gamma_i$).

Our analysis applies to protons and neutrons separately. BCS theory has no mechanism for their mixing; they could, however, mix by coupling to density modes. In this case the degeneracy v would effectively double and the coefficient in Eq. (49) would be reduced by half. Also, we have no mechanism for generating the $O(3)$ piece spontaneously. There must always be an intrinsic $O(3)$ piece present in this and, by extension, in the $U(3)$ Hamiltonian.

6. Fermions and Supersymmetry

In order to demonstrate the supersymmetry inherent in the BCS formalism [18] let us express the $O(6)$ to $O(5)$ symmetry-breaking in the equivalent language of $SU(4)(\sim O(6))$ breaking to $\mathrm{Sp}(4)(\sim O(5))$. As before, we have six bosonic fields, their conjugate momenta, and four fermionic fields corresponding to a spin $\frac{3}{2}$ object. We represent the bosonic fields as the six independent components of a 4×4 antisymmetric matrix $\mathbf{\Phi}$ and the conjugate momenta similarly as $\mathbf{\Pi}$. The fermions form a four-component spinor $\mathbf{\Psi}$. Then the SU(4) BCS Hamiltonian equivalent to Eq. (28) is

$$H = \mathrm{Tr}[\mathbf{\Pi}^\dagger \mathbf{\Pi}] + 2G^2 \mathrm{Tr}[(\mathbf{\Phi}^\dagger \mathbf{\Phi} - c^2)^2] + \sqrt{2}G[\mathbf{\Psi}\mathbf{\Phi}^\dagger \mathbf{\Psi} + \mathbf{\Psi}^\dagger \mathbf{\Phi} \mathbf{\Psi}^\dagger]. \tag{50}$$

In the above, $\mathbf{\Psi}$ and $\mathbf{\Psi}^\dagger$ are taken to be row (or column) matrices according as they are to the left (or right) of $\mathbf{\Phi}$. Symmetry-breaking to the ground state

$$\mathbf{\Phi} \to c\eta, \tag{51}$$

$$\eta = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{Sp}(4) \tag{52}$$

leads us to an equivalent description of the nucleus as in Section 2.

Let us form the hybrid spinors

$$\mathbf{Q} = \mathbf{\Pi}^\dagger \mathbf{\Psi} + i\sqrt{2}G(\mathbf{\Phi}^\dagger \mathbf{\Phi} - c^2)\mathbf{\Psi}^\dagger,$$
$$\mathbf{Q}^\dagger = \mathbf{\Psi}^\dagger \mathbf{\Pi} - i\sqrt{2}G\mathbf{\Psi}(\mathbf{\Phi}^\dagger \mathbf{\Phi} - c^2). \tag{53}$$

Then it is easily seen that

$$H = \mathrm{Tr}\{\mathbf{Q}, \mathbf{Q}^\dagger\}, \tag{54}$$

where the curly bracket denotes anticommutation. So the Hamiltonian can be written in a supersymmetric form. The algebra after linearisation corresponds to that of osp $(\frac{8}{12})$ [18], of which the more familiar $SU(\frac{4}{6})$ is a subalgebra. However, since the $\mathbf{Q}$'s are not nilpotent, we do not have a true supersymmetry. This is reminiscent of the physical observation that we have two bosons per fermion, and their masses are not the same either, in contrast to the usual supersymmetry.

Since the starting Hamiltonian exhibits a kind of supersymmetry, we might expect that some relic of this will be evident even after we have corrected for second-order effects. With this hope, we calculate the fermion tree diagrams (we already have the bosonic contribution). We revert to the notation of Section 2. The calculation is simpler in this notation since the Majorana condition on the Ψ's requires that

$$[\Psi^\dagger, \mathcal{O}\Psi] = 0 \tag{55}$$

if $\mathcal{O}$ is the unit matrix or the product of three or four different Γ's. We give the fermions a small kinetic energy K to facilitate delicate corrections at the limit $K \to 0$ which we take in the end. As before, we find that ultimately divergences cancel and we are left only with small finite corrections from potentially divergent terms.

