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Approximating a Function by Choosing a Covering of Its Domain and k Points from Its Range*

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A. INTRODUCTION

1. *Preliminary Remarks*

Suppose that we would like to compute the value of a real-valued function φ for every point θ that might be selected from some given set E, but that this is infeasible and that we have to settle for approximation in the following manner: a finite covering[1] of E, denoted Σ and containing j sets—to be called *cells*—is chosen. To each cell of Σ we assign a real number, called an *outcome*, from a k-element set $A \subset \mathbb{R}$ of possible outcomes. Given the θ that is currently of interest we find a cell containing that θ, and we take as our *estimate* of $\varphi(\theta)$ the outcome assigned to that cell. We define the *error* of this procedure as the supremum, over all θ in E, of the possible distances between the outcome so obtained and the true value $\varphi(\theta)$. Our objective is to minimize error by a suitable choice of Σ, of A, and of the assignment rule.

Now given Σ and A, the choice of assignment rule is a simple matter. It

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[1] Our reasons for not requiring the covering to be a partitioning are given in footnote 6 below.

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is easily shown[2] that as long as φ has a supremum and an infinum on every cell of Σ, a best (error-minimizing) assignment of the outcomes in the given set A to the cells in Σ is given by the *closest-to-midvalue outcome function*. To make this precise, let each cell of Σ be indexed by an element of a set M. Thus the typical cell can be denoted σ_m, with m in M. If the elements of A are assigned to the cells of Σ by an outcome function $h: A \to M$, then the error of our approximation procedure is $\sup\{|\varphi(\theta) - h(m)| : m \in M, \theta \in \sigma_m\}$. For every m in M, let

$$u_m \equiv \inf\{\varphi(\theta) : \theta \in \sigma_m\}$$
$$v_m \equiv \sup\{\varphi(\theta) : \theta \in \sigma_m\}$$
$$w_m \equiv \tfrac{1}{2}(u_m + v_m).$$

Then the *closest-to-midvalue outcome function for* (φ, Σ, A) is denoted h_A (with reference to φ and Σ suppressed) and is defined by

$h_A(m) \equiv$ that element of A closest to w_m, with ties broken downward.[3]

For any outcome function h, error is not less than it is for the outcome function h_A. We assume from now on that for any given pair (φ, Σ), the function h_A is the assignment rule (outcome function) that is used.

Figure 1 below provides an example with a number of special features that are useful in subsequent sections. (Figure 1 will be discussed in more detail later on.) Here E is the unit square; a point θ is a pair (θ_1, θ_2); the covering Σ is a uniform 64-cell grid (a point on the boundary of several cells lies in all those cells); $\varphi(\theta) = \theta_1 + \theta_2$; and A is the three-element set $\{\frac{3}{8}, 1, \frac{13}{8}\}$. For a given cell, the midvalue is the value taken by φ at the cell's center. The three solid diagonal contour lines marked $\frac{3}{8}$, 1, $\frac{13}{8}$ are midvalue contour lines for certain cells, as are the 12 broken diagonal contours. The closest-to-midvalue function h_A assigns to each cell that element of $A = \{\frac{3}{8}, 1, \frac{13}{8}\}$ which is written inside the cell.[4]

Now suppose we associate two distinct kinds of *cost* with the approximation procedure. One cost is j, the number of cells in the covering Σ; and the other cost is k, the number of elements in A, the set of available outcomes. These would appear to be the simplest costs that one can suggest without giving any additional structure to E or to Σ. We shall argue below (in Section A2) that they are natural cost measures in two particular settings: (i) Σ and h_A describe a "mechanism" to be used by an

[2] See Hurwicz and Marschak (1984).

[3] In an alternative definition ties would be broken upward; that would achieve the same error.

[4] The shaded cells are explained in Section A4 below.

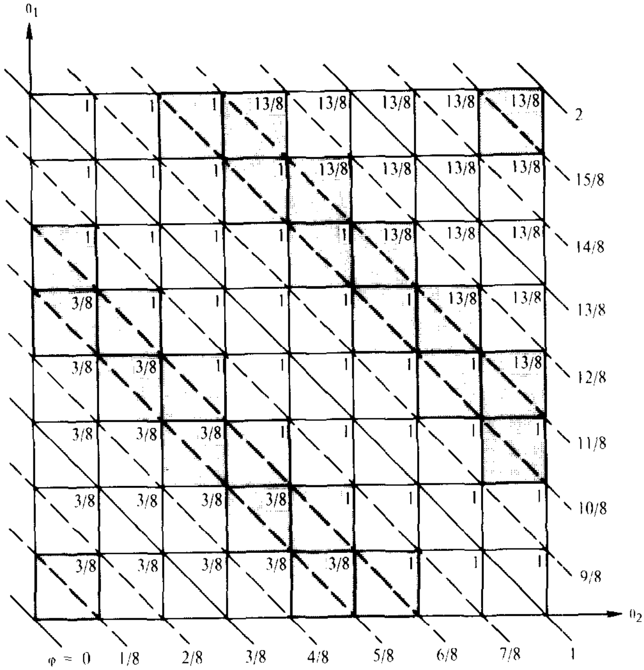

FIG. 1. $\varphi = \theta_1 + \theta_2$. The diagonal lines are φ-contours.

organization each of whose members observes a different coordinate of θ; and (ii) Σ and h_A describe a certain model of distributed computing.

In general, if k is fixed and j is increased, and a best fresh choice of a k-cell covering and a j-element available-outcome set is made, then error is improved; error can certainly not be made worse. Similarly, if j is fixed and k is increased then error improves or stays the same. One would like, accordingly, to find pairs (Σ, A) that are efficient with regard to the two costs and with regard to error. More precisely, if Σ is a covering of E with $\#\Sigma = j$ (# means "number of elements in"), and if A is an outcome set with $\#A = k$, then call the pair (Σ, A) a *(j, k)-pair*. Call a (j^*, k^*)-pair (Σ^*, A^*) *not larger than a* (j, k)-pair (Σ, A) if $k^* \leq k, j^* \leq j$. A (j, k)-pair (Σ, A) is *efficient* if every not larger pair has an error at least as high as the error of (Σ, A) when in all cases the assignment rule used is the closest-to-midvalue function. Our ultimate goal is to solve the following general problem:

For given φ, E, j, k find the efficient (j, k)-pairs (Σ, A). $\quad (*)$

For certain pairs (φ, E) we have obtained (in Hurwicz and Marschak, 1984) solutions to Problem $(*)$ when the coverings that can be chosen are required to have certain properties, in particular the "grid" property, which we shall define below (in Section A3). But, in general, Problem (*) is very difficult. As a step toward the solution of Problem (*) we consider in the present paper the following simpler problem.

> Given φ, E, and a particular j-cell covering Σ of E, find a set A of k outcomes that minimizes error when outcomes are assigned to cells according to h_A, the closest-to-midvalue function for (φ, Σ, A). (**)

We shall develop two alternative sufficient conditions on a k-element set A—to be called *equal-error conditions*—that guarantee A to be a solution to Problem (**). We shall apply these conditions to obtain an explicit solution to Problem (**) in one case: E is a rectangle in $\mathbb{R}^n$, Σ is a uniform "grid," and φ is linear.

Now it is clear that if there is no restriction on the coverings that can be chosen, then any (j, k)-pair (Σ, A) for which $k \neq j$ is inefficient. One does not want to pay for unused outcomes, and if two distinct cells of a j-cell covering are assigned the same outcome, then one can merge them to obtain a $(j - 1)$-cell covering. Nevertheless, we do not impose the requirement that $k \neq j$ in the statement of the two general problems. The reason is that there may be sharp restrictions on the j-cell coverings that are choosable. In particular, we may want to restrict attention to coverings whose cells are *cartesian products*. That is to say, there are n sets, $E_1, \ldots, E_n$, such that $E = E_1 \times E_2 \times \cdots \times E_n$, and the typical cell of the covering, say σ_m, can be written $\sigma_m^1 \times \cdots \times \sigma_m^n$, with[5] $\sigma_m^i \subseteq E_i$, $i = 1, \ldots, n$. In that case, Problem (*), with the cartesian-product restriction added, becomes quite natural. For it may well be that one cannot coarsen the given j-cell covering into a k-cell covering, with $k < j$—by merging all cells that are assigned a given outcome—without thereby destroying the required cartesian-product property.[6] Solving Problem (**)—with the

[5] More generally, we may define Σ, with index set M, to be a cartesian-product covering of E if $E \subseteq E_1 \times E_2 \times \cdots \times E_n$ and if, for all m in M, $\sigma_m = E \cap (\sigma_m^1 \times \cdots \times \sigma_m^n)$, where $\sigma_m^i \subseteq E_i$, all i.

[6] For greater generality, we have not required the covering to be a partitioning. In particular, we may be interested in a particular j-cell cartesian-product covering (because of its low error or or other reasons) and that covering may have the following property: the covering cannot be replaced by a j-cell cartesian-product *partitioning* such that any two points lying in a cell of the partitioning also lie in some cell of the original covering. That is to say, the "overlap" of the original covering cannot be removed without increasing the number of cells or destroying the cartesian-product property. Examples of such coverings are easy to construct.

Note also that if a covering (with or without the cartesian-product property) has overlap at *boundary points only*, then even if the overlap were removed, doing so would not, in general, change the overall error attained for a given outcome set. That is the case since our definition of error involves suprema; the supremum of the errors in a cell σ_m (i.e., the supremum of the distances $|h(m) - \varphi(\theta)|$ over all θ in σ_m) is independent, in general, of the errors at boundary points.

cartesian-product restriction added—is again a step toward solving the restricted version of Problem (*).

2. *Plan of the Paper*

We proceed in the next section (A3) to two specific motivations for Problems (*) and (**), with the cartesian-product restriction added. In Section A4 we illustrate our main results by means of two examples. These results are (i) two "equal-error" theorems (Theorems 1 and 2), asserting that if a k-element set A satisfies one of the "equal-error" conditions (and several additional conditions are met as well) then A solves Problem (**); and (ii) Theorem 3, which applies Theorems 1 and 2 to obtain an explicit solution to Problem (**) for the special case mentioned above: E is a rectangle in $\mathbb{R}^n$, E is a uniform grid, and φ is linear. In Part B, we establish Theorems 1 and 2. In those theorems we do *not* require the covering Σ_M of the set E to have the cartesian-product property. The theorems extend, moreover, to certain cases of an infinite M, as long as φ is bounded on E. In Part C, we establish Theorem 3. In Part D we apply Theorem 3 to the study of the informational efficiency of a certain finite-message price mechanism in a very simple class of exchange economies. In Part E we formulate an open question.

3. *Two Motivations*

3.1. *A motivation from the economic theory of mechanisms.* Consider an organization composed of n persons, called $1, \ldots, n$. Person i observes the *local environment* θ_i, which lies in a set E_i. Let $\theta \equiv (\theta_1, \ldots, \theta_n)$ and $E \equiv E_1 \times \cdots \times E_n$. The organization would like to take an appropriate *action*, namely $\varphi(\theta)$, in response to the current environment φ; $\varphi(\theta)$ lies in some set A. We shall call φ the *desired-outcome function.*[7] Since information about θ is initially dispersed among the n persons, we have to design a scheme in which some suitable communication among agents occurs. We suppose, in particular, that the organization will use a *mechanism on E*, which we have to design. A mechanism on E is a triple $\pi = [M, (\mu_1, \ldots, \mu_n), h]$, where M is a set called the *message space*, with typical element m, called a *message*; μ_i is a correspondence from E_i to M; and h, called (as above) the *outcome function*, is from M to A. One way to interpret the operation of the mechanism is as follows: a "trial" message m in M is announced to all n persons. Person i inspects his current local environment θ_i to determine whether or not $m \in \mu_i(\theta_i)$. If so, person i signals "Yes." If at least one person fails to signal "Yes," then a new trial message is announced. But if all n persons signal "Yes," then m has been found to be an *equilibrium message for* θ, i.e.,

[7] Also often called the *performance function.*

$$m \in \mu(\theta), \qquad \text{where } \mu(\theta) \equiv \cap_i \mu_i(\theta_i).$$

In that case, the action or *outcome* $h(m)$ takes place. The mechanism π *realizes* the desired-outcome function φ if and only if the following holds for all θ in E:

(i) $\mu(\theta) \neq \emptyset$ and

(ii) for all m in $\mu(\theta)$, $h(m) = \varphi(\theta)$.

The designer wishes to find a mechanism that realizes φ and is informationally least costly among all mechanisms realizing φ. The most widely studied measures of cost are various measures of the size of the message space M.

Particularly well studied has been the case in which the organization is a pure exchange economy with l commodities; the n persons are consumers; θ_i describes person i's current preferences and initial commodity bundle (endowment); and $\varphi(\theta)$ is a Pareto-optimal trade, i.e., an exchange of commodities leading to a new commodity bundle for each consumer, the bundles so obtained having the property that no consumer can be made better off without making some other consumer worse off. Of particular interest is the "price" or "competitive" mechanism, wherein the announced trial message comprises a proposed trade together with a price for each commodity; person i signals "Yes" if, at the announced prices, the income generated by the sale of his initial bundle could not purchase a better bundle for him than what he would end up with under the announced proposed trade. The mechanism realizes the required function φ. It has, moreover, been shown to do so with the smallest possible message space (when appropriate regularity conditions are imposed on the admissible mechanisms) for a number of alternative concepts of message-space size.[8]

In the bulk of the mechanisms in the literature—including the work on the informational minimality of the competitive mechanism—both the message space and the set of outcomes are *continua*. But real communication and computing technologies do not in fact permit a message or an outcome to be a point of a continuum. Messages and outcomes must be rounded off, and the benefits of greater precision have to be weighed against the costs. Accordingly, it is of considerable interest to study mechanisms in which message space and outcome set are *not* continua but are, rather, (i) infinite but discrete (e.g., they are the set of all integers, or the set of all integer pairs); or (ii) finite. Infinite-but-discrete mecha-

[8] For a recent and extensive survey, see Hurwicz (1986).

nisms were studied (in Hurwicz and Marschak, 1985); we there sought, in particular, discrete analogs to the results about informational minimality of the continuum price mechanism. Such discrete analogs deal with suitably constructed discrete *approximations* to the continuum price mechanism.

