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A Necessary Condition for Decentralization and an Application to Intertemporal Allocation*

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A necessary condition for realization by a decentralized finite-dimensional mechanism is proved. The concepts of the realization of a time sequence of allocations by a decentralized temporal process and by the particular more realistic class of decentralized evolutionary temporal processes are defined. The above condition shows that the optimal allocation in a simple model for intertemporal production and consumption can be realized by a decentralized temporal process but not by a decentralized evolutionary process which uses either a finite number of verification conditions or a finite-dimensional message space. *Journal of Economic Literature* Classification Numbers: 022, 024, 111, 113.

1. INTRODUCTION

This work is concerned with the question of whether a time sequence of resource allocations can be realized by means of a decentralized mechanism. More precisely, we are interested in welfare maximization in an infinite-horizon intertemporal (or intergenerational) economy of the type

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studied by Malinvaud [6], Koopmans [4], and many others since. The welfare criterion used by Malinvaud was production efficiency, and the mechanism he considered was period-by-period profit maximization, with prices treated parametrically. Malinvaud showed that, in contrast to the finite-horizon case, profit maximization does not guarantee efficiency when the horizon is infinite. An additional "transversality condition" (e.g., that the sequence of present values of the inputs converges to zero) is needed, and such a condition cannot be implemented by individual decision makers with finite lives. Therefore profit maximization does not provide a decentralized mechanism which realizes efficiency. Koopmans conjectured that it would not be possible to realize efficiency in the infinite horizon problem with any decentralized mechanism.

It is natural to inquire whether it is possible to design any decentralized mechanism, not necessarily involving perfectly competitive behavior, which would maximize some well-defined measure of welfare over some class of allocations. We shall consider an economy with a single commodity in which the rate of increase of utility is of the form $u(c_t)$, where u is a utility function and c_t is the rate of consumption in the tth time interval, while an investment of the remaining stock x_t at the beginning of this time interval results in an amount $f(x_t)$ at the end of the interval. The economy is thus described by the utility function $u(c)$ and the production function $f(x)$. As the measure of welfare we use the sum of discounted one-period utilities

$$W = \sum \delta^t u(c_t)$$

with $0 < \delta < 1$.

In order to discuss the question of whether the allocation of the optimal consumption stream $\{c_t\}$ for this model can be realized by a decentralized mechanism, we need a number of definitions. A *goal function* (or *performance function*) Q is a mapping from a set $\mathfrak{E}$ of admissible economies into an allocation space $\hat{\mathfrak{A}}$, which is a normed linear vector space.

The set $\mathfrak{E}$ is a subset of a linear vector space $\hat{\mathfrak{E}}$, and we assume that $\mathfrak{E}$ is open in the rather weak sense that its intersection with any finite-dimensional linear subspace is open. Correspondingly, we shall assume that Q is twice continuously differentiable in the sense that its restriction to the intersection of $\mathfrak{E}$ with any finite-dimensional hyperplane is twice continuously differentiable. In particular, for any e_0 in $\mathfrak{E}$ and any η in $\hat{\mathfrak{E}}$ the function $\tilde{Q}(\rho) \equiv Q(e_0 + \rho\eta)$ is a twice continuously differentiable function of ρ for small values of ρ. The derivative $\tilde{Q}'(0)$ is called the derivative (or Gâteaux derivative) of Q at e_0 in the direction η, and is denoted by $Q_e(e_0; \eta)$.

These definitions will be discussed in greater detail at the beginning of Section 2.

A *mechanism* consists of a message space[1] $\mathfrak{M}$, which is an open subset of a linear vector space $\hat{\mathfrak{M}}$; a verification function G, which is a twice continuously differentiable mapping from $\mathfrak{W} \equiv \mathfrak{E} \times \mathfrak{M}$ into a Euclidan space R^k; and an outcome function H, which is a twice continuously differentiable mapping from $\mathfrak{M}$ into the allocation space $\hat{\mathfrak{A}}$.

The differentiable mapping $G: \mathfrak{W} \to R^k$ is said to be *nondegenerate* at $w_0 \in \mathfrak{W}$ if the range (image) of its derivative $G_w(w_0; \omega)$ as ω varies is the whole space R^k.

DEFINITION 1.1. The mechanism $(\mathfrak{M}, G, H)$ is said to *realize*[2] the goal function Q if

(a) $H(m) = Q(e)$ whenever $G(e, m) = 0$;

(b) For each e in $\mathfrak{E}$ there is at least one $m \in \mathfrak{M}$ so that

  (i) $G(e, m) = 0$;

  (ii) G is nondegenerate at (e, m).