In order to illustrate these ideas, let us examine the pair of diagrams in Fig. 6, in which a σ meson and a fermion of incident energy $E_\Delta = \sqrt{K^2 + \Delta^2}$ scatter off each other. Their contribution, with that of their conjugate processes, is

$$\frac{iG^2}{2m_\sigma V^2}\left[\Gamma_0 \frac{1}{E_\Delta + m_\sigma - \Delta\Gamma_0 - K\Gamma_7}\Gamma_7\Gamma_i + \Gamma_7\Gamma_i \frac{1}{E_\Delta - m_\sigma - \Delta\Gamma_0 - K\Gamma_7}\Gamma_0\right](a_0 a_i^\dagger + a_i a_0^\dagger), \tag{56}$$

where the fermion propagator is written in a matrix form reminiscent of Eq. (6) (E_f is the energy of the propagating fermion):

$$G_\Delta(K) = \frac{1}{E_f - \Delta\Gamma_0 - K\Gamma_7}. \tag{57}$$

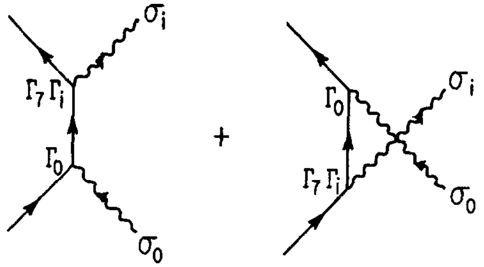

FIG. 6. Typical meson-fermion scattering diagrams.

After some manipulation Eq. (56) yields as the result before projection

$$\frac{1}{K}\times(-i)\frac{G^2}{2m_\sigma V^2}\Gamma_0\Gamma_i(a_0 a_i^\dagger + a_i a_0^\dagger). \tag{58}$$

In spite of the K in the denominator, this expression is actually finite because the numerator $\Gamma_0\Gamma_1$ is also of order K for on-shell states. Incidentally, if we had not allowed for a small but non-vanishing K, the diagrams would have cancelled by the Majorana condition above. Thus we see there is a discontinuity at $K=0$ and extreme care has to be taken to retain all the terms till after the projection, when we can take the limit $K\to 0$. Note also that the coefficient $1/c^2V^2$ of the fermionic contribution is the same as for the bosons. The sum of the terms of the above mixed boson-fermion type (henceforth called $\mathbf{L}\cdot\mathbf{s}$ type for reasons that will be clear soon) are

$$\frac{G^2}{m_\sigma^2 V^2}\Gamma_i\Gamma_j(a_i^\dagger a_j - b_i^\dagger b_j) + iG^2\left[\frac{3\Gamma_7\Gamma_i}{m_\sigma^2 V^2} - \frac{\Gamma_0\Gamma_i}{2m_\sigma K V^2}\right](a_0^\dagger a_i + a_i^\dagger a_0). \tag{59}$$

The fermion mass terms are fairly straightforward, so we will not detail the calculation.

The projection to particle-number-diagonal space is somewhat tricky. We first project the operators $\mathcal{O}$ onto states of positive (or negative) energy and then onto positive (or negative) particle number and sum over the two projections:

$$\mathcal{O}\to\sum_{\pm}\frac{1\pm\Gamma_7}{2}\frac{|E|\pm H}{2|E|}\mathcal{O}\frac{|E|\pm H}{2|E|}\frac{1\pm\Gamma_7}{2}. \tag{60}$$

This yields, for example,

$$\begin{aligned}\Gamma_0\Gamma_i &\to \Gamma_0\Gamma_i K/2\Delta,\\ \Gamma_7\Gamma_i &\to \Gamma_0\Gamma_i/2,\end{aligned} \tag{61}$$

and so on. Adding together all the second-order contributions and putting in the volume correction C we get as the final result for the fermionic pieces

$$H_{bf'} + H'_f = \frac{1}{2} \Delta N_f + \frac{C}{8c^2 V} \left[-\frac{1}{2} N_f N_6 + \sum_i L_{0i} s_{0i} + \frac{1}{4} \sum_i s_{0i} s_{0i} \right]. \tag{62}$$