In studying infinite-but-discrete or finite mechanisms, we continue to permit the set E to be a continuum and also the set of possible desired outcomes, i.e., the image $\varphi(E)$. In particular, suppose that φ is real-valued and that $\varphi(E)$ contains the closed interval $[a, a + 1]$, where a is an integer. Then a mechanism that has, say, the integers as its message space, and also has the integers as its outcome set, (the set $h(M)$) *cannot* realize φ in our previous sense. For some θ, the true $\varphi(\theta)$ equals $a + \frac{1}{2}$. But the closest available outcomes are a and $a + 1$. Hence the mechanism's *error relative to* φ, namely $\sup\{|\varphi(\theta) - h(m)| : m \in \mu(\theta); \theta \in E\}$, must be at least $\frac{1}{2}$. If, on the other hand, the outcome set $h(M)$ consists of the integers plus all numbers of the form $x + \frac{1}{2}$, where x is an integer, then the analogous lower bound on error becomes $\frac{1}{4}$. Thus if one is given the desired-outcome function φ and wants to argue that an infinite-but-discrete mechanism $\bar{\pi} = [\bar{M}, (\bar{\mu}, \ldots, \bar{\mu}_n), \bar{h}]$ is informationally efficient, then it is not enough—as it was in the continuum case—to study only the size of the message space. Rather one has to argue that no other mechanism that has a message space no larger than $\bar{M}$ *and* has the outcome set $\bar{h}(\bar{M})$— or an outcome set no more costly than $\bar{h}(\bar{M})$ in some appropriate sense—can achieve an error relative to φ that is not smaller than the error of $\bar{\pi}$. In particular, one may want to study an infinite-but-discrete mechanism that approximates some continuum mechanism π, where π realizes φ and does so with the smallest message space among all (suitably regular) continuum mechanisms realizing φ. If one wants to obtain a discrete analog to π's informational minimality, then the best one can hope for is the following result: there is a sequence of infinite-but-discrete mechanisms, each approximating π and each with the same message space M^* *and* the same set of permissible outcomes, say A, such that by choosing an appropriate mechanism in the sequence one can get as close as desired to the *lower bound on error implied by the common outcome set*. Moreover, the error achieved by any member of the sequence cannot be matched by any (regular) mechanism whose permissible outcome set is again A but whose message space is smaller than M^*. Results of that sort were in fact obtained (in Hurwicz and Marschak, 1985) for a certain infinite-but-discrete approximation to the price mechanism.

Suppose one turns to *finite* mechanisms. Then again both message space *and* outcome set enter the assessment of a proposed mechanism; both of these sets are now finite. Given a desired outcome function φ, a finite mechanism $[\bar{M}, \bar{\mu}_1, \ldots, \bar{\mu}_n), \bar{h}]$ on E is efficient if there is no other mechanism $[M, (\mu_1, \ldots, \mu_n), h]$ such that

$$\#M \leq \#\overline{M}$$

$$\#h(M) \leq \#h(\overline{M})$$

$$\sup\{|\varphi(\theta) - h(m)| : m \in \mu(\theta);\ \theta \in E\} \leq \sup\{|\varphi(\theta) - \overline{h}(m)| : m \in \overline{\mu}(\theta),\ \theta \in E\}$$

and one of these inequalities is strict (# means "number of elements in"). The cost measures $\#M$ and $\#h(M)$ may be thought of as simple measures of the various efforts made by the organization's members as the sequence of trial messages unfolds and a final action (outcome) is generated: efforts of observing the local environment, communicating, and choosing the action.

Now a mechanism $\pi = [M, (\mu_1, \ldots, \mu_n), h]$ can be interpreted in a more compact way. For π defines a *cartesian-product covering of E*, say Σ_M, whose cells are indexed by the elements of M. The typical cell is

$$\sigma_m = \mu^{-1}(m) = \{\theta \in E : m \in [\mu_1(\theta_1) \cap \mu_2(\theta_2) \cap \cdots \cap \mu_n(\theta_n)]\}.$$

Moreover, each set σ_m is the cartesian product $\sigma_m^1 \times \cdots \times \sigma_m^n$, where

$$\sigma_m^i = \mu_i^{-1}(\theta) = \{\theta_i \in E_i : m \in \mu_i(\theta)\}.$$

One can also go in the reverse direction: a mechanism is completely defined by a cartesian-product covering Σ_m, with typical cell $\sigma_m = \sigma_m^1 \times \cdots \times \sigma_m^n$, together with an outcome function h on the index set M. The compact form (Σ_M, h) is natural for finite mechanisms (i.e., for M finite) but is well defined also when M is infinite-but-discrete. Reinterpreting the "trial announcement" procedure: a cell index (a message) m in M is announced; person i examines θ_i to see whether $\theta_i \in \sigma_m^i$; if all persons signal "Yes," then a cell containing θ—namely the cell σ_m—has been found. The outcome $h(m)$ thereupon takes place. Thus h may now be viewed as a function that assigns an outcome to each cell of the covering Σ_M. The pair (Σ_M, h) is precisely the object of study in our Problem (**). In Problem (**)—restated in the terminology just developed—we are given Σ_M and an integer $k \leq \#M$. We are to find a k-element outcome set A, so that the pair (Σ_M, h_A) is error-minimizing, where h_A is the closest-to-midvalue outcome function for (φ, Σ_m, A) and "error" means $\sup\{|\varphi(\theta) - h_A(m)| : \theta \in \sigma_m;\ m \in M\}$.

Of particular interest are coverings Σ_M which have not only the cartesian-product property but are also *grids*. Σ_M is a grid if given any $m_1, \ldots, m_n$ in M, there exists m^* in M such that $\sigma_m^* = \sigma_{m_1}^1 \times \sigma_{m_2}^1 \times \cdots \times \sigma_{m_n}^n$. If $E = E_1 \times \cdots \times E_n$ is, say, a rectangle in $\mathbb{R}^n$, then such a covering Σ_M defines a grid in the ordinary geometric sense. For a grid mechanism

(Σ_M, h), the sequence of "trial announcements" can, in principle, be dispensed with: person i determines a set $\sigma_{m_i} \subseteq E_i$ containing the current θ_i and reports m_i to a Center. Having received n such reports, the Center knows that $\theta = (\theta_1, \ldots, \theta_n)$ lies in $\sigma_{m^*} = \sigma^1_{m_1} \times \sigma^1_{m_2} \times \cdots \times \sigma^n_{m_n}$ and thereupon takes the action $h(m^*)$.

3.2. *A distributed-computing motivation.* Suppose we would like to compute—for every point e in a set $\bar{E}$—the real-valued function $\bar{\varphi}(e)$. Suppose $\bar{\varphi}(e)$ can be written

$$\bar{\varphi}(e) = \varphi[\theta_1(e), \theta_2(e), \ldots, \theta_n(e)].$$

Let $Z_i \equiv \{\theta_i(e) : e \in \bar{E}\}$. Suppose we have $n + 1$ processors. Processor i, $i = 1, \ldots, n$, approximately computes $\theta_i(e)$ and reports the result to a central processor, $n + 1$, who (approximately) computes φ.

We can vary the precision of these computations. In particular, we may enable Processor i to compute $\theta_i(e)$ to the accuracy of a fixed number of digits, or—if $\theta_i(e)$ is a vector—to compute each of its components to a given accuracy. That means that we partition Z_i into a finite number of sets; let σ^i denote the typical set of that partitioning. Processor i reports to the central processor the set in which $\theta_i(e)$ lies. Suppose the reports are, for example, $\bar{\sigma}^1, \ldots, \bar{\sigma}^n$. Having received these n reports, the central processor knows that $\theta(e) \equiv (\theta_1(e), \ldots, \theta_n(e))$ lies in the set $E \cap (\bar{\sigma}^1 \times \cdots \times \bar{\sigma}^n)$, where $E \equiv \{\theta(e) : e \in \bar{E}\}$. The central processor then proceeds to compute its approximation to φ. But the central processor also has a chosen precision, i.e., its output must be one of a finite number—say k—of numbers. The k numbers might, for example, be the set of all 8-binary-digit numbers. We may, however, let each of those k numbers be a *coding* of a suitably chosen *final answer*. The final answers so coded comprise the k-element set A.

We now have all the elements of Problem (**). We have a covering (in fact a partitioning) Σ_M of E with cells indexed by the elements of a set M. That covering has the cartesian-product property[9] and, as it happens, the grid property as well. Its typical cell is $\sigma_m = E \cap (\sigma^1_m \times \ldots \times \sigma^n_m)$, $m \in M$, where σ^i_m is a set in processor i's partitioning of Z_i. Given Σ_M, and given the integer k, Problem (**) is to find an error-minimizing k-tuple A, where error means $\sup\{|\varphi(\theta(e)) - h_A(m)| : \theta(e) \in \sigma_m;\ m \in M\}$ and h_A is, as before, the closest-to-midvalue function for (φ, Σ_M, A).

To summarize: *given* the precision to which each processor i in $\{1, \ldots, n\}$ computes θ_i, the solution to (**) yields a best set of k final answers, where k is determined by the central processor's precision.

[9] In the generalized sense of footnote 5 above.

4. *Two Examples Illustrating the Main Results*

Consider the example of Problem (**) provided by Fig. 1. Here E is the nonnegative unit square; θ is the pair (θ_1, θ_2); $\varphi(\theta) = \theta_1 + \theta_2$; and the covering Σ_M is a 64-cell uniform grid. Thus $\inf\{\varphi(\theta) : \theta \in E\} = 0$; $\sup\{\varphi(\theta) : \theta \in E\} = 2$. We may let the index set M be $\{(x, y) : x, y \in \{1, \ldots, 8\}\}$, and we may let $\sigma_{(x,y)}$ denote the cell which we reach by counting x cell-intervals from left to right and y intervals from bottom to top. Given the integer $k = 3$, Problem (**) is to find an error-minimizing outcome triple. A solution turns out to be $A = \{\frac{3}{8}, 1, \frac{13}{8}\}$, which, as one can readily check, achieves an error of $\frac{3}{8}$.[10] The number shown in each cell is the element of A assigned to the cell by h_A, the closest-to-midvalue outcome function for (φ, Σ_M, A).

To prove that $\{\frac{3}{8}, 1, \frac{13}{8}\}$ is a solution, we apply Theorem 1, the first of two general *equal-error theorems*. Both of these theorems deal with the *errors on the belts defined by the k elements of A*. The triple $\{\frac{3}{8}, 1, \frac{13}{8}\}$ defines a $(0, \frac{3}{8})$-belt, i.e., the set $\{\theta : 0 \leq \varphi(\theta) \leq \frac{3}{8}\}$; a $(\frac{13}{8}, 2)$-belt, i.e., the set $\{\theta : \frac{13}{8} \leq \varphi(\theta) \leq 2\}$; and two "interior" belts: the $(\frac{3}{8}, 1)$-belt ($\{\theta : \frac{3}{8} \leq \varphi(\theta) \leq 1\}$) and the $(1, \frac{13}{8})$-belt ($\{\theta : 1 \leq \varphi(\theta) \leq \frac{13}{8}\}$). Now suppose that by "error on a belt" we mean the largest value taken by $|\varphi(\theta) - h_A(m)|$ for θ in the belt and for all m with $\theta \in \sigma_m$. Then one sees that the error on the $(0, \frac{3}{8})$-belt is $\frac{3}{8}$. That error occurs at the point (0, 0). The error on the $(\frac{3}{8}, 1)$-belt is also $\frac{3}{8}$; it occurs, for example, at $(\frac{5}{8}, \frac{1}{8})$; for that point, $\varphi(\theta) = \frac{6}{8}$, but the point lies in four cells, including the cell $\sigma_{(5,1)}$ which is assigned the outcome $\frac{3}{8}$. The error in the $(1, \frac{13}{8})$-belt is $\frac{3}{8}$ as well. That error occurs, for example, at the point $(\frac{7}{8}, \frac{3}{8})$ for which $\varphi(\theta) = \frac{10}{8}$; but that point lies in four cells, including $\sigma_{(3,4)}$, which is assigned $\frac{13}{8}$. Finally the error on the $(\frac{13}{8}, 2)$-belt is also $\frac{3}{8}$ (it occurs at the point (1, 1)).