We now assume that the economies in $\mathfrak{E}$ are defined by two independent pieces of information, which are held by two separate agents. That is, we suppose that

$$\mathfrak{E} = \mathfrak{U} \times \mathfrak{F}.$$

In our application the two agents are the consumer and the producer, an element of $\mathfrak{U}$ represents the consumer's utility function u, and an element of $\mathfrak{F}$ is the production function f.

DEFINITION 1.2. The mechanism $(\mathfrak{M}, G, H)$ with

$$G: \mathfrak{U} \times \mathfrak{F} \times \mathfrak{M} \to R^k,$$
$$H: \mathfrak{M} \to \hat{\mathfrak{A}}$$

is said to be *decentralized* (or *privacy preserving*) if G has the form

$$G(u, f, m) = R(u, m) + S(f, m). \tag{1.1}$$

The decomposition (1.1) states that the condition $G = 0$ can be verified by an information processor (a person or a computer able to generate messages, receive responses, and compute the outcome function) which

[1] Note that the message space need not be finite-dimensional, which strengthens our impossibility result.

[2] The term *implement*, which might also be used here, is customarily reserved for mechanisms derived from a game theoretic approach.

elicits the response $R(u, m)$ to the message m from an agent who only has the information u, and the response $S(f, m)$ from a second agent who only has the information f.

When each component of R is a difference between a functional of u and a functional of m while each component of S is the difference between a functional of f and a functional of m, the mechanism consists of determining the outcome from the values of finitely many linear functionals elicited from each of the agents. If R and S do not have this form, the equilibrium state $G=0$ must usually be reached by an iterative process such as a bidding process. Such a process is called a *tâtonnement.*

We remark that it is more usual to define a decentralized mechanism by requiring one of the agents to verify a condition $g_1(u, m)=0$ and the other agent to verify another condition $g_2(f, m)=0$, where the range of g_1 lies in a Euclidean space R^{j_1} and that of g_2 lies in R^{j_2}. If we let $k=j_1+j_2$ and think of R^k as a direct product, we can set

$$G(u, f, m)=\begin{pmatrix} g_1(u, m) \\ g_2(f, m) \end{pmatrix}=\begin{pmatrix} g_1(u, m) \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ g_2(f, m) \end{pmatrix}$$

to obtain an allocation function of the form (1.1). Conversely, if we have a G of the form (1.1), we can define the new message space $\mathfrak{M}^*=\mathfrak{M}\times R^k$ and define $g_1(u, m, m')=R(u, m)+m'$ and $g_2(f, m, m')=S(f, m)-m'$ to reduce the equation $G=0$ to the form $g_1=0$, $g_2=0$.

We now consider the case where the allocation space $\mathfrak{A}$ consists of time sequences $\{a_t\}$, where t ranges over a finite or infinite interval of nonnegative integers. In this case, the goal function Q has components Q_t, the desired allocations at times t, and if $(\mathfrak{M}, G, H)$ is a mechanism which realizes Q, then H has components H_t.

We shall call the triple $(\mathfrak{M}, G, H)$ consisting of a message space, a verification function, and an outcome function a *temporal process* if the verification function G is the direct sum of mappings G_t with finite-dimensional ranges R^{k_t}. That is, the equation $G=0$ is equivalent to the sequence of equations $G_t=0$. For the infinite-horizon problem the range of G may be infinite-dimensional, so that the concept of a temporal process is an extension of the concept of a mechanism.

We define the realization of the goal function Q by the temporal process exactly as in Definition 1.1. We use Definition 1.2 to define a decentralized temporal process.[3] This definition is, of course, equivalent to saying that each of the mappings $G_t(u, f, m)$ is of the form $R_t(u, m)+S_t(f, m)$.

[3] If the time sequence is interpreted as an intergenerational model, one thinks of different agents in different time intervals. Our concept of decentralization takes account of the fact that in each generation there are two agents who possess different private information.

We observe that "in general" a temporal process for the infinite horizon problem will have the property that the component H_0 of the outcome function cannot be determined without verifying all the conditions $G_t = 0$. That is, for some e in $\mathfrak{E}$ and for every nonnegative integer s there are two elements m and m' in $\mathfrak{M}$ such that

$$G_t(e, m) = G_t(e, m') = 0 \qquad \text{for} \quad t = 0, ..., s$$

but $H_0(m) \neq H_0(m')$.

For an intergenerational model this means that no allocation can occur at time 0 until all the future agents, even those who are as yet unborn, have verified their conditions. Such a process is thus a prescription for economic paralysis rather that a realistic model for economic behavior. For this reason we introduce a definition of a more restricted but also more reasonable class of processes.