In the above we have made a phase rotation $a_0 \to i a_0$ in order to write the bosonic pieces in the standard form L_{0i} for angular momenta. The Γ's have been similarly disguised as spin: $s_{0i} = (i/2)\, \Gamma_0 \Gamma_i$. By this notation we have betrayed our fond hope that the fermionic contributions can be coupled to the bosonic contributions Eq. (48) in such a way as to yield Casimirs of Spin (6) and Spin (5):

$$\begin{aligned} C_{\text{Spin}(5)} &= 1/2 \sum_{ij} (L_{ij} + s_{ij})^2, \\ C_{\text{Spin}(6)} - C_{\text{Spin}(5)} &= O_6 - O_5 + 2 \sum_i L_{0i} s_{0i}. \end{aligned} \tag{63}$$

However, this is not to be. The fermionic contributions turn out to be too small [19, 20] by about a factor of 2 because of extensive cancellations. This is clear from a glance at Eq. (59)—the first term drops out completely because of the equivalence of the a's and b's after projection.

The result is not surprising because there was no strict supersymmetry between bosons and fermions from the beginning. At the same time, we are reminded that our σ-model Hamiltonian is in fact an approximation to the underlying BCS theory. In particular, there is a difference in BCS in the coupling of the σ and π modes to the fermions which we have neglected in our model. It is also possible, of course, that the assumed fermion space is inadequate and needs to be enlarged to describe the platinum region.

7. The $SU(3)$ Chain

Let us now turn to the other prominent IBM chain, that of $U(6) \supset U(3) \supset O(3)$ which describes rotational nuclei. The $U(3)$ to $O(3)$ symmetry-breaking is also of spontaneous origin, as the following analysis shows. We assume a starting $U(3)$ symmetry and adapt it to the BCS system with the help of a pseudospin [21, 22]: in particular, our valence fermions consist of six ($\mathrm{l} = 1, \mathrm{s} = \frac{1}{2}$) states which couple into spin singlets and spin 0 and 2 bosons (the spin here simply takes care of the antisymmetry of the fermions):

$$(1 \times \tfrac{1}{2}) \otimes (1 \times \tfrac{1}{2}) = (1 \times 1 = 2 \oplus 0) \otimes (\tfrac{1}{2} \times \tfrac{1}{2} = 0). \tag{64}$$

The bosons can be represented as the six independent components of 3×3 symmetric matrices $\mathbf{\Phi}$ and the fermions as the spinor $\Psi = (\psi_{1\uparrow}\, \psi_{0\uparrow}\, \psi_{-1\uparrow}\, \psi^\dagger_{1\downarrow}\, \psi^\dagger_{0\downarrow}\, \psi^\dagger_{-1\downarrow})$ in analogy whith Eq. (2) (the subscripts are the z-components of l and s).

The Hamiltonian is written in a somewhat mixed notation as

$$H_{U(3)} = \mathrm{Tr}[\mathbf{\Pi}^\dagger \mathbf{\Pi}] + 4G^2\, \mathrm{Tr}[\mathbf{\Phi}^\dagger \mathbf{\Phi} \mathbf{\Phi}^\dagger \mathbf{\Phi}] - \frac{16c^2G^2}{3} \mathrm{Tr}[\mathbf{\Phi}^\dagger \mathbf{\Phi}] + G\Psi^\dagger[\Gamma_m \sigma_m + \Gamma'_m \pi_m]\Psi. \tag{65}$$