So $\{\frac{3}{8}, 1, \frac{13}{8}\}$ achieves equality of the four belt errors defined by the three outcomes, where "belt error" is given the straightforward meaning just used. One also verifies that $A = \{\frac{3}{8}, 1, \frac{13}{8}\}$ has the following *no-alien property*. In Fig. 1, we have shaded the *critical* cells in each belt, i.e., all the cells containing a point at which the belt error occurs. We note that every such shaded cell is assigned one of the outcomes defining the belt, rather than an "alien" outcome. (E.g., the shaded cells in the $(\frac{3}{8}, 1)$-belt are assigned $\frac{3}{8}$ or 1 but not $\frac{13}{8}$; the shaded cells in the $(0, \frac{3}{8})$-belt are assigned $\frac{3}{8}$ and not 1 or $\frac{13}{8}$.) Theorem 1 asserts that the equality of belt errors in the

[10] A naive (and wrong) conjecture might be that a "solution" is $A^* = \{\frac{1}{3}, 1, \frac{5}{3}\}$, since one then always has available an outcome that is within $\frac{1}{3}$ of the correct value of φ. But A^* has, in fact, an error higher than $\frac{3}{8}$. Consider the point $\theta^* = (\frac{5}{8}, \frac{1}{8})$, for which $\varphi = \frac{3}{4}$. That point lies in several cells, including the cell $\sigma_{(5,1)}$, whose midvalue is $\frac{5}{8}$. Since $\frac{5}{8}$ is closer to $\frac{1}{3}$ than to 1, the closest-to-midvalue rule assigns the outcome $\frac{1}{3}$ (out of the set A^*) to $\sigma_{(1,5)}$. Thus the error at θ^* (and hence also the overall error) is at least $\frac{3}{4} - \frac{1}{3} = \frac{5}{12} > \frac{3}{8}$.

sense just given, together with the no-alien property, imply that A is error-minimizing.

Turn now to Fig. 2. This time $E = \{\theta = (\theta_1, \theta_2): 0 \leq \theta_1 \leq 64; 0 \leq \theta_2 \leq 16\}$. Σ_M is a 16-cell grid composed of side-8 squares. An error-minimizing outcome triple turns out to be {40, 72, 104} which achieves an error of 40. This triple, however, *violates* the no-alien property. On the (72, 104) belt, the belt error—as we have so far defined it—is quickly verified to be 40; 40 is the error, for example, at the point $\theta^* = (40, 8)$. (Since $\varphi(\theta^*) = 80$, the error at θ^*—which lies in four cells—is max(80-40, 80-72, 104-80) = 40.) The point θ^*, however, lies, in particular, in $\varphi_{(5,1)}$ (using the previous indexing system) and that cell is assigned neither 72 nor 104, but rather the "alien" outcome 40.

So we cannot appeal to Theorem 1 in arguing that {40, 72, 104} is error-minimizing. Instead, we use a new definition of "belt error" and we appeal to Theorem 2. The new definition rests on a definition of the *(m, A)-error* on a belt, i.e., the contribution of the cell σ_m to the belt error.[11]

Let $\alpha = \inf\{\varphi(\theta): \theta \in E\}$, $\beta = \sup\{\varphi(\theta): \theta \in E\}$, and let $a_1, \ldots, a_k$ be the elements of an outcome set A, ordered from lowest to highest. Let r, s be two successive members of the $(k + 2)$-tuple $(\alpha, a_1, a_2, \ldots, a_k, \beta)$. Then

(i) The (m, A)-error on the (r, s)-belt equals the entire *cell error*, namely $\sup\{|\varphi(\theta) - h_A(m)| : \theta \in \sigma_m\}$ if $u_m < r \leq w_m < s < v_m$. (Recall that $u_m \equiv \inf\{\varphi(\theta): \theta \in \sigma_m\}$, $v_m \equiv \sup\{\varphi(\theta): \theta \in \sigma_m\}$; $w_m \equiv \frac{1}{2}(u_m + v_m)$.) [Thus in Fig. 2, consider the (40, 72)-belt and the cell $\sigma_{(5,1)}$. Only a portion of the cell lies in the (40, 72)-belt. We have $u_m = 32$, $v_m = 80$, $w_m = 56$. Since $32 < 40 < 56 < 72 < 80$, the (m, A)-error equals the entire cell error, which is 40 (that cell error occurs at the point (40, 8), for which $\varphi = 80$).]

(ii) If $h_A(m)$ is neither r nor s then the (m, A)-error is zero.

(iii) In all other cases the (m, A)-error is simply the supremum of the quantities $|h_A(m) - \varphi(\theta)|$ for all cells σ_m having points in the belt.[12]

The new definition of belt error is the maximum of the (m, A)-errors as just defined. For this new definition one readily verifies that the four belts defined by $A = \{40, 72, 104\}$ have equal belt errors. Theorem 2 asserts that if a k-element set A displays equality of belt errors in the new sense, then A is error-minimizing among all k-element sets.

[11] In its rigorous form—not the informal version just sketched—the first definition of belt error rests on a different definition of "(m, A)-error on a belt." See Section B1 below.

[12] Some further detail is required to cover points on the boundary of two belts and lying in more than one cell.

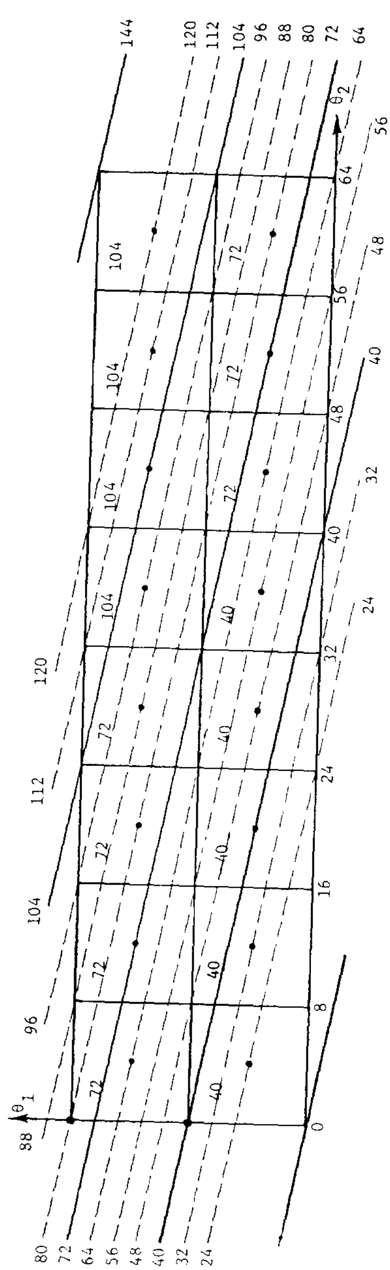

FIG. 2. $\varphi = \theta_1 + 5\theta_2$. The diagonal lines are φ-contours.

B. Two Equal-Error Theorems

1. *General Concepts*

We are given

(i) a set E;

(ii) a covering of E, with cells indexed by the elements of the set M; we now always use the symbol Σ_M for the covering; $\Sigma_M = \{\sigma_m \subseteq E : m \in M\}$;

(iii) a function $\varphi : E \to \mathbb{R}$, with $\inf\{\varphi(\theta) : \theta \in E\} = \alpha$ and $\sup\{\varphi(\theta) : \theta \in E\} = \beta$;

(iv) an integer $k > 0$.

Our problem is: find a k-element set $A \subset \mathbb{R}$ such that for every k-element set $A' \subset \mathbb{R}$ we have

$$\sup\{|\varphi(\theta) - h_A(m)| : \theta \in \sigma_m,\ m \in M\}$$
$$\leq \sup\{|\varphi(\theta) - h_{A'}(m)| : \theta \in \sigma_m,\ m \in M\},$$

where h_A is the closest-to-midvalue function for (φ, Σ_M, A), and $h_{A'}$ is the closest-to-midvalue function for (φ, Σ_M, A').

Clearly, we can confine our search to sets A contained in the closed interval $[\alpha, \beta]$. Note also that for given φ *the only aspect of* Σ_M *that enters the problem is the set* $\{(u_m, v_m) : m \in M\}$. If that set is common to two distinct coverings and if A solves the problem for the first covering, then it also solves the problem for the second covering.

We shall be dealing with *belts*. For $\alpha \leq X < Y \leq \beta$, the (X, Y)-*belt* is the set $\{\theta \in E : X \leq \varphi(\theta) \leq Y\}$. In defining the *error on the* (X, Y)-*belt*, we have to pay attention to points θ that lie on the boundary of a belt and also lie in more than one cell of the covering, say in $\sigma_{m'}$ and $\sigma_{m''}$. We have to decide which of the distances $|\varphi(\theta) - h_A(m')|$, $|\varphi(\theta) - h_A(m'')|$ should count toward the (X, Y)-belt's error and which toward the adjoining belt's error.

We shall define belt error in two distinct senses; for each sense there will be a theorem. For *both* error definitions, we need sets B^m_{XY} defined as follows:

Definition B1. For X, Y with $\alpha \leq X < Y \leq \beta$ and for any[13] m in M

$$B^m_{XY} \equiv \begin{cases} \emptyset & \text{if } u_m = Y < v_m \text{ or } v_m = X > u_m \text{ or } \varphi(\sigma_m) \cap [X, Y] = \emptyset \\ \{\theta : \theta \in \sigma_m ; X \leq \varphi(\theta) \leq Y\} & \text{otherwise} \end{cases}$$

In Figs. 3 and 4, the set B^m_{XY} is portrayed for several cases.

[13] The symbol $\varphi(\sigma_m)$ denotes the image of σ_m under φ, i.e., $\{\varphi(x) : x \in \sigma_m\}$. The symbol $]G, H[$ to be used later, denotes the open interval $\{x : G < x < H\}$.

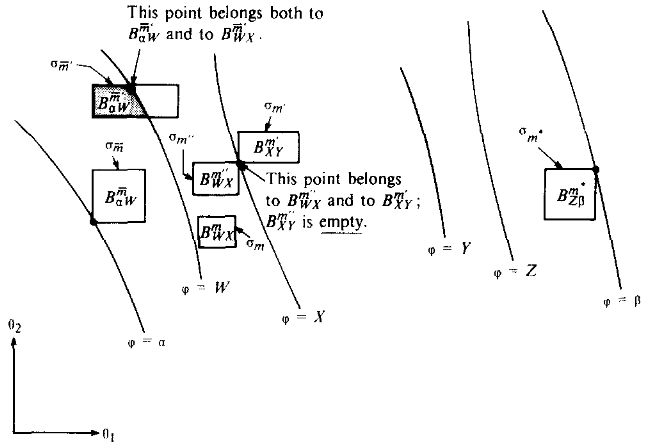

FIGURE 3

2. *The First Equal-Error Theorem*

DEFINITION B2. The *(m,a)-error on the (X,Y)-belt, first sense* is denoted $f_{mA}(X, Y)$ and is defined only when $B^m_{XY} \neq \emptyset$. We have

$$f_{mA}(X, Y) \equiv \sup\{|h_A(m) - \varphi(\theta)| : \theta \in B^m_{XY}\}.$$

The *A-error (or, for brevity, the error) on the (X, Y)-belt, first sense* is denoted $f_A(x, y)$, and is defined as follows:

$$f_A(X, Y) \equiv \sup\{f_{mA}(X, Y) : m \in M;\ B^m_{XY} \neq \emptyset\}.$$

Note that $f_A(\alpha, \beta)$ equals the *overall error*, namely $\sup\{|h_A(m) - \varphi(\theta)| : m \in M;\ \theta \in \sigma_m\}$. The k-element outcome set we seek minimizes overall error; since $f_A(\alpha, \beta)$ is a convenient compact symbol for overall error, we shall use it in connection with *both* theorems.

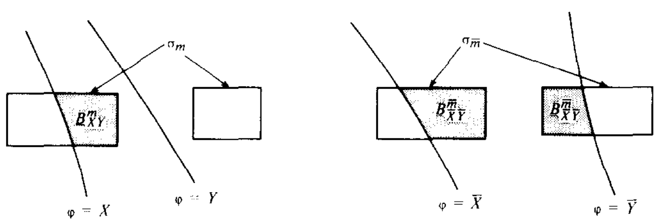

FIGURE 4

Our first theorem imposes the "no-alien" condition which we illustrated in Part A for the example of Fig. 2.

DEFINITION B3. Writing the set A as an ordered k-tuple $\{a_1, a_2, \ldots, a_k\}$, with $\alpha \le a_1 < a_2 < \ldots < a_k \le \beta$, we say that A *has the no-alien property for* (Σ_M, φ) if and only if, for any two successive members—say, r, s—of the ordered $(k + 2)$-tuple $\{\alpha, a_1, \ldots, a_k, \beta\}$, we have

$$\text{"}f_{mA}(r, s) = f_A(r, s)\text{" implies "}h_A(m) \in \{r, s\}.\text{"}$$

THEOREM 1. *Let M be finite and let A be a k-element set with elements $a_1, \ldots, a_k$, where $k \ge 1$ and $\alpha \le a_1 < a_2 < \ldots < a_k \le \beta$. Let A have the no-alien property for (Σ_M, φ). For every i in $\{1, \ldots, k - 1\}$, let there exist m in M such that $B^m_{a_i,a_{i+1}} \ne \emptyset$. Let A satisfy the equal-error condition*

$$f_A(\alpha, a_1) = f_A(a_1, a_2) = f_A(a_2, a_3) = \ldots = f_A(a_{k-1}, a_k) = f_A(a_k, \beta).$$

Then $f_A(\alpha, \beta) = a_1 - \alpha$ and for every k-element set A' we have

$$f_A(\alpha, \beta) \le f_{A'}(\alpha, \beta).$$

A Sketch of the Proof.[14] It will be convenient to use the statement "m is *A-critical on* (X, Y)." That will mean "$f_A(X, Y) = f_{mA}(X, Y)$." We also need the *cell error for σ_m given A*. This is denoted $\eta_A(m)$ and is defined by $\eta_A(m) \equiv \sup\{|\varphi(\theta) - h_A(m)| : \theta \in \sigma_m\}$.

In the proof we suppose that for some k-element set A', we have

$$f_{A'}(\alpha, \beta) < f_A(\alpha, \beta) \qquad (1_B)$$

and we obtain a contradiction. Let the k-elements of A' be $a'_1, \ldots, a'_k$, with $\alpha \le a'_1 < \ldots < a'_k \le \beta$. We obtain our contradiction in the following four steps.