DEFINITION 1.3. A temporal process is said to be an *evolutionary process* if it has the property that

$$G_t(e, m) = G_t(e, m') = 0 \qquad \text{for} \quad t = 0, ..., s$$

imply that

$$H_s(m) = H_s(m').$$

We note that if the goal function Q is realized by the evolutionary process $(\mathfrak{M}, G, H)$, then Q_t is realized by the mechanism

$$(\mathfrak{M}, \{G_0, G_1, ..., G_t\}, H_t).$$

Thus, it is not necessary to know the future[4] in order to make the allocation at time t.

Since we are interested in proving the impossibility of implementation with a decentralized evolutionary process, we have introduced a rather weak definition of such a mechanism. While Definition 1.3 precludes verification by future agents, it permits requiring verification by all past agents. A class of evolutionary processes which prohibit this unrealistic

[4] If one wishes to model an economy which changes in time, the space $\mathfrak{E}$ of admissible economies will also consist of time sequences, and one will need to require that G_s only depend on the components e_t with $t \leq s$. It is clear that one cannot realize a goal function in which Q_s depends on components e_t with $t > s$ by such a mechanism. In fact, Hurwicz and Majumdar [2] have proven the impossibility of realizing such a function by a temporal proces of a class of proposed action processes.

feature as well is obtained by letting the space $\mathfrak{M}$ consist of time sequences $\{m_t\}$ and choosing the functions G and H in the form

$$H_t(m) = h_t(m_t)$$

$$G_t(e, m) = g_t(e, m_t, h_0(m_0), h_1(m_1), ..., h_{t-1}(m_{t-1})).$$

If this process is evolutionary and if it realizes the goal function Q, then the functions $h_s(m_s)$ must have the values $Q_s(e)$. Thus if at time t the historic allocations $a_s = Q_s(e)$ for $s < t$ have been observed, the mechanism $(\mathfrak{M}_t, g_t(e, m_t, a_0, ..., a_{t-1}), h_t)$ realizes Q_t. Here $\mathfrak{M}_t$ is the range of the tth component of m as m varies over members of $\mathfrak{M}$ such that $h_s(m_s) = a_s$ for $s < t$. In particular, Q_0 is realized by the mechanism $(\mathfrak{M}_0, g_0(e, m_0), h_0)$.

In Section 2 we shall derive a rather general necessary condition which must be satisfied by a goal function Q if it is to be realized by a decentralized mechanism. The condition is that the mixed second derivative Q_{uf}, which is a bilinear form in the displacements of u and f, must vanish whenever each of these displacements satisfies a certain finite set of linear equations.

The detailed one-commodity model which we have mentioned above is formulated and analyzed in Section 3. We show that there is a unique optimal investiment and consumption stream $(\{x_t\}, \{c_t\})$, and that this goal function can be realized by an explicit decentralized temporal process which is not an evolutionary process.[5]

In Section 4 we combine the fact that if the optimal consumption stream is realized by an evolutionary process then Q_0 is realized by the mechanism $(\mathfrak{M}, G_0, H_0)$ with the results of Sections 2 and 3 to show that the infinite-horizon problem cannot be realized by a decentralized evolutionary process.[6] Theorem 4.1 gives the lower bound $k \geqslant T/2$ for the dimension of the range of the verification function G_0 in any decentralized mechanism which realizes the first allocation x_0 in the corresponding T-horizon problem. Theorem 4.2 uses the same proof to show that there is no decentralized evolutionary process which realizes x_0 in the infinite-horizon problem.

[5] Related propositions on the properties of optimal programs are found in Sobel [8] and in Majumdar and Nermuth [5]. The paper by Brock and Majumdar [1] contains independently obtained results on temporal processes. (An early version of the latter paper was circulated in 1984.)

[6] We observe that this result is only proved when the goal function is defined by maximizing the discounted total utility. In fact, V. Bala, M. Majumdar, and T. Mitra [Decentralized evolutionary realization in intertemporal economies: A possibility result, mimeo, Cornell University, July 23, 1989] have recently found that another optimality criterion for intertemporal production and consumption leads to a goal function which can be realized by a decentralized evolutionary process.

We conclude that for this model Koopmans' view is too pessimistic if one is willing to accept any temporal process, but that it is valid if one restricts oneself to evolutionary processes.

We show in Section 5 that if the verification function G satisfies a sufficiently strong solvability condition, then the dimension of the message space is at least as great as that of the range of G. In this case, the lower bound for the latter in the finite-horizon problem also gives a lower bound for the former.