In the above, the Yukawa piece is given in terms of real fields σ and π, where

$$\mathbf{\Phi} = (\sigma + i\pi)/\sqrt{2}, \tag{66}$$

with σ and π further expanded as

$$\sigma = \sum_m \sigma_m \lambda_m/\sqrt{2}, \qquad \pi = \sum_m \pi_m \lambda_m/\sqrt{2}. \tag{67}$$

Here λ_m, $m = 0, 1, ..., 5$ are the Gell–Mann matrices of SU(3) (or any unitarily equivalent set) with $\lambda_0 = \sqrt{2/3}\mathbf{1}$. The λ's are normalised so that the trace of their square is two. The spinors are coupled to the bosons using the 6×6 matrices

$$\begin{aligned} \Gamma_m &= \tau_1 \otimes \lambda_m = \begin{pmatrix} 0 & \lambda_m \\ \lambda_m & 0 \end{pmatrix} \\ \Gamma'_m &= \tau_2 \otimes \lambda_m = \begin{pmatrix} 0 & -i\lambda_m \\ i\lambda_m & 0 \end{pmatrix}, \end{aligned} \tag{68}$$

where the τ's are Pauli matrices. The Γ's are not Clifford algebra matrices but serve to couple the fermions into pairs such as $\psi_{1\uparrow}\psi_{1\downarrow}$ etc.

We set the condensate proportional to the unit matrix

$$\langle \mathbf{\Phi} \rangle = \sqrt{2/3}c\mathbf{1} \tag{69}$$

and note that the mass conditions Eq. (21) are satisfied, with

$$m_\sigma = 4\sqrt{2/3}Gc. \tag{70}$$

Finally we break up the σ and π fields as

$$\begin{aligned} \sigma &= (a + a^\dagger)/\sqrt{2m_\sigma V} \\ \pi &= (b + b^\dagger)/\sqrt{2m_\pi V} \end{aligned} \tag{71}$$

and calculate the diagrams. For the bosonic diagrams we do not need to expand the fields in terms of the λ matrices, but we do have to be careful with divergences. As before we take $m_\pi \to 0$ in the end and divergent diagrams cancel. The following definitions for the Casimirs of $U(3)$ and $O(3)$ are used to rewrite the result for the bosonic part:

$$\begin{aligned} U_3(\sigma) &= \sum_i \mathrm{Tr}[a^\dagger \lambda_i a]\,\mathrm{Tr}[a^\dagger \lambda_i a], \qquad \lambda_i \in U(3) \\ O_3(\sigma) &= 4 \sum_i \mathrm{Tr}[a^\dagger \lambda_i a]\,\mathrm{Tr}[a^\dagger \lambda_i a], \qquad \lambda_i \in O(3). \end{aligned} \tag{72}$$

In the above, the antisymmetric λ_i are the elements of the $O(3)$ subgroup of $U(3)$. The $U(3)$ Casimir is three times smaller than in the IBM, and the $O(3)$ Casimir is given an extra factor of four to conform to the usual convention.

The following identities are similarly useful for the fermionic calculations:

$$\begin{aligned} [\Gamma_j, \Gamma_i]\, a_j^\dagger a_i &= -\Gamma_k\, \mathrm{Tr}[a^\dagger \lambda_k a - a \lambda_k a^\dagger] \\ \{\Gamma_j', \Gamma_i'\} b_j^\dagger b_i &= \Gamma_k'\, \mathrm{Tr}[b^\dagger \lambda_k b + b \lambda_k b^\dagger]. \end{aligned} \tag{73}$$

We identify the above as $\mathbf{L}\cdot\mathbf{s}$ type generators. $\mathbf{L}$ and $\mathbf{s}$ are defined as

$$\begin{aligned} L_i(\sigma) &= \mathrm{Tr}\left[a^\dagger \frac{\lambda_i}{2} a \pm a \frac{\lambda_i}{2} a^\dagger\right], \\ s_i &= \Gamma_i/2, \end{aligned} \tag{74}$$

where the $\pm$ signs are for the symmetric ($\in U(3)/O(3)$) and antisymmetric ($\in O(3)$) λ's respectively. With these definitions, and after projection, the result is

$$\begin{aligned} H'_{U_3} = {} & 1/2m_\sigma N_6 + 1/2\Delta N_f + 3C/32c^2 V[-U_3 + 3/8 O_3 \\ & - 3\mathbf{L}\cdot\mathbf{s}(U_3) + 2\mathbf{L}\cdot\mathbf{s}(O_3) - 4\mathbf{s}\cdot\mathbf{s}(U_3/O_3)]. \end{aligned} \tag{75}$$