Step 1. $a'_1 < a_1$ and $a'_k > a_k$. To show this, we use (1_B) and both the equal-error and no-alien properties of A. The argument implies, in addition, that $f_A(\alpha, a_1) = a_1 - \alpha$, $f_A(a_k, \beta) = \beta - a_k$, from which it follows (using the equal-error property of A), that $f_A(\alpha, \beta) = a_1 - \alpha$.

Step 2. For some $i \in \{1, \ldots, k - 1\}$, either $a'_i < a_i$ and $a'_{i+1} \ge a_{i+1}$, or $a'_i \le a_i$ and $a'_{i+1} > a_{i+1}$. This follows from Step 1.

Step 3. Let some $i \in \{1, \ldots, k - 1\}$ satisfy $a'_i \le a_i$, $a'_{i+1} \ge a_{i+1}$ (Step 2 guarantees the existence of such an i). Then *if* m^* is A-critical on (a_i, a_{i+1}), either

[14] Those details of this proof, and of subsequent proofs, that are not provided here can be found in Hurwicz and Marschak, 1984, an unpublished document.

$$\eta_{A'}(m^*) \geq \eta_A(m^*) \qquad (2_B)$$

or

$$f_{m^*A'}(a_i, a_{i+1}) \geq f_{m^*A}(a_i, a_{i+1}). \qquad (3_B)$$

In addition

$$f_{m^*A}(a_i, a_{i+1}) = f_A(a_i, a_{i+1}) = \eta_A(m^*) = f_A(\alpha, \beta). \qquad (4_B)$$

In arguing that (2_B) or (3_B) must hold, we use the no-alien property of A.

Step 4. By hypothesis, M is finite and for every i in $\{1, \ldots, k-1\}$, we have $B^m_{a_i,a_{i+1}} \neq \emptyset$ for some m in M. That implies that there indeed exists m^* in M such that m^* is A-critical on (a_i, a_{i+1}), where i is the element of $\{1, \ldots, k-1\}$ considered in Step 3. But, by Step 3, m^* satisfies (2_B) or (3_B) as well as (4_B). We also have

$$f_{A'}(\alpha, \beta) \geq \eta_{A'}(m^*) \geq f_{m^*A'}(a_i, a_{i+1}). \qquad (5_B)$$

But if m^* satisfies (2_B), then (4_B), (5_B) imply $f_{A'}(\alpha, \beta) \geq f_A(\alpha, \beta)$, in contradiction to ($1_B$); and that is true as well if m^* satisfies (3_B).

That concludes the proof.

An Extension of Theorem 1. The requirement that M be finite can be dropped. The conclusion of the theorem holds if one replaces the finiteness of M by the following two conditions on the pair (Σ_M, φ): (i) A "no-gap" condition: for every m in M, $]u_m, v_m[\subseteq \{\varphi(e) : e \in \sigma_m\}$; and (ii) the set $\{(u_m, v_m) : m \in M\}$ is closed in $\mathbb{R}^2$. One can show that for every (a_i, a_{i+1}), these two conditions (together with our requirement that for some m, $B^m_{a_i,a_{i+1}} \neq \emptyset$) imply the existence of a cell σ_m such that m is A-critical on (a_i, a_{i+1}). The preceding four-step argument can then be used again.

3. *The Second Equal-Error Theorem*

For the second equal-error theorem, we start with a second definition of the (m, A)-error on a belt. This definition—unlike the first one—*is confined to the belts defined by successive elements of the ordered* $(k+2)$-*tuple* $\{\alpha, a_1, a_2, \ldots, a_k, \beta\}$, where $a_1, a_2, \ldots, a_k$ are the elements of A, written in increasing order. In many cases, the second definition of (m, A)-error on a belt coincides with the first one. Just as in the case of the first definition, the second definition of (m, A)-error on the (r, s)-belt is confined to triples (m, r, s) such that $B^m_{rs} \neq \emptyset$.

DEFINITION B4. If r, s are successive elements of the ordered $(k+2)$-tuple $\{\alpha, a_1, a_2, \ldots, a_k, \beta\}$ and if $B^m_{rs} \neq \emptyset$, then the (m, A)-*error on the*

(r, s)-belt, second sense, denoted $g_{mA}(r, s)$, is defined by

$$g_{mA}(r, s) = \begin{cases} 0 & \text{if } h_A(m) \neq r \text{ and } h_A(m) \neq s \\ \eta_a(m) & \text{if } u_m < r \leq w_m < s < v_m \\ f_{mA}(r, s) & \text{in all other cases.} \end{cases}$$

The *error on the (r, s)-belt, second sense* is denoted $g_A(r, s)$ and is defined by

$$g_A(r, s) \equiv \sup\{g_{mA}(r, s) : m \in M;\ B_{rs}^m \neq \emptyset\}.$$

Note that if the cell σ_m has points in the (r, s)-belt but is assigned an "alien" outcome—neither r nor s—then the (m, A)-error on the belt is now zero. The counterpart of the no-alien condition of Theorem 1 would now be: "$g_A(r, s) = g_{mA}(r, s)$" implies "$h_A(m) \in \{r, s\}$." But that requirement is now automatically met as long as the belt error $g_A(r, s)$ is not zero. For that reason, our second theorem does not impose the no-alien requirement as an extra condition.

THEOREM 2. *Let A be a k-element set with elements $a_1, \ldots, a_k$, where $\alpha \leq a_1 < a_2 < \cdots < a_k \leq \beta$. Let M be finite. For every i in $\{1, \ldots, k-1\}$, let there exist m in M such that $B_{a_i, a_{i+1}}^m \neq \emptyset$. Let A satisfy the equal-error condition*

$$g_A(\alpha, a_1) = g_A(a_1, a_2) = \ldots = g_A(a_{k-1}, a_k) = g_A(a_k, \beta).$$

Then $f_A(\alpha, \beta) = a_1 - \alpha$ and for every k-element set A', we have

$$f_A(\alpha, \beta) \leq f_{A'}(\alpha, \beta).$$

A Sketch of the Proof. We first define, for any k-element outcome set $\bar{A}$, with ordered elements $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k$,

$$\delta_A \equiv \max[g_{\bar{A}}(\alpha, \bar{a}_1), g_{\bar{A}}(\bar{a}_1, \bar{a}_2), \ldots, g_{\bar{A}}(\bar{a}_{k-1}, \bar{a}_k), g_{\bar{A}}(\bar{a}_k, \beta)]$$

and we prove (in a rather lengthy argument requiring study of a number of cases) that

$$\text{for any } k\text{-element outcome set } \bar{A} \subset [\alpha, \beta],\ \delta_{\bar{A}} = f_{\bar{A}}(\alpha, \beta). \tag{6_B}$$

In view of (6_B), it is enough to show that for the sets A and A' in the statement of the theorem, $\delta_{A'} \geq \delta_A$. Suppose, to the contrary that

$$\delta_{A'} < \delta_A. \tag{7_B}$$

We obtain a contradiction in four steps.

Step 1. $a_1' < a_1$ and $a_k' > a_k$. To show this, we use (7_B) and the equal-error property of A.

Step 2. For some $i \in \{1, \ldots, k-1\}$, either $a_i' < a_i$ and $a_{i+1}' \geq a_{i+1}$; or $a_i' \leq a_i$ and $a_{i+1}' > a_{i+1}$. This follows from Step 1 and is identical with Step 2 in the proof of Theorem 1.

Step 3. Let some $i \in \{1, \ldots, k-1\}$ satisfy $a_i' \leq a_i$, $a_{i+1}' \geq a_{i+1}$. (Step 2 guarantees the existence of such an i). Suppose $g_{m^*A}(a_i, a_{i+1}) = g_A(a_i, a_{i+1})$. Then $g_{m^*A'}(a_i', a_{i+1}') \geq g_{m^*A}(a_i, a_{i+1})$. (The argument requires study of a number of cases; they differ with regard to the ordering of a_i, u_{m^*}, w_{m^*}, and a_{i+1}.)

Step 4. By hypothesis, M is finite and for every i in $\{1, \ldots, k-1\}$, we have $B^m_{a_ia_{i+1}} \neq \emptyset$ for some m in M. That implies that there indeed exists m^* in M such that $g_{m^*A}(a_i, a_{i+1}) = g_A(a_i, a_{i+1})$, where i is the element of $\{1, \ldots, k-1\}$ considered in Step 3. We have (using Step 3): $\delta_{A'} \geq g_{m^*A'}(a_i', a_{i+1}') \geq g_{m^*A}(a_i, a_{i+1}) = g_A(a_i, a_{i+1})$. But since A has the equal-error property (for belt error in the second sense), we have $g_A(a_i, a_{i+1}) = \delta_A$. Thus $\delta_{A'} \geq \delta_A$, which contradicts ($7_B$). It is also easily shown that for any ordered k-tuple $\bar{A} = (\bar{a}_1, \ldots, \bar{a}_k)$, we have $g_{\bar{A}}(\alpha, a_1) = \bar{a}_1 - \alpha$, $g_{\bar{A}}(\bar{a}_k, \beta) = \beta - \bar{a}_k$. From that it follows that for our equal-error k-tuple A, we have $f_A(\alpha, \beta) = \delta_A = a_1 - \alpha$.

That concludes the proof.

An Extension. Just as for Theorem 1, one can drop the requirement that M be finite provided one adds the "no-gap" and closedness conditions given above in connection with the extension of Theorem 1.

4. *The Relation between the Two Theorems*

The hypothesis of Theorem 1 does not imply that of Theorem 2, and vice versa. One can find a set $\{(u_m, v_m) : m \in M\}$ and a k-element set A such that A has both the first equal-error property and the no-alien property but lacks the second equal-error property. One can also construct those objects so that A has the second equal-error property but not the first.

C. An Application of Theorem 1 and 2: E Is a "Rectangle" in $\mathbb{R}^n$, φ Is Linear, and Σ_M Is a Uniform Grid

1. *Statement of the Problem*

Suppose E is a set in $\mathbb{R}^n$, namely the "rectangle"

$$E_{G\bar{G}} \equiv \{\theta = (\theta_1, \ldots, \theta_n) : G_i \leq \theta_i \leq \bar{G}_i,\ i = 1, \ldots, n\}.$$

We write $G = (G_1, \ldots, G_n)$, $\overline{G} = (\overline{G}_1, \ldots, \overline{G}_n)$. Let the covering Σ_M of $E_{G\overline{G}}$ be obtained by intersecting $E_{G\overline{G}}$ with a grid composed of cubes of side L. The grid is the set $\{D_L(J_1, \ldots, J_n): J_1, \ldots, J_n \text{ are integers}\}$, where

$$D_L(J_1, \ldots, J_n) \equiv \{(\theta_1, \ldots, \theta_n): J_iL \leq \theta_i \leq (J_i + 1)L,\ i = 1, \ldots, n\}.$$

Every cell σ_m of the covering E_M satisfies $\sigma_m = E_{G\overline{G}} \cap D_L(J_1, \ldots, J_n)$ for some integers $J_1, \ldots, J_n$. Finally, let

$$\varphi(\theta) = \gamma_1\theta_1 + \gamma_2\theta_2 + \ldots + \gamma_n\theta_n.$$

We write $\gamma = (\gamma_1, \ldots, \gamma_n)$.

We shall consider a *regular problem* of finding an error-minimizing outcome set. The problem is defined by G, $\overline{G}$, L, γ, and an integer $k > 0$. In a regular problem:

(i) The vertices of $E_{G\overline{G}}$ are at "grid points", i.e., for some integers $P_1, \overline{P}_1, P_2, \overline{P}_2, \ldots, P_n, \overline{P}_n$, we have $G_i = LP_i$, $\overline{G}_i = L\overline{P}_i$, $i = 1, \ldots, n$.

(ii) $\gamma_1 = 1$ and $\gamma_2, \ldots, \gamma_n$ are positive integers.

(iii) For the function φ defined by γ, the set of possible midvalues $w_m = \frac{1}{2}(u_m + v_m)$ of the covering Σ_M is *equally spaced*, that is to say, they can be ordered from lowest to highest, each separated from the next by a constant distance.

Under conditions (i) and (ii), condition (iii) means that the set of possible midvalues is precisely the λ-element set

$$\Lambda_{G\overline{G}L\gamma} \equiv \left\{\alpha + \frac{L}{2}(1 + \zeta),\ \alpha + \frac{L}{2}(1 + \zeta) + L,\ \alpha + \frac{L}{2}(1 + \zeta) + 2L,\ \ldots,\ \alpha + \frac{L}{2}(1 + \zeta) + (\lambda - 1)L\right\},$$

where $\zeta \equiv \gamma_2 + \gamma_3 + \ldots + \gamma_n$ and, as before, α denotes $\inf\{\varphi(\theta): \theta \in E_{G\overline{G}}\} = \varphi(G)$. The final element of $\Lambda_{G\overline{G}L\gamma}$ can also be written $\beta - L/2(1 + \zeta)$, where, as before, β denotes $\sup\{\varphi(\theta): \theta \in E_{G\overline{G}}\} = \varphi(\overline{G})$.

EXAMPLE. In Fig. 1, $G = (0, 0)$, $\overline{G} = (1, 1)$, $\gamma_1 = \gamma_2 = \zeta = 1$, $\alpha = 0$, $\beta = 2$, $L = \frac{1}{8}$. There are 15 equally spaced midvalues (i.e., $\lambda = 15$), namely $\{\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \ldots, \frac{15}{8}\}$. In Fig. 2, $G = (0, 0)$, $\overline{G} = (64, 16)$, $\gamma_1 = 1$, $\gamma_2 = \zeta = 5$, $\alpha = 0$, $\beta = 144$, $L = 8$. There are 13 equally spaced midvalues (i.e., $\lambda = 13$), namely $\{24, 32, 40, 48, \ldots, 120\}$.