Our one-commodity model is, of course, exceedingly simple. It was investigated in the expectation that if such a simple model cannot be decentralized, then, in general, a more complex one is even less likely to be capable of being decentralized. One way of tying this model into a more realistic (but still quite restricted) one is to think of $\mathfrak{E}$ as a small subeconomy of a much larger market economy and to identify the one commodity with money. In this case $f(x)-x$ could represent the return in one time period of an investment x in an optimal portfolio, while $u(c)$ would represent the utility of spending an amount c on an optimal bundle of goods and services.

2. A Necessary Condition for Decentralization

In this section we shall derive a condition which a goal function $Q(u, f)$ must satisfy if it can be realized by a decentralized mechanism. Note that we are here discussing a single mechanism rather than a temporal process.

The only information one has about the functions u and f in the mechanism problem is that they lie in certain sets $\mathfrak{U}$ and $\mathfrak{F}$ of functions. Therefore our condition will involve derivatives of mappings between spaces which may be infinite-dimensional, and we begin with a brief discussion of the differentiation process.

A *linear vector space* $\hat{\mathfrak{S}}$ is defined to be a collection of elements such as vectors, functions, or vector-valued functions with the property that if s_1 and s_2 are in $\hat{\mathfrak{S}}$ and α and β are any real numbers, then $\alpha s_1 + \beta s_2$ is also in $\hat{\mathfrak{S}}$, and the usual laws of vector addition and multiplication by a scalar hold.

We define a weak topology in any normed linear vector space $\hat{\mathfrak{S}}$ by saying that a subset $\mathfrak{B}$ of $\hat{\mathfrak{S}}$ is open if for any integer l, any collection $\eta_1, ..., \eta_l$ of elements of $\hat{\mathfrak{S}}$, and any point $b \in \mathfrak{B}$ there is an $\varepsilon > 0$ such that the linear combination

$$b + \sum_{j=1}^{l} \rho_j \eta_j$$

also lies in $\mathfrak{B}$ whenever $\rho_1^2 + \cdots + \rho_l^2 < \varepsilon^2$.

We have assumed that the allocation space is a normed linear vector space. A *normed linear vector space* is a linear vector space $\mathfrak{R}$ in which there is a real-valued function $\|r\|$, the norm of r, with the properties that $\|r\| > 0$ for $r \neq 0$, that $\|\alpha r\| = |\alpha| \, \|r\|$ for all real α, and that $\|r+s\| \leqslant \|r\| + \|s\|$. The quantity $\|r_1 - r_2\|$ is thought of as the distance between r_1 and r_2, and convergence is defined in terms of it in the obvious way.

Any Euclidean space is, of course, a normed linear vector space.

A function $R(\rho)$ of the real variable ρ with values in the normed linear space $\mathfrak{R}$ is said to have the *limit* r as $\rho \to \sigma$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $|\rho - \sigma| < \delta$ implies that $\|R(\rho) - r\| < \varepsilon$. With this definition we can define continuity and differentiation of such a function.

As usual, we say that a mapping $R(s)$ from an open subset $\mathfrak{B}$ of the linear vector space $\hat{\mathfrak{S}}$ into the normed linear vector space $\hat{\mathfrak{R}}$ is *continuous* if for every $b \in \mathfrak{B}$ and every positive ε there is an open subset $\mathfrak{C}$ of $\mathfrak{B}$ which contains b and has the property that $\|R(c) - R(b)\| < \varepsilon$ for all $c \in \mathfrak{C}$.

It is easily seen that because of our definition of open set this is equivalent to saying that $R(s)$ is continuous if for every $b \in \mathfrak{B}$, every integer l, and every collection $\eta_1, \ldots, \eta_l$ of elements of $\hat{\mathfrak{S}}$ the function

$$\tilde{R}(\rho_1, \ldots, \rho_l) \equiv R\left(b + \sum_{j=0}^{l} \rho_j \eta_j\right)$$

is continuous. Similarly, we say that R is *continuously differentiable* or *twice continuously differentiable* in B if for each $b \in B$ and finite collection $\eta_j \in \hat{\mathfrak{S}}$ this function $\tilde{R}$ of the finitely many variables $\rho_1, \ldots, \rho_l$ has this property.

It is easily verified that if R is continuously differentiable, then the mapping

$$R_s(b; \eta) \equiv \left.\frac{\partial R(b + \rho_1 \eta)}{\partial \rho_1}\right]_{\rho_1 = 0}$$

is linear in η. It is continuous in both b and η in the sense defined above. The functional derivative R_s defined in this way is called the Gâteaux derivative. If the mapping R happens to have a Fréchet derivative, it will coincide with the Gâteaux derivative.