The condensate $c = 65$ MeV ($v = 6$). As noted earlier, the $U(3)$ Casimir here is three times as small as in the IBM. So in the IBM notation the ratio of the coefficients of U_3 and O_3 is $-\frac{8}{9}$. This agrees roughly with the available data [4] in which the ratio varies from about -1 to $-\frac{1}{3}$. The $1/A$ dependence is also consistent. It is to be expected that as in the $O(6)$ case an intrinsic $O(3)$ piece is always present.

The $U(5)$ chain is not easily amenable to a BCS interpretation. While a pseudospin picture with $(\mathrm{l} = 2, \mathrm{s} = \frac{1}{2})$ and a 5×5 symmetric bosonic matrix seems natural, the spin 4 part would have to be suppressed to conform to the IBM. It could alternately be regarded as an $O(6)$ chain with explicit $U(5)$ symmetry-breaking.

8. Conclusions and Speculations

We find that two important group chains of the IBM are of the spontaneous symmetry-breaking type, reflecting an underlying BCS mechanism. The magnitude of the resulting Hamiltonian is derived from ballpark values of the nuclear density and pairing energy while the relative sizes of its different terms are uniquely determined by the infinite-medium field theory. Heavier nuclei have smaller Hamiltonians—they scale as $1/A$, A being the atomic number.

This is a parameter-free theory, and its limitations directly reflect the limitations of the basic assumptions. The fuzziness of the overall scale results from the adaptation of infinite-medium results to the finite medium. The relative sizes of the different terms are taken for precision to be that of the infinite medium. However, since the fuzzfactor C is actually different for different diagrams, the *relative* sizes of the terms are also a little fuzzy. In addition, there are questions about the accuracy with which the σ-model reflects the BCS theory—as discussed earlier, the Yukawa coupling constant G is not necessarily constant or even the same for σ's and π's except in the limit of zero momentum-transfer.

On the plus side, the overall results are insensitive to details of the starting Hamiltonian and even to details of the adaptations to finiteness. They are applicable to all nuclei of the type discussed, with the single identifier A changing the scale of the effect for different nuclei. While the fermionic results depend on the assumption of a valence shell with a specific symmetry, the bosonic part is determined only by the starting symmetry (whether $O(6)$ or $U(3)$); the valence symmetry provides merely the rationale for the latter. The spontaneous effects are, in short, surprisingly robust. On the whole, BCS seems to be the secret of IBM.

One might remark on the coincidence that the coherence length of the Cooper pairs is so close to the nuclear radius—does this mean then that they are one and the same thing, and that the nucleus is a set of Cooper pairs held together by surface tension? In this case, there remains only one basic parameter, the saturation nuclear density. The pairing energy would then die as $1/A^{1/3}$ and disappear for an infinite medium. The relevance of BCS theory in the nucleus inspite of the large coherence length leads one to ask if other aspects of BCS might also be important. For example, are there vortices in nuclei? One of the effects of a vortex is that it allows rotational spectra about an axis of symmetry. This it does by effectively breaking the symmetry about the axis—all directions are no more equivalent but distinguished by an integrable phase. While the anomalous spectra would probably have a shell-model explanation, a BCS description would be simple and universal.

Finally, it is curious that the two different cases of symmetry-breaking that occur in the nucleus, the chirality pairing in pion physics and the nucleon pairing we have been discussing, have similar values for their condensate. Perhaps in some devious fashion this derives from their common root, QCD.

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