It is easy to show that if conditions (i) and (ii) are satisfied, then condition (iii)—the equal spacing of midvalues—*must be satisfied for a sufficiently large set* $E_{G\overline{G}}$. More precisely, (iii) is satisfied if $\overline{P}_1 - P_1 \geqq \zeta + 1$.

For G, $\overline{G}$, L, γ satisfying (i)–(iii), and for an integer $k > 0$, the regular problem—which we now denote $P^k_{G\overline{G}L\gamma}$—is to find an error-minimizing k-element outcome set A. In a *nonregular* problem, G, $\overline{G}$, L, γ fail to satisfy one or more of these conditions. A wide class of nonregular problems can in fact be solved by first solving a suitably constructed regular problem and then slightly perturbing the solution. In particular, one can so solve any problem in which $\gamma_1, \ldots, \gamma_n$ are rational numbers and the set $E_{G\overline{G}}$ is sufficiently large. By solving an appropriate regular problem one can also solve a problem in which γ satisfies (ii) but the set E is not a rectangular $E_{\overline{G}G}$ at all; E does, however, have the property $[\alpha, \beta] \subseteq \varphi(E)$. Specifically, one considers the smallest rectangle $E_{G\overline{G}}$ that satisfies (i) and contains E, and one solves a regular problem associated with that rectangle.

2. *Some Tools and Intermediate Results*

We proceed now to the solution of the regular problem $P^k_{G\overline{G}L\gamma}$. We can confine our search for a solution to k-element outcome sets $\{a_1, \ldots, a_k\}$, where $\alpha < a_1 < a_2 < \cdots < a_k \leq \beta$. We start by defining three conditions that A may satisfy for given G, $\overline{G}$, L, γ. Recall that once L is specified, the covering Σ_M of $E_{G\overline{G}}$—which we shall now call the *L-grid covering*—is determined.

DEFINITION C1. A is said to satisfy *condition* Γ if and only if, for every cell σ_m in the L-grid covering of $E_{G\overline{G}}$,

$$]u_m, v_m[\text{ contains at most one element of } A.$$

A is said to satisfy *Condition* Δ if and only if for every pair $\{r, s\}$ of successive elements of the ordered $(k + 2)$-tuple $\{\alpha, a_1, a_2, \ldots, a_k, \beta\}$, there exists m', m'' in M such that

$$B^{m'}_{rs} \neq \emptyset, \qquad h_A(m') = r; \qquad B^{m''}_{rs} \neq \emptyset,\ h_A(m'') = s.$$

A is said to satisfy *Condition* Ψ if and only if, for every pair r, s of successive elements of $\{\alpha, a_1, a_2, \ldots, a_k, \beta\}$, there exist $\overline{m}$, $\overline{\overline{m}}$ in M such that

$$B^{\overline{m}}_{rs} \neq \emptyset, \qquad h_A(\overline{m}) = r, \qquad r \leq w_{\overline{m}} \leq s$$
$$B^{\overline{\overline{m}}}_{rs} \neq \emptyset, \qquad h_A(\overline{\overline{m}}) = s, \qquad r \leq w_{\overline{\overline{m}}} \leq s.$$

Next we need to associate two critical midvalues with any pair of successive outcomes in A. We continue to let ζ denote $\gamma_2 + \gamma_3 + \cdots + \gamma_n$.

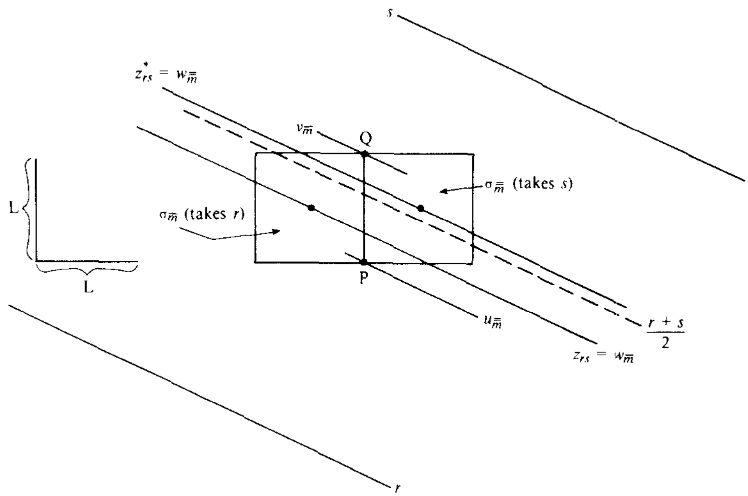

FIG. 5. The diagonal lines are φ-contours.

DEFINITION C2. Given any two numbers r, s with $r < s$,

$$z_{rs} \equiv \text{the largest midvalue in } \Lambda_{G\bar{G}L\gamma} \text{ not greater than } \tfrac{1}{2}(r + s)$$

$$z^*_{rs} \equiv \text{the smallest midvalue in } \Lambda_{G\bar{G}L\gamma} \text{ greater than } \tfrac{1}{2}(r + s)$$

$$\rho_{rs} \equiv z_{rs} + (L/2)(1 + \zeta) - r$$

$$\rho^*_{rs} \equiv s - z^*_{rs} + (L/2)(1 + \zeta).$$

Finally, we shall need several lemmas.[15]

LEMMA 1. *Let r, s be two successive elements*[16] *of A.*

(i) If A satisfies Conditions Γ and Δ, then

$$f_A(r, s) = \max(\rho_{rs}, \rho^*_{rs}).$$

(ii) If A satisfies Condition Ψ, then

$$g_A(r, s) = \max(\rho_{rs}, \rho^*_{rs}).$$

To visualize Lemma 1, it is helpful to inspect Figs. 5–7. Figure 5 portrays

[15] The proofs are found in Hurwicz and Marschak, 1984.

[16] "Two successive elements" means two successive members of the ordered k-tuple $\{a_1, a_2, \ldots, a_k\}$.

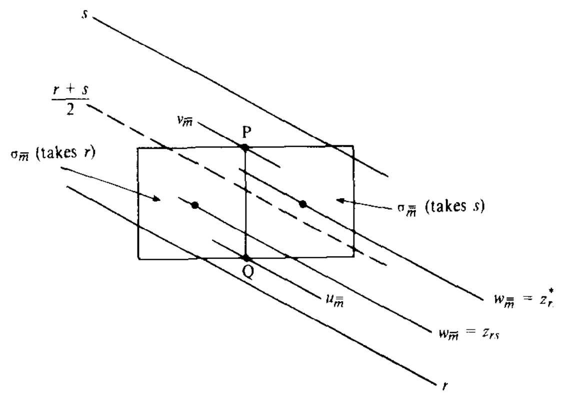

FIGURE 6

two cells, $\sigma_{\bar m}$ and $\sigma_{\bar{\bar m}}$, lying in the interior of the (r, s)-belt. The diagonal lines are φ-contours. The cell $\sigma_{\bar m}$ *takes* r, i.e., $h_A(\bar m) = r$. Moreover, its midvalue is maximal among all midvalues taking r. In view of the definition of the closest-to-midvalue function h_A (ties are broken downward), that is equivalent to saying that its midvalue is the largest midvalue not greater than $(\frac{1}{2})(r + s)$. Therefore, applying Definition C2, we denote the midvalue of $\sigma_{\bar m}$ by z_{rs}. Analogously, $\sigma_{\bar{\bar m}}$ takes s and its midvalue is minimal among all cells taking s. That is to say, its midvalue is the smallest midvalue greater than $\frac{1}{2}(r + s)$, and so we label it z^*_{rs}. The figure is consistent with A satisfying all three of Conditions Γ , Δ, Ψ. Hence Lemma 1 implies that the *error on the* (r, s)*-belt, first sense*—i.e., $f_A(r, s)$—is the larger of $v_{\bar m} - r$ and $s - u_{\bar{\bar m}}$. We may call $v_{\bar m} - r$ the *left error for* (r, s) and $s - u_{\bar{\bar m}}$ the *right error for* (r, s). The left error occurs, for example, at point Q; $\varphi(Q) = v_{\bar m}$ is the φ-value *furthest from* r *among all points taking* r *(i.e., lying in cells that take* r*)*. The right error occurs, for example, at point P; $\varphi(P) = u_{\bar{\bar m}}$ is the φ-value *furthest from* s *among all points taking* s. Examining the cells $\sigma_{\bar m}$, $\sigma_{\bar{\bar m}}$ of Fig. 5, and checking Definition B4, we see that $f_{mA}(r, s) = g_{mA}(r, s)$ for $m = \bar m$, $\bar{\bar m}$. Hence, in Fig. 5, *the error on the* (r, s)*-belt, second sense*, coincides with the error in the first sense. So the error on the (r, s)-belt, second sense (i.e., $g_A(r, s)$) also equals the

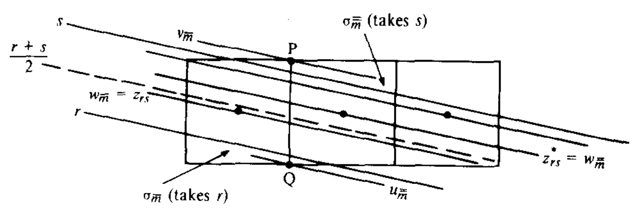

FIGURE 7

larger of the right and left errors. But the left error $v_{\overline{m}} - r$ can alternatively be written $z_{rs} + L/2(1 + \zeta) - r = \rho_{rs}$. For if one starts at the center of $\sigma_{\overline{m}}$—which lies on the midvalue contour z_{rs}—one reaches the point Q, which lies on the $v_{\overline{m}}$ contour, by adding $L/2$ to each coordinate. Since $\gamma_1 = 1$, that increases φ by $L/2$ (from increasing the first coordinate) plus $(L/2) \cdot (\zeta)$ (from increasing the remaining coordinates). Similarly, the right error $s - u_{\overline{\overline{m}}}$ can be written $s - [z^*_{rs} - (L/2)(1 + \zeta)] = \rho^*_{rs}$.

In Fig. 6, A continues to obey Conditions Γ, Δ, Ψ, but now it is no longer the case that the entire cells $\sigma_{\overline{m}}$, $\sigma_{\overline{\overline{m}}}$ are in the interior of the (r, s)-belt. The belt error in both senses is again the larger of the left and right errors; it occurs again, either at P or at Q.

In Fig. 7, Condition Γ fails, since the interval $]u_{\overline{\overline{m}}}, v_{\overline{m}}[$ contains both r and s. Condition Ψ, however, holds. For $m = \overline{m}, \overline{\overline{m}}$, the (m, A)-errors on the (r, s)-belt, second sense, are (according to Definition B4) the entire cell errors $\eta_A(\overline{m})$, $\eta_A(\overline{\overline{m}})$, respectively. The belt error, second sense, is the larger of these two cell errors. Either that occurs at P and equals the left error $v_{\overline{m}} - r = z_{rs} + (L/2)(1 + \zeta) - r = \rho_{rs}$, or it occurs at Q and equals the right error $s - u_{\overline{\overline{m}}} = s - [z^*_{rs} - (L/2)(1 + \zeta)] = \rho^*_{rs}$.

There are several other possible cases, not illustrated. The proof of Lemma 1 deals with all of them.

Henceforth, we shall routinely refer to the quantities ρ_{rs}, ρ^*_{rs} as the *left error for* (r, s) and the *right error for* (r, s), respectively. We shall also write

$$\tau(r, s) \equiv \max(\rho_{rs}, \rho^*_{rs}).$$

We shall require four further relatively minor results.

LEMMA 2. *Let r, s be two successive elements of A satisfying $r = \alpha + (L/2)(1 + \zeta) + BL/2$, $s = \alpha + (L/2)(1 + \zeta) + CL/2$, where B, C are integers. Then* $(1/2)(r + s) = \alpha + (L/2)(1 + \zeta) + ((B + C)/4)L$, *and*

(i) *if $(B + C)/4$ is an integer, then $\tau(r, s)$ equals ρ_{rs} but not ρ^*_{rs} (the left error exceeds the right error)*;

(ii) *if $(B + C)/4 - 1/2$ is an integer, then $\tau(r, s) = \rho_{rs} = \rho^*_{rs}$ (the left and right errors are equal).*

LEMMA 3. *If A satisfies Condition Γ, then A has the no-alien property with respect to (Σ_M, φ).*

LEMMA 4. *If A satisfies Condition Γ, then (writing A as the ordered k-tuple $a_1, \ldots, a_k$),*

$$f_A(\alpha, a_1) = a_1 - \alpha, \qquad f_A(a_k, \beta) = \beta - a_k.$$

If A satisfies condition Ψ, then $g_A(\alpha, a_1) = a_1 - \alpha$, $g_A(a_k, \beta) = \beta - a_k$.

LEMMA 5. *If, for every pair r, s of successive elements of A, we have* $r - s \geq 2L$ *then A satisfies Conditions* Δ *and* Ψ.

3. *Explicit Solutions for the Regular Problem*

A regular problem $P^k_{G\bar{G}L\gamma}$ is of two types. In *Type I, k is less than half the number of distinct midvalues*, i.e., $k < \lambda/2$. In *Type II, k is at least half the number of distinct midvalues*, i.e., $k \geq \lambda/2$.

3.1. *Solving the Type I Problem.* Type I is by far the more difficult of the two types. It falls into six cases. In each of the six cases, we present a k-element set A with elements $a_1, \ldots, a_k$ such that $\alpha < a_1 < a_2 < \ldots < a_k < \beta$. We then proceed to the following four steps.