If R is twice continuously differentiable, we define the second Gâteaux derivative

$$R_{ss}(b; \eta_1, \eta_2) \equiv \left.\frac{\partial^2 R(b + \rho_1 \eta_1 + \rho_2 \eta_2)}{\partial \rho_1 \, \partial \rho_2}\right]_{\rho_1 = \rho_2 = 0},$$

which is linear in η_1 and in η_2.

We shall assume that the elements u and f which define a decentralized economy lie in an open subset $\mathfrak{U}$ of a normed linear vector space $\hat{\mathfrak{U}}$ and an open subset $\mathfrak{F}$ of a normed linear vector space $\hat{\mathfrak{F}}$, respectively. We are given a twice continuously differentiable goal function $Q(u, f)$ from $\mathfrak{U} \times \mathfrak{F}$ into a normed linear vector space $\hat{\mathfrak{A}}$, the allocation space. We suppose that this goal function is realized by the decentralized mechanism $(\mathfrak{M}, G, H)$, where $\mathfrak{M}$ is an open subset of a normed linear vector space $\hat{\mathfrak{M}}$, G is a twice continuously differentiable mapping from $\mathfrak{W} = \mathfrak{U} \times \mathfrak{F} \times \mathfrak{M}$ into the Euclidean space R^k, and H is a twice continuously differentiable mapping from $\mathfrak{M}$ into $\hat{\mathfrak{A}}$. The statement that the mechanism realizes the goal function and is decentralized means that Definitions 1.1 and 1.2 are satisfied.

We shall sometimes write the points of $\mathfrak{W}$ as (u, f, m). We shall work in a neighborhood of a point $w_0 = (u_0, f_0, m_0) \in \mathfrak{W}$ at which the verification function G vanishes and is nondegenerate. (The existence of such a point for each (u_0, f_0) is assured by the assumptions contained in Definition 1.1.)

Suppose that $G(w)$ is nondegenerate at the point w_0. This means that the range of the derivative $G_w(w_0; v)$ as v varies over the space $\hat{\mathfrak{W}}$ is the whole space R^k. Therefore, for each of the coordinate unit vectors ε_j of R^k there is an element v_j such that $G_w(w_0; v_j) = \varepsilon_j$. That is,

$$[G_w(w_0; v_j)]_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.1}$$

THEOREM 2.1. *Suppose that the mechanism* $(\mathfrak{M}, G, H)$ *is decentralized and that it realizes the goal function*

$$Q : \mathfrak{U} \times \mathfrak{F} \to \hat{\mathfrak{A}}.$$

Suppose further that G *vanishes and is nondegenerate at the point* $w_0 = (u_0, f_0, m_0)$, *and that the maps* G, Q, *and* H *are twice continuously differentiable. Then* Q *has the following property*:

For every $\zeta \in \hat{\mathfrak{U}}$ *such that*

$$G_u(w_0; \zeta) \equiv G_w(w_0; (\zeta, 0, 0)) = 0 \tag{2.2}$$

and every $\eta \in \hat{\mathfrak{F}}$ *such that*

$$G_f(w_0; \eta) \equiv G_w(w_0; (0, \eta, 0)) = 0 \tag{2.3}$$

the derivatives of Q *satisfy*

$$Q_u(u_0, f_0; \zeta) = Q_f(u_0, f_0; \eta) = 0 \tag{2.4}$$

and

$$Q_{uf}(u_0, f_0; \zeta, \eta) = 0. \tag{2.5}$$

Proof. Since G is nondegenerate at w_0, there exist $v_1, ..., v_k \in \hat{\mathfrak{W}}$ such that the equations (2.1) hold. Choose any $\zeta \in \hat{\mathfrak{U}}$ such that (2.2) is satisfied and any $\eta \in \hat{\mathfrak{F}}$ such that (2.3) is valid.

We note that the expression $G(w_0 + (\sigma\zeta, \rho\eta, 0) + \sum_{j=1}^{k} a_j v_j)$ is a twice continuously differentiable k-vector-valued function of the finite set of variables $\sigma, \rho, a_1, ..., a_k$. It vanishes when all these variables are zero, and by (2.1) its Jacobian determinant with respect to the a_j is not zero at this point. Therefore, by the implicit function theorem there exist $\varepsilon > 0$ and twice continuously differentiable functions $\alpha_1(\sigma, \rho), ..., \alpha_k(\sigma, \rho)$ such that

$$\begin{aligned} G\left(w_0 + (\sigma\zeta, \rho\eta, 0) + \sum_{j=1}^{k} \alpha_j v_j\right) &= 0 \qquad \text{for} \quad \rho^2 + \sigma^2 < \varepsilon^2 \\ \alpha_j(0, 0) &= 0 \qquad \text{for} \quad j = 1, ..., k. \end{aligned} \tag{2.6}$$