(i) We verify that the indicated spacing of the k elements is feasible.

(ii) Any two successive elements of A are at least $2L$ apart. Hence, by Lemma 5

$$A \text{ satisfies Conditions } \Delta \text{ and } \Psi. \tag{1_C}$$

We apply Lemma 2 in order to calculate $\tau(a_i, a_{i+1})$ for $i = 1, \ldots, k - 1$. We verify that

$$a_1 - \alpha = \tau(a_i, a_2) = \tau(a_2, a_3) = \ldots = \tau(a_{k-1}, a_k) = \beta - a_k. \tag{2_C}$$

(iii) We next consider the first of two subcases. In the first subcase, A satisfies Condition Γ because it satisfies

$$L(1 + \zeta) \leq a_{i+1} - a_i, \qquad i = 1, \ldots, k - 1. \tag{3_C}$$

[Recall that for every cell σ_m of the L-grid covering of $E_{G\bar{G}}$, we have $v_m - u_m = L(1 + \zeta)$. Hence, (3_C) implies Condition Γ.] That means, by (1_C) and Lemma 1, that every quantity $\tau(a_i, a_{i+1})$, $i = 1, \ldots, k - 1$, calculated in Step (ii), indeed equals $f_A(a_i, a_{i+1})$. It also means, by Lemma 3, that A satisfies the no-alien condition and, by Lemma 4, that

$$f_A(\alpha, a_1) = a_1 - a, \qquad f_A(a_k, \beta) = \beta - a_k. \tag{4_C}$$

In view of (2_C) and (4_C), we conclude that

the k-tuple A has the first equal-error property and the no-alien property.

Therefore, by virtue of Theorem 1, A solves $P^k_{G\bar{G}G\gamma}$.

(iv) We turn, finally, to the second subcase. Here (3_C) no longer holds and Condition Γ may fail. However, as noted in Step (ii), successive elements of A are at least $2L$ apart and so A satisfies Condition Ψ. Hence, by Lemma 1, every quantity $\tau(a_i, a_{i+1})$, $i = 1, \ldots, k-1$, calculated in Step (ii), equals $g_A(a_i, a_{i+1})$. In view of (2_C), we conclude, in the second subcase that

A has the second equal-error property.

Hence, by virtue of Theorem 2, A solves $P^k_{G\bar{G}L\gamma}$.

We now proceed to the six cases. They depend on the value taken by T, defined by

$$T = \begin{cases} \dfrac{\lambda}{2} & \text{if } \lambda \text{ is even} \\[2ex] \dfrac{\lambda + 1}{2} & \text{if } \lambda \text{ is odd.} \end{cases}$$

Case 1. λ is even, so that $T = \lambda/2$; T/k is an integer Q.

We claim that a solution to the regular problem $P^k_{G\bar{G}L\gamma}$ is the k-tuple $A = \{a_1, \ldots, a_k\}$ defined by Fig. 8 and that $f_A(\alpha, \beta) = L(Q + \zeta/2)$. To justify the claim we follow the four steps just summarized.

(i) We first show that the indicated spacing of the k-points $a_1, \ldots, a_k$ is feasible. Recall that the highest of the equally spaced midvalues can be written either $\alpha + (L/2)(1 + \zeta) + (\lambda - 1)L$ or $\beta - (L/2)(1 + \zeta)$. Equating these two expressions, we get

$$\alpha + L(\lambda + \zeta) = \beta. \tag{5_C}$$

Now in the present case we have $a_k = \alpha + L(Q + \zeta/2) + (k - 1)(2QL)$. Using ($5_C$) and the fact that $\lambda = 2kQ$, we have

$$\begin{aligned} a_k &= \beta - L\lambda - L\zeta + LQ + \frac{L\zeta}{2} + L\zeta - 2QL \\ &= \beta - L\left(Q + \frac{\zeta}{2}\right). \end{aligned}$$

So the spacing in the figure is feasible.

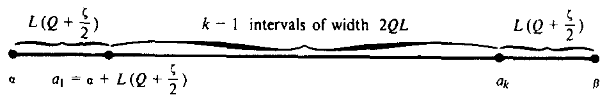

FIGURE 8

(ii) Consider next the $k - 1$ interior intervals. Since $Q \geq 1$, all intervals are at least $2L$ in width and hence (by Lemma 5), A satisfies conditions Δ and Ψ. We now apply Lemma 2 to calculate $\tau(a_i, a_{i+1})$ for $i = 1, \ldots, k - 1$. For the typical pair of successive outcomes, we have, for some integer $R > 0$,

$$(a_i, a_{i+1}) = \left[\alpha + L\left(Q + \frac{\zeta}{2}\right) + 2QLR,\ \alpha + L\left(Q + \frac{\zeta}{2}\right) + 2QL(R + 1)\right]$$

and

$$\frac{1}{2}(a_i + a_{i+1}) = \alpha + L\left(Q + \frac{\zeta}{2}\right) + (2QR + Q)L = \alpha + \frac{L}{2}(1 + \zeta) + L\left(2QR + 2Q - \frac{1}{2}\right).$$

We are in case (ii) of Lemma 2, and $\tau(a_i, a_{i+1})$ equals both left and right errors. Further, since the highest midvalue not exceeding $\frac{1}{2}(a_i + a_{i+1})$ is $\alpha + (L/2)(1 + \zeta) + L(2QR + 2Q - 1)$, we have

$$\begin{aligned}\tau(a_i, a_{i+1}) &= \alpha + L(1 + \zeta) + L(2QR + 2Q - 1) \\ &\quad - \left[\alpha + L\left(Q + \frac{\zeta}{2}\right) + 2QLR\right] \\ &= L\left(Q + \frac{\zeta}{2}\right) = a_1 - \alpha = \beta - a_k. \end{aligned} \tag{6C}$$

(iii) For any two successive outcomes a_i, a_{i+1}, we have $a_{i+1} - a_i = 2QL$. Hence (3_C) and therefore condition Γ hold if

$$\zeta + 1 \leq 2Q$$

or equivalently (since $Q = T/k = \lambda/2k$), $k \leq \lambda/(\zeta + 1)$.

So if $k \leq \lambda/(\zeta + 1)$, then *the set A of Fig.* 8, in view of Lemmas 1 and 4, has the first equal-error property. In view of Lemma 3, A has the no-alien property. Hence *A solves $P^k_{G\bar{G}L\gamma}$ by virtue of Theorem* 1, and $f_A(\alpha, \beta) = L(Q + \zeta/2)$.

(iv) If $k > \lambda/(\zeta + 1)$, then Condition Γ may fail. Since, however, A satisfies Condition Ψ, it follows from (6_C) and Lemmas 1 and 4 that A has the second equal-error property. Hence *A solves $P^k_{G\bar{G}L\gamma}$ by virtue of Theorem* 2, and $f_A(\alpha, \beta) = L(Q + \zeta/2)$.

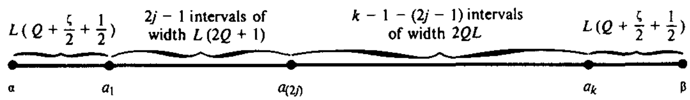

FIGURE 9

Case 2. $T = \lambda/2$; Q, the largest integer not exceeding T/k, satisfies $T/k = Q + j/k$; and $0 < j < (k + 1)/2$. We claim that the k-tuple $\{a_i, \ldots, a_k\}$ portrayed in Fig. 9 solves $P^k_{B\bar{B}\gamma L}$.

(i) We show that the indicated spacing is feasible. We have, using (5_C) and the fact that $\lambda = 2kQ + 2j$,

$$\begin{aligned}
a_k &= \alpha + L\left(Q + \frac{\zeta}{2} + \frac{1}{2}\right) + (2j-1)(2Q+1)L \\
&\quad + [(k-1) - (2j-1)](2QL) \\
&= \beta - L(\lambda + \zeta) + L\left(Q + \frac{\zeta}{2} + \frac{1}{2}\right) + (2j-1)(2Q+1)L \\
&\quad + [k - 1 - (2j-1)](2QL) \\
&= \beta - L\gamma - L\zeta + LQ + \frac{L\zeta}{2} + \frac{L}{2} + L\lambda - L - 2QL \\
&= \beta - L\left(Q + \frac{\zeta}{2} + \frac{1}{2}\right).
\end{aligned}$$

So the spacing shown in the figure is feasible.

(ii) Now consider the interior intervals. For $i = 1, \ldots, 2j - 1$, $[a_i, a_{i+1}]$ equals $[\alpha + L(Q + \zeta/2 + \frac{1}{2}) + RL(2Q + 1), \alpha + L(Q + \zeta/2 + \frac{1}{2}) + (R + 1)L(2Q + 1)]$, where $R > 0$ is some integer; $\frac{1}{2}(a_i + a_{i+1}) = \alpha + L(Q + \zeta/2 + \frac{1}{2}) + RL(2Q + 1) + (Q + \frac{1}{2})L = \alpha + L/2(1 + \zeta) + L(2Q + 2QR + R + \frac{1}{2})$. We are in case (ii) of Lemma 2 and $\tau(a_i, a_{i+1})$ equals both left and right errors.

Further, since the largest midvalue not exceeding $\frac{1}{2}(a_i, a_{i+1})$ is $\alpha + L/2(1 + \zeta) + L(2Q + 2QR + R)$, we have, for $i = 1, \ldots, 2j - 1$,

$$\begin{aligned}
\tau(a_i, a_{i+1}) &= \alpha + L(1 + \zeta) + L(2Q + 2QR + R) \\
&\quad - \left[\alpha + L\left(Q + \frac{\zeta}{2} + \frac{1}{2}\right) + RL(2Q + 1)\right] \\
&= L\left(Q + \frac{\zeta}{2} + \frac{1}{2}\right) = a_1 - \alpha = \beta - a_k.
\end{aligned}$$

For $i = 2j, \ldots, k-1$, (a_i, a_{i+1}) equals, say, $[\alpha + L(Q + \zeta/2 + \frac{1}{2}) + (2j - 1)(2Q + 1)L + 2QRL, \alpha + L(Q + \zeta/2 + \frac{1}{2}) + (2j - 1)(2Q + 1)L + 2Q(R + 1)L]$, where $R > 0$ is some integer; $\frac{1}{2}(a_i + a_{i+1}) = \alpha + L(Q + \zeta/2 + \frac{1}{2}) + (2j - 1)(2Q + 1)L + (2QR + Q)L = \alpha + L/2(1 + \zeta) + L(2Q + 2QR + (2j - 1)(2Q + 1))$. We are in case (i) of Lemma 2; $\tau(a_i, a_{i+1})$ equals the left error; and the largest midvalue not exceeding $\frac{1}{2}(a_i + a_{i+1})$ equals $\frac{1}{2}(a_i + a_{i+1})$ itself. Hence,

$$\tau(a_i, a_{i+1}) = \alpha + L(1 + \zeta) + L[2Q + 2QR + (2j-1)(2Q + 1)]$$
$$- \left[\alpha + L\left(Q + \frac{\zeta}{2} + \frac{1}{2}\right) + (2j - 1)(2Q + 1)L - 2QRL\right]$$
$$= L\left(Q + \frac{\zeta}{2} + \frac{1}{2}\right) = a_1 - \alpha = \beta - a_k.$$

(iii) For any two successive outcomes (a_i, a_{i+1}), we have $a_{i+1} - a_i \geq 2QL$. Hence (3_C) and therefore condition Γ hold if

$$\zeta + 1 \leq 2Q. \qquad (7_C)$$

But $2Q = (2(T - j))/k = (\lambda - 2j)/k$. So (7_C) holds if $k \leq (\lambda - 2j)/(\zeta + 1)$. But $j < (k + 1)/2$ implies $2j \leq k$. Hence (7_C) and condition Γ hold if $k \leq (\lambda - k)/(\zeta + 1)$ or

$$k \leq \frac{\lambda}{\zeta + 2}.$$

So if $k \leq \lambda/(\zeta + 2)$, then the set A of Fig. 9 solves $P^k_{G\bar{G}L\gamma}$ by virtue of Theorem 1 and $f_A(\alpha, \beta) = L(Q + \zeta/2 + \frac{1}{2})$.

(iv) If, on the other hand, $k > \lambda/(\zeta + 2)$, then A solves $P^k_{G\bar{G}L\gamma}$ by virtue of Theorem 2 and again $f_A(\alpha, \beta) = L(Q + \zeta/2 + \frac{1}{2})$.

Case 3. λ is even, so that $T = \lambda/2$; Q, the largest integer not exceeding T/k, satisfies $T/k = Q + j/k$; and $(k + 1)/2 \leq j \leq k - 1$. The k-tuple A portrayed in Fig. 10 solves $P^k_{G\bar{G}L\gamma}$.

For this case and the remaining ones, we omit the four steps of the argument.

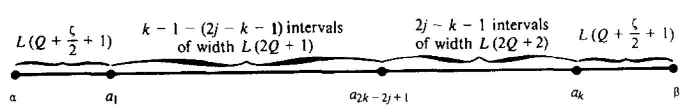

FIGURE 10

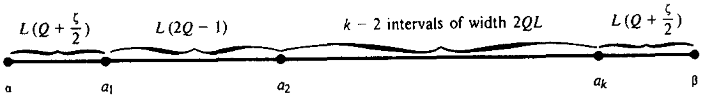

FIGURE 11

Case 4. λ is odd, so that $T = (\lambda + 1)/2$; T/k is an integer Q. $P^k_{G\bar{G}L\gamma}$ is solved by the k-tuple $A = \{a_1, \ldots, a_k\}$ portrayed in Fig. 11.