We take the partial derivative of (2.6) with respect to σ, set $\sigma = \rho = 0$, and use (2.1) and (2.2) to find that

$$\frac{\partial \alpha_j}{\partial \sigma}(0, 0) = 0 \qquad \text{for} \quad j = 1, ..., k. \tag{2.7}$$

Similarly we find that

$$\frac{\partial \alpha_j}{\partial \rho}(0, 0) = 0 \qquad \text{for} \quad j = 1, ..., k. \tag{2.8}$$

We now take the mixed second partial derivative of (2.6) with respect to σ and ρ and set $\sigma = \rho = 0$. By using the definition (1.1) of a decentralized mechanism together with (2.1), (2.7), and (2.8), we see that

$$\frac{\partial^2 \alpha_j}{\partial \rho \, \partial \sigma}(0, 0) = 0 \qquad \text{for} \quad j = 1, ..., k. \tag{2.9}$$

Since the mechanism realizes Q and since the outcome function H is independent of u and f, we have

$$Q\left(u_0 + \sigma\zeta + \sum_{j=1}^{k} \alpha_j u_j, f_0 + \rho\eta + \sum_{j=1}^{k} \alpha_j f_j\right) = H\left(m_0 + \sum_{j=1}^{k} \alpha_j m_j\right), \tag{2.10}$$

where we have written

$$w_0 = (u_0, f_0, m_0)$$

and

$$v_j = (u_j, f_j, m_j).$$

We take the derivative of both sides with respect to ρ and set $\sigma = \rho = 0$. We conclude from (2.6) that

$$Q_u(u_0, f_0; \zeta) = 0.$$

Similarly we conclude from (2.10) and (2.8) that $Q_f(u_0, f_0; \eta) = 0$. Finally, we take the mixed derivative of (2.10) and use (2.7), (2.8), and (2.9) to obtain (2.5).

Thus the theorem is established.

Remarks. 1. In mechanism design theory the mapping G is usually an unknown of the problem. Theorem 2.1 is important because it shows that Q cannot be realized by a decentralized mechanism in which the range of the verification function has dimension k unless there are k linear homogeneous equations in ζ and k in η which together imply (2.5).

2. If one makes the stronger hypothesis that the range of the linear transformation $(G_u\ F_f)$ is all of R^k, one can choose the v_i in (2.1) so that the m-components m_i are all zero. Then the right-hand side of (2.0) is independent of the α_j and hence of σ and ρ, so that the conclusion of Theorem 2.1 follows without any assumption about the smoothness of the outcome function H. This result for the case in which all the spaces are finite-dimensional was found by Hurwicz, Reiter, and Saari [3, Theorem 4 (Theorem 4′ in the 1980 version)]. See also Saari [7].

3. A Model for Intertemporal Investment

In this section we shall derive some properties of the goal function of the following standard model for intertemporal investment. There is a production function f such that an input of the amount x of a single commodity produces an output of $f(x)$ of the same commodity after one unit of time. The commodity may be consumed or used as input for further production. There is also a utility function u such that the consumption of amount c of the commodity in one unit of time produces utility $u(c)$. The initial amount y_0 of the commodity is prescribed and the goal is to find the schedule of consumptions c_t and reinvestments x_t at the integer values of the time t which maximizes the discounted sum

$$\sum_{t=0}^{\infty} \delta^t u(c_t), \tag{3.1}$$

where δ is a given number in the interval (0, 1). The usual assumptions on the functions u and f can be written in the form

$$u \in C^2([0, X]),\ u(0) = 0, \qquad u' > 0,\ u'' < 0,$$

$$f \in C^2([0, X]),\ f(0) = 0, \qquad f' > 0,\ f'' < 0,\ \delta f'(0) > 1,\ f(X) < X.$$

Here X is a number larger than any quantity of the commodity that can occur in the problem, and $C^2([0, X])$ is the usual set of functions which are twice continuously differentiable on the interval $[0, X]$. It follows from the conditions on f that there is a positive number $x^* = x^*(f, \delta)$ such that

$$\delta f'(x^*) = 1. \tag{3.2}$$

One seeks the maximum of the sum (3.1) under the constraints

$$\begin{aligned} x_{t+1} + c_{t+1} &= f(x_t) \\ x_t &\geqslant 0, \\ c_t &\geqslant 0, \\ x_0 + c_0 &= y_0, \end{aligned} \tag{3.3}$$

where the initial stock y_0 with $0 < y_0 < X$ is prescribed.

We begin with an easier problem with a finite time horizon T. This is the problem of maximizing the sum

$$\sum_{t=0}^{T} \delta^t u(c_t), \tag{3.4}$$

under the constraints (3.3) and the additional constraint that x_T is a prescribed nonnegative number[7] x_T^*. We shall always assume that x_T^* is *admissible* in the sense that

$$x_T^* < f(f(\dots f(y_0) \dots)), \tag{3.5}$$

the Tth iterate of f applied to y_0. This inequality states that x_T^* is less than the amount of the commodity one can amass by the time T by not consuming anything before this time. Because we are maximizing a continuous function over a bounded and closed finite-dimensional set, the maximum certainly exists. Because u is increasing, one finds that replacing the first equality in the constraints (3.3) by the inequality

$$x_{t+1} + c_{t+1} \leqslant f(x_t)$$

does not change the problem. In this form one is maximizing a strictly concave functional over a convex set, and it follows that there is exactly one optimizing sequence.

[7] If one does not prescribe x_T, one obtains the natural boundary condition $x_T = 0$ from the maximization. We prefer to leave the prescribed value of x_T free, because this will enable us to handle the infinite horizon problem.

By solving (3.3) for c_t and substituting in the sum (3.4), we easily derive the Euler conditions

$$-u'(c_t)+\delta f'(x_t)\,u'(c_{t+1})\begin{cases}=0 & \text{if } x_t>0, c_t>0\\ \leqslant 0 & \text{if } x_t=0\\ \geqslant 0 & \text{if } c_t=0\end{cases} \qquad \text{for } t=0, 1, ..., T-1. \tag{3.6}$$

In order to establish properties of solutions of the system (3.3), (3.6), we shall use the following lemma:

LEMMA 3.1. *Let* $(\{x_t\}, \{c_t\})$ *and* $(\{\hat{x}_t\}, \{\hat{c}_t\})$ *be two solutions of the system* (3.3), (3.6) *with* $x_0<\hat{x}_0$. *Then*

$$\begin{aligned}x_t&<\hat{x}_t,\\ c_t&>\hat{c}_t\end{aligned} \qquad \textit{for} \quad t=1, ..., T.$$

Proof. We see from the conditions $\delta f'(0)>1$ and (3.6) that since $x_t=0$ implies that $c_{t+1}=0$, it also implies that $u'(c_t)>u'(0)$. This contradicts the condition $u''<0$. Thus we always have

$$x_t>0 \qquad \text{for} \quad t<T.$$

We now prove the Lemma by induction. Clearly

$$c_0=y_0-x_0>y_0-\hat{x}_0=\hat{c}_0.$$

Suppose now that $x_t<\hat{x}_t$ and $c_t>\hat{c}_t$. Then by (3.6) and the facts that f' and u' are decreasing and $x_t>0$, $c_t>\hat{c}_t\geqslant 0$

$$u'(c_{t+1})=\frac{u'(c_t)}{\delta f'(x_t)}<\frac{u'(\hat{c}_t)}{\delta f'(\hat{x}_t)}\leqslant u'(\hat{c}_{t+1})$$

so that $c_{t+1}>\hat{c}_{t+1}$. By (3.3) and the fact that f is increasing

$$x_{t+1}=f(x_t)-c_{t+1}<f(\hat{x}_t)-\hat{c}_{t+1}=\hat{x}_{t+1}.$$

Thus the lemma is established.

It is clear from this lemma that the solution of (3.3), (3.6) with prescribed x_T is unique, and that x_t increases and c_t decreases for all t when x_T is increased. The next lemma presents another important property of solutions of this system. We recall the definition (3.2) of x^*.

LEMMA 3.2. *Let* $(\{x_t\}, \{c_t\})$ *be a solution of the system* (3.3), (3.6). *If* $x_{t-1} \leqslant x_t \leqslant x^*$, *then*

$$x_{t+1} \geqslant x_t,$$

$$c_{t+1} \geqslant c_t.$$

If $x_{t-1} \geqslant x_t \geqslant x^*$, *then*

$$x_{t+1} \leqslant x_t,$$

$$c_{t+1} \leqslant c_t.$$

Moreover, if one of the inequalities in the hypotheses is strict, the first inequality in the conclusions is strict.