Case 5. λ is odd, so that $T = (\lambda + 1)/2$; Q, the largest integer not exceeding T/k, satisfies $T/k = Q + j/k$; and $0 < j \le (k + 1)/2$. The k-tuple $A = \{a_1, \ldots, a_k\}$ portrayed in Fig. 12 solves $P^k_{G\bar{G}L\gamma}$.

Case 6. λ is odd, so that $T = (\lambda + 1)/2$; Q, the largest integer not exceeding T/k, satisfies $T/k = Q + j/k$; and $(k + 1)/2 < j \le k - 1$. The k-tuple $A = \{a_1, \ldots, a_k\}$ portrayed in Fig. 13 solves $P^k_{G\bar{G}L\gamma}$.

Summarizing, we have

THEOREM 3, FIRST PART. *If k is less than half the number of midvalues, then a solution to the regular problem* $P^k_{G\bar{G}L\gamma}$ *is provided by the appropriate k-tuple* $(a_1, \ldots, a_k)$ *among the equal-error k-tuples portrayed in Figs.* 8 *to* 13. *The error for that k-tuple is* $a_1 - \alpha$.

The solution $\{\frac{3}{8}, 1, \frac{13}{8}\}$ to the three-outcome problem of Fig. 1 is obtained from Fig. 12 (Case 5). We have 15 midvalues, so $\lambda = 15$ and $T = (\lambda + 1)/2 = 8$. The largest integer not exceeding $T/k = \frac{8}{3}$ is 2 and $\frac{8}{3} = 2 + \frac{2}{3}$. So $Q = 2$ and $j = 2$, which is not larger than $(k + 1)/2 = 2$. The requirements of case 5 are met. Figure 12 tells us that the three elements of the solution are spaced $L(2Q + 1)$ apart, since $2j - 2 = 2$ and $(k - 1) - (2j - 2) = 0$. Since $L = \frac{1}{8}$, we have $L(2Q + 1) = \frac{5}{8}$. To find a_1, we compute $\alpha + L(Q + \zeta/2 + \frac{1}{2})$. Since $\alpha = 0$ and $\zeta = 1$, we have $a_1 = \frac{3}{8}$. Hence, $a_2 = 1$ and $a_3 = \frac{13}{8}$.

The solution {40, 72, 104} to the three-outcome problem of Fig. 2 is also obtained from Fig. 12 (Case 5). This time we have 13 midvalues, i.e., $\lambda = 13$ and $T = 7$. The largest integer not exceeding $t_k = \frac{7}{3}$ is 2, and $\frac{7}{3} = 2 + \frac{1}{3}$. So $Q = 2$ and $j = 1$, which is not larger than $(k + 1)/2 = 2$. The requirements of Case 5 are met. The elements of the solution are spaced $2QL$ apart, since $2j - 2 = 0$. Since $L = 8$, we have $2QL = 32$. To find a_1 we compute $\alpha + L(Q + \zeta/2 + \frac{1}{2})$. Since $\alpha = 0$ and $\zeta = 5$, we have $a_1 = 40$. Hence $a_2 = 72$ and $a_3 = 104$.

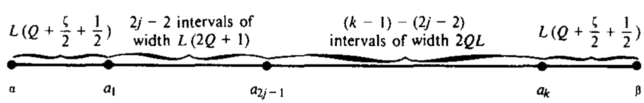

FIGURE 12

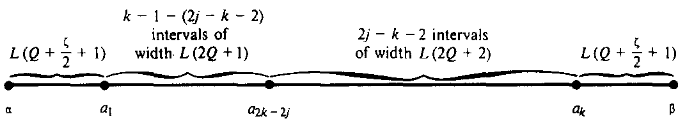

FIGURE 13

Note that in the Fig. 2 example we have $L(1 + \zeta) = 48 > 32 = 72 - 40$. Thus Condition ($3_C$) of our five-step proof pattern fails to hold for (40, 72, 104). In the Fig. 1 example, on the other hand, (3_C) holds for the triple ($\frac{3}{8}$, 1, $\frac{13}{8}$). That means that while the Case 5 formula is a solution in the Fig. 1 example by virtue of Theorem 1, it is a solution in the Fig. 2 example by virtue of Theorem 2. That accords with our observation in Section A4, namely that (40, 72, 104) lacks the no-alien property and hence it is Theorem 2—not Theorem 1—that shows (40, 72, 104) to be error-minimizing.

3.2 *Solving a Problem of Type* 2. *If* $k = \lambda$, then we have as many outcomes available as there are midvalues and our solution is simply the midvalue set itself. Now *suppose* $k = \lambda - 1$. Consider the k-tuple A defined by

$$a_1 = \alpha + L\left(1 + \frac{\zeta}{2}\right)$$

$$a_2 = \alpha + L\left(1 + \frac{\zeta}{2}\right) + L$$

$$\vdots$$

$$a_i = \alpha + L\left(1 + \frac{\zeta}{2}\right) + (i - 1)L$$

$$\vdots$$

$$a_{k-1} = \alpha + L\left(1 + \frac{\zeta}{2}\right) + (\lambda - 3)L$$

$$a_k = \alpha + L\left(1 + \frac{\zeta}{2}\right) + (\lambda - 2)L. \qquad (8_C)$$

It is easy to show that A has the second equal-error property; A therefore solves $P^k_{G\bar{G}L\gamma}$ by virtue of Theorem 2. Next *suppose* $k = T$. For λ odd, so that $T = (\lambda + 1)/2$, consider A defined by

$$a_1 = \alpha + L\left(1 + \frac{\zeta}{2}\right)$$

$$a_2 = \alpha + L\left(1 + \frac{\zeta}{2}\right) + 2L$$

$$a_3 = \alpha + L\left(1 + \frac{\zeta}{2}\right) + 4L$$

$$\vdots$$

$$a_i = \alpha + L\left(1 + \frac{\zeta}{2}\right) + (2i - 2)L$$

$$\vdots$$

$$a_{k-2} = \alpha + L\left(1 + \frac{\zeta}{2}\right) + (\lambda - 5)L$$

$$a_{k-1} = \alpha + L\left(1 + \frac{\zeta}{2}\right) + (\lambda - 3)L$$

$$a_k = \alpha + L\left(1 + \frac{\zeta}{2}\right) + (\lambda - 2)L. \qquad (9_C)$$

For λ even, so that $T = \lambda/2$, consider A defined by

$$a_1 = \alpha + L\left(1 + \frac{\zeta}{2}\right)$$

$$a_2 = \alpha + L\left(1 + \frac{\zeta}{2}\right) + L$$

$$a_3 = \alpha + L\left(1 + \frac{\zeta}{2}\right) + 2L$$

$$\vdots$$

$$a_i = \alpha + L\left(1 + \frac{\zeta}{2}\right) + (2i - 2)L$$

$$\vdots$$

$$a_{k-1} = \alpha + L\left(1 + \frac{\zeta}{2}\right) + (\lambda - 4)L$$

$$a_k = \alpha + L\left(1 + \frac{\zeta}{2}\right) + (\lambda - 2)L. \qquad (10_C)$$

Again, in both cases we can show A to have the second equal-error property so that A solves the problem by virtue of Theorem 2.

Now notice that for *both* $k = \lambda - 1$ and $k = T$, the error achieved by the solutions—hence the lowest attainable error—is $a_{\text{l}} - \alpha = L(1 + \zeta/2)$. It is clear, moreover, that if a $\bar{k}$-element set $\bar{A}$ solves $P^{\bar{k}}_{G\bar{G}L\gamma}$, a $\bar{\bar{k}}$-element set $\bar{\bar{A}}$ solves $P^{\bar{\bar{k}}}_{G\bar{G}L\gamma}$, and $\bar{k} < \bar{\bar{k}}$, then[17]

$$f_{\bar{A}}(\alpha, \beta) \leq f_{\bar{\bar{A}}}(\alpha, \beta).$$

Now we have shown that

$$\min\{f_A(\alpha, \beta) : A \text{ has } \lambda - 1 \text{ elements}\} = \min\{f_A(\alpha, \beta) : A \text{ has } T \text{ elements}\}$$
$$= L\left(1 + \frac{\zeta}{2}\right).$$

It follows that we then also have, for $T < k < \lambda - 1$,

$$\min\{f_A(\alpha, \beta) : A \text{ has } k \text{ elements}\} = L\left(1 + \frac{\zeta}{2}\right).$$

Thus if $k > T$ outcomes are available, then adding more of them achieves no further reduction in error *until* one reaches $k = \lambda$.

THEOREM 3, SECOND PART. *If* $k = \lambda - 1$; *or if* λ *is odd and* $k = (\lambda + 1)/2$; *or if* λ *is even and* $k = \lambda/2$; *then* $P^k_{G\bar{G}L\gamma}$ *is solved by the k-element outcome set A given, respectively, in* (8_C), (9_C), *and* (10_C). *Each of these has the second equal-error property and for each of them* $f_A(\alpha, \beta) = L(1 + \zeta/2)$. *If* $T < k < \lambda - 1$ (*where* $T = (\lambda + 1)/2$ *if* λ *is odd and* $\lambda/2$ *if* λ *is even*), *then* $\min\{f_A(\alpha, \beta) : A$ *has k elements*$\}$ *again equals* $L(1 + \zeta/2)$.

D. AN APPLICATION OF THEOREM 3 TO THE INFORMATIONAL EFFICIENCY OF FINITE PRICE MECHANISMS

Consider a pure-exchange economy with two persons—1 and 2—and two goods, X and Y. Person i, $i = 1, 2$, has the utility function

$$u^i = \alpha_i(w_i^x + x_i) - \tfrac{1}{2}\beta_i(w_i^x + x_i)^2 + w_i^y + y_i,$$

where w_i^x, w_i^y denote endowments of X and Y (initial holdings prior to trade); x_i, y_i are net trades; $\alpha_i > 0$, $\beta_i > 0$, $w_i^x > 0$, $w_i^y > 0$. We assume that α_i, β_i, w_i^x, w_i^y, $i = 1, 2$, are such that

[17] That follows from the fact that one can always construct a $\bar{\bar{k}}$-element set A^* by adding to the $\bar{k}$-element set $\bar{A}$, $\bar{\bar{k}} - \bar{k}$ elements which are "unused": they are placed, for example, below the lowest midvalue and at a distance D from the lowest midvalue, where D exceeds the distance between the lowest midvalue and min $\bar{A}$. Then $f_{A^*}(\alpha, \beta) = f_{\bar{A}}(\alpha, \beta)$ and, since $f_{\bar{\bar{A}}}(\alpha, \beta) \leq f_{A^*}(\alpha, \beta)$, we have $f_{\bar{\bar{A}}}(\alpha, \beta) \leq f_{\bar{A}}(\alpha, \beta)$.

u^1 is nondecreasing in x_1 for all $x_1 \in [-w_1^x, w_2^x]$

u^2 is nondecreasing in x_2 for all $x_2 \in [-w_2^x, w_1^x]$. $\quad (1_D)$

Now let $x_1 = x$, $x_2 = -x$, $y_1 = y$, $y_2 = -y$. Then the set of Pareto-optimal net trades for the economy is readily shown to be given by the set

$$\{(x, y) : x = \overline{\varphi}(\theta, \beta);\ y \in [-w_1^y, w_2^y]\},$$

where $\beta = (\beta_1, \beta_2)$, $\theta = (\theta_1, \theta_2)$; $\theta_i \equiv \alpha_i - \beta_i w_i^x$, $i = 1, 2$ and

$$\overline{\varphi}(\theta) \equiv \frac{\theta_1 - \theta_2}{\beta_1 + \beta_2}.$$

Thus we may parametrize the economy by the pair (θ, β) and we may take $\overline{\varphi}(\theta, \beta)$ as our desired outcome. That is appropriate, since for any economy that corresponds to a given (θ, β), we reach a Pareto optimum if we combine the X-trade given by $x_1 = \overline{\varphi}(\theta, \beta)$, $x_2 = -x_1$ with *any* arbitrarily selected Y-trade—i.e., we choose any y in $[-w_1^y, w_2^y]$ and let $y_1 = y$, $y_2 = -y$.

The *continuum price mechanism* realizes the desired-outcome function φ (in the language of Section A3.1 above) on the set of all (θ, β) that are consistent with (1_D); it does so, moreover, with a message space that is minimal among all mechanisms that achieve the same thing and obey appropriate regularity conditions. In the continuum price mechanism, the typical message is (x, p) where x defines a proposed X-trade (i.e., $x_1 = x$, $x_2 = -x$) and p is a price for the X-good, the price of the Y-good being one. Taking (θ_i, β_i) as i's local environment, the continuum price mechanism is—in the terminology given in 3.1 of Part A—$\pi = [M, (\mu_1, \mu_2), h]$, where $M = \{(x, p) : (x, p) \in \mathbb{R}^2; p \geq 0\}$; $\mu_1[(\theta_2, \beta_2)] = \{(x, p) : \theta_1 - \beta_1 x = p\}$; $\mu_2[(\theta_2, \beta_2)] = \{(x, p) : \theta_2 + \beta_2 x = p\}$; and $h[(x, p)] = x$. One sees from the definitions of μ_1, μ_2 that person i signals "Yes" to a proposed (x, p) if and only if the ratio of i's marginal utilities for the two goods—should the proposed trade take place—equals the ratio of their prices.

In Hurwicz and Marschak (1985) we studied certain *infinite-but-discrete* mechanisms that *approximate* the continuum price mechanism π; we showed that for certain unbounded sets of parametrized economies (θ, β), such approximations are informationally efficient (relative to φ) in the sense sketched in 3.1 of Part A.

Can one also obtain analogs to the informational minimality of the continuum price mechanism for mechanisms that are *finite* approximations to π? We shall use Theorem 3 to obtain one rather special analog. To do so, we now fix $\beta_1 = \beta_2 = 1$, so that our parametrized economy is simply $\theta = (\theta_1, \theta_2)$. We shall study a set E of parametrized economies θ,

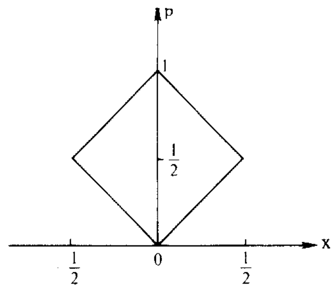

FIGURE 14

namely the nonnegative unit square:[18] $E = E_1 \times E_2$ and $E_i \equiv \{\theta_i : 0 \leq \theta_i \leq 1\}$, $i = 1, 2$. Thus person i's local environment is some $\theta_i \in [0, 1]$. Since $\beta_1 = \beta_2 = 1$, our desired-outcome function becomes $\varphi = \frac{1}{2}(\theta_1 - \theta_2)$.

We next consider a uniform lattice of points in $\mathbb{R}^2$; the points are spaced $1/t$ apart—where $t > 0$ is an integer—and they include the point (0, 0). Our finite mechanism will have as its message space—denoted M_t—the set of all such lattice points contained in the set $\mu(E) = \{(x, p) : (x, p) \in \mu_i(\theta_i, 1), i = 1, 2; (\theta_1, \theta_2) \in E\}$; the set $\mu(E)$ is the set of all pairs (x, p) that are equilibrium messages for some $(\theta_1, \theta_2) \in E$ in the continuum mechanism. The set $\mu(E)$ is readily verified to be the rotated square of Fig. 14. To avoid trivial complications, we henceforth require that t be even and $t \geq 6$. For $t = 6$, the set M_t consists of the 25 points shown as dots in Fig. 15; in general the procedure yields a set M_t consisting of $t^2/2 + t + 1$ lattice points.

In our finite approximate price mechanism, person i divides $E_i = [0, 1]$ into equal intervals (overlapping at boundary points). The *centers* of these intervals will serve as *potential surrogates* for the true current local environment θ_i. That is to say, person i signals "Yes" to a proposed lattice point in M_t, say $(\bar{x}, \bar{p})$, if the potential surrogate closest to θ_i (with ties broken downward) is, say, $\bar{\theta}_i$, and if for that $\bar{\theta}_i$ he would have signaled "Yes" to $(\bar{x}, \bar{p})$ in the continuum price mechanism. We choose the set of potential surrogates for the local environments in E_i in the "tightest" possible way, i.e., we divide $E_i = [0, 1]$ into the *largest* number of equal intervals such that for every pair of interval centers $(\bar{\theta}_1, \bar{\theta}_2)$, there exists a lattice point $\bar{m} = (\bar{x}, \bar{p})$ in M_t for which $\bar{m} \in \mu_1(\bar{\theta}_1)$, $\bar{m} \in \mu_2(\bar{\theta}_2)$. For $i = 1$, 2, the tightest set of potential surrogates for θ_i turns out to be the set

$$S_t = \left\{\frac{1}{t}, \frac{3}{t}, \frac{5}{t}, \ldots, \frac{t-1}{t}\right\}.$$

[18] Condition (1_D) can be shown to hold for any economy that is parametrized by a pair (θ, β) with $\beta_1 = \beta_2 = 1$ and $\theta \in E$.

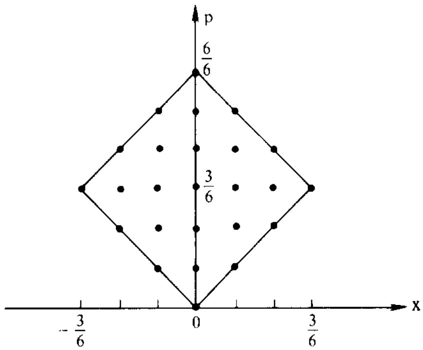

FIGURE 15

For a fixed integer $t > 0$ our finite approximation to the continuum price mechanism π is, then, the mechanism $\pi_t = [M_t, (\overline{\mu}_1, \overline{\mu}_2), \overline{h}]$, where for $i = 1, 2$

$$\overline{\mu}_i(\theta_i) = \{\overline{m} \in M_t : \text{for the element } \overline{\theta}_i \text{ of } S_t \text{ closest to } \theta_i, \text{ with ties broken downward, we have } \overline{m} \in \mu_i(\overline{\theta}_i)\}$$

and

$$\text{for } (\overline{x}, \overline{p}) \text{ in } M_t, \ \overline{h}[(\overline{x}, \overline{p})] = \overline{x}.$$

Now call a lattice point $\overline{m}$ in M_t a "used" point if for some θ in E, we have $\overline{m} \in \mu(\theta) \equiv \mu_1(\theta_1) \cap \mu_2(\theta_2)$. Straightforward calculation shows that

(i) of the $t^2/2 + t + 1$ lattice points in M_t only $t^2/4$ are "used";

(ii) the number of distinct outcomes that are "used"—i.e., the number of elements in the set $\overline{h}(\overline{\mu}(E))$—is $t - 1$;

(iii) the error of π_t—i.e., $\sup\{|\overline{h}(m) - \varphi(\theta)| : m \in \overline{\mu}(\theta), \theta \in E\}$—equals $1/t$.

The mechanism π_t is a *grid* mechanism: its associated covering[19] divides E_i into $t/2$ equal intervals and divides E into $t^2/4$ *squares* of side $4/t^2$ (with overlap at boundaries). We now compare the error achieved by π_t with the lowest error achievable by any *square grid* mechanism whose number of possible messages—i.e., the number of cells in the associated grid covering—does not exceed $t^2/2 + t + 1$, the number of lattice points in M_t, *and* whose number of distinct outcomes does not exceed $t - 1$, the

[19] See the discussion in 3.1 of Part A concerning the covering associated with a finite mechanism.

number of outcomes used in π_t. Thus our comparison preserves the dynamically desirable grid property of π_t and in fact preserves its "squareness," which essentially means that in choosing their surrogates both persons i approximate all of the environments in E_i with the same precision.

It turns out that if at most $t - 1$ outcomes are available and if the square grid mechanism to be compared with π_t has J^2 cells, each of them a side-$1/J$ square—where $t/2 < J \leq V_t$ and $V_t \equiv \sqrt{t^2/2 + t + 1}$—then the square grid mechanism's error is *higher* than the error of π_t. The "fineness" of such a J-by-J grid is inappropriate when only $t - 1$ outcomes are available; π_t, on the other hand, covers E with the coarser $t/2$-by-$t/2$ grid and thereby achieves a good balance between the number of messages (lattice points) that are used (namely $t^2/4$) and the number of available outcomes (namely $t - 1$).[20]

More precisely, consider any integer J with $t/2 < J \leq V_t \equiv \sqrt{t^2/2 + t + 1}$. We consider a grid covering of E in which each cell is a square of side $1/J$. We wish to find an error-minimizing $(t - 1)$-tuple of outcomes. Define $\tilde{\theta}_1 \equiv \frac{1}{2}\theta_1$, $\tilde{\theta}_2 \equiv -\frac{1}{2}\theta_2$, $\tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2)$. Then $\varphi(\theta) = \frac{1}{2}(\theta_1 - \theta_2) = \tilde{\varphi}(\tilde{\theta})$, where $\tilde{\varphi}(\tilde{\theta}) \equiv \tilde{\theta}_1 + \tilde{\theta}_2$. Since, for our set E (the unit square), θ_1 and θ_2 have $[0, 1]$ as their domain, the two new variables have as their domains $[0, \frac{1}{2}]$ and $[-\frac{1}{2}, 0]$, respectively. For each of the side-$1/J$ squares in E, the variable $\tilde{\theta}_i$ $(i = 1, 2)$ varies over an interval of width $1/2J$. Accordingly, we wish to solve the regular problem $P^{t-1}_{G\bar{G}L\gamma}$, where $G = (0, -\frac{1}{2})$, $\bar{G} = (\frac{1}{2}, 0)$, $L = 1/2J$, $\gamma = (1, 1)$. Thus $\zeta = 1$. A $(t - 1)$-tuple which solves this problem will also be an error-minimizing $(t - 1)$-tuple of outcomes for $\varphi = \frac{1}{2}(\theta_1 - \theta_2)$, when these outcomes are assigned to the side-$1/J$ squares covering the unit square E. The lowest value of $\tilde{\theta}_1 + \tilde{\theta}_2$ on $E_{G\bar{G}}$ is $\alpha = -\frac{1}{2}$ and the highest value is $\beta = \frac{1}{2}$.

The lowest midvalue in the set $\Lambda_{G\bar{G}L\gamma}$ is

$$\alpha + \frac{L}{2}(1 + \zeta) = -\frac{1}{2} + \left(\frac{1}{4J}\right)(2) = -\frac{1}{2} + \frac{1}{2J}.$$

[20] It is easy to see, in simple examples, that when the number of outcomes is fixed and exceeds half the numbers of midvalues, then an increase in grid fineness may actually make the error larger if the fixed number of outcomes continues to exceed half the number of midvalues for the new grid. The increase in the number of the grid's cells must exceed a certain threshold if the error is to be made smaller. Suppose E is again the unit square and our desired-outcome function is $\theta_1 + \theta_2$. Then for the nine-cell (i.e., 3-by-3) square grid, there are five distinct midvalues. If five outcomes are available, then we choose them to be the five midvalues and we achieve an error of $\frac{1}{3}$. If we now move to the 16-*cell square grid* (the 4-by-4 grid), which has *seven* distinct midvalues, but we continue to permit *five* outcomes, then the smallest attainable error becomes *larger*. It becomes $\frac{3}{8}$. That smallest error is achieved by the 4-element outcome set $\{\frac{3}{8}, \frac{7}{8}, \frac{11}{8}, \frac{13}{8}\}$, which follows the formula in (9_C) for $k = T = (\lambda + 1)/2 = 4$. According to the second part of Theorem 3, that is an error-minimizing 4-element outcome set and no 5-element set can achieve a higher error.

The highest midvalue is

$$\beta - \frac{L}{2}(1 + \zeta) = \frac{1}{2} - \left(\frac{1}{4J}\right)(2) = \frac{1}{2} - \frac{1}{2J}.$$

The midvalues in $\Lambda_{G\bar{G}L\gamma}$ are spaced $1/2J$ apart, and the number of midvalue is λ, where (from (5_C))

$$\lambda = \frac{\beta - \alpha}{L} - \zeta = 2J - 1.$$

Hence we have that $T = J$, where, as in Part C

$$T \equiv \begin{cases} \dfrac{\lambda}{2} & \text{if } \lambda \text{ is even} \\[2ex] \dfrac{\lambda + 1}{2} & \text{if } \lambda \text{ is odd.} \end{cases}$$

The number of outcomes available is $t - 1$ and for $t > 6$ we have

$$t - 1 > V_t \geq J = T.$$

The two parts of Theorem 3, taken together, tell us that for our J-by-J square grid, T outcomes achieve a smaller error than k outcomes for $k < T$, but T outcomes achieve the *same* error as k outcomes for $T < k \leq \lambda - 1 = 2J - 2$. (The lowest error attainable for the J-by-J grid is the error achieved by $\lambda = 2J - 1$ outcomes, namely the midvalues themselves; but that lowest error is not attainable for $t - 1$ outcomes, since $J > t/2$, and hence $2J - 1 > t - 1$.) According to the second part of Theorem 3, the lowest error for $k = T = J$ outcomes is

$$L\left(1 + \frac{\zeta}{2}\right) = \left(\frac{3}{2}\right)\left(\frac{1}{2J}\right) = \frac{3}{4J}.$$

This error decreases as J rises, and for the highest possible J, namely $J = V_t$, the error equals $3/4V_t$. But the error of π_t is $1/t$ and

$$\frac{3}{4J} \geq \frac{3}{4V_t} = \frac{3}{4\sqrt{t^2/2 + t + 1)}}.$$

The last term exceeds $1/t$, the error of the price mechanism π_t, if $t^2 > 16t + 16$, i.e., for $t > 16$. Since the error $3/4J$ is decreasing in J, it is of interest to consider t such that V_t is an integer (the lowest such t is 6 and the next is 40) and to study the ratio of $1/t$ (the error of π_t) to the error of the finest square grid (the largest J) permitted by our constraints on J, namely the V_t-by-V_t grid. That error is $3/4V_t$ and the ratio of $1/t$ to that error can be written $\frac{4}{3}\sqrt{\frac{1}{2} + 1/t + 1/t^2}$, which has the limit $(\sqrt{8})/3$ as t increases without limit. We may conclude:

For $t > 16$ and t even, the error of the finite price mechanism π_t is less than the lowest error attainable for square grid mechanisms that have no more distinct outcomes than does π_t and have more cells than the number of lattice points (messages) "used" in π_t but not more cells than there are lattice points in M_t. As the even integer t increases without limit, the error of π_t gets arbitrarily close to $(\sqrt{8})/3$ times the lowest error achievable by square grid mechanisms that have no more distinct outcomes than π_t and have exactly as many cells as there are lattice points in M_t.

E. Concluding Remarks

For the case when φ is linear and the covering Σ_M is a uniform grid, we were able to find an explicit solution to the problem of finding an error-minimizing k-element outcome set. A central question for further study is: what broader class of pairs (φ, Σ_M) admits an explicit solution rather than requiring a (nontrivial) algorithm? If, on the other hand, a (nontrivial) algorithm is required, then how fast can such an algorithm be?

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