Proof. Because f is strictly concave, $x_t \leqslant x^*$ implies that $\delta f'(x_t) \geqslant 1$. Then (3.6) implies that if $c_t > 0$, $u'(c_{t+1}) \leqslant u'(c_t)$ so that $c_{t+1} \geqslant c_t$. If $c_t = 0$, this is obviously also true. By (3.3) and because f is increasing, we then see that

$$x_{t+1} = f(x_t) - c_{t+1} \geqslant f(x_{t-1}) - c_t = x_t.$$

The second part of the lemma is proved in the same way.

Remark. If we define the number x_{-1} by the equation $f(x_{-1}) = y_0$, we see that the lemma is true for $t = 0$.

If we think of the sequence $\{x_t\}$ as a function of t, Lemma 3.2 states that a local minimum value of this function must lie above x^* and a local maximum value must lie below x^*. Thus any solution can have at most one local maximum or minimum, and cannot have both. In particular any value of x_t can be taken on at most twice. More exact qualitative information can be obtained from the boundary values. For example, if $y_0 < f(x^*)$ so that $x_{-1} < x^*$ and if $x_T^* < x^*$, then the sequence $\{x_t\}$ lies below x^* and has at most one maximum and no minimum.

We are now in a position to prove the existence and properties of the maximizing sequence.

THEOREM 3.1. *If the boundary value* x_T^* *is admissible in the sense that the strict inequality* (3.5) *is satisfied, the* finite-horizon *problem of maximizing the sum* (3.4) *under the constraints* (3.3) *and with* $x_T = x_T^*$ *has a unique solution* $(\{x_t\}, \{c_t\})$. *This solution satisfies the inequalities*

$$x_t > 0, c_t > 0 \qquad \textit{for} \quad t = 0, ..., T-1,$$

and is the unique solution of the constraints (3.3) *together with the equation*

$$-u'(c_t) + \delta f'(x_t)\, u'(c_{t+1}) = 0 \qquad \textit{for} \quad t = 0, 1, ..., T-1. \tag{3.7}$$

Moreover,

(a) *if* $y_0 \leqslant f(x^*)$, $x_T \leqslant x^*$ *but not both equalities hold, then* $x_t < x^*$, x_t *has no local minimum, and* c_t *is increasing;*

(b) *if* $y_0 \geqslant f(x^*)$, $x_T \geqslant x^*$ *but not both equalities hold, then* $x_t > x^*$, x_t *has no local maximum, and* c_t *is decreasing;*

(c) *if* $y_0 \leqslant f(x^*)$, $x_T \geqslant x^*$ *or* $y_0 \geqslant f(x^*)$, $x_T \leqslant x^*$, *but if not both* $y_0 = f(x^*)$ *and* $x_T = x^*$, *then* x_t *is monotone and* c_t *can have at most one maximum or minimum;*

(d) *if* $y_0 = f(x^*)$ *and* $x_T = x^*$, *then* $x_t = x^*$ *and* $c_t = f(x^*) - x^*$ *for all* t.

Proof. The existence and uniqueness of the maximizer as well as of the solution of the system (3.3), (3.6) follow from the above discussion. The property $x_t > 0$ was shown in the proof of Lemma 3.1. The monotonicity properties come directly from Lemmas 3.1 and 3.2.

The Euler conditions (3.6) will thus reduce to the Euler equation (3.7) once we prove the inequality $c_t > 0$. In order to prove this inequality, we observe that since the strict inequality (3.5) holds, a slightly larger $\hat{x}_T^*$ is also admissible. If $c_0 = 0$ in the solution of the original problem, then because y_0 is fixed, the solution $(\{\hat{x}_t\}, \{\hat{c}_t\})$ for $\hat{x}_T^*$ would have to have $\hat{c}_0 \geqslant c_0$ and hence $\hat{x}_0 \leqslant x_0$. But then Lemma 3.1 would imply that $\hat{x}_T \leqslant x_T$, so that the problem with $\hat{x}_T > x_T$ would have no solution. Since we already have proved that there is a solution, we conclude that $c_0 > 0$. We can now apply the same argument to the problem where for any $t_0 \in (0, T)$ we start with the initial stock $f(x_{t_0 - 1})$ to conclude that $c_{t_0} > 0$. This completes the proof.

We obtain a similar result for the infinite-horizon problem.

THEOREM 3.2. *The* infinite-horizon *problem of maximizing the sum* (3.1) *under the constraints* (3.3) *has a unique solution* $(\{x_t\}, \{c_t\})$. *This solution satisfies the constraints* (3.3), *the inequalities*

$$x_t > 0, c_t > 0 \qquad \text{for} \quad t \geqslant 0,$$

and the equation (3.7). *It is also the only positive solution of the equations* (3.3) *and* (3.7) *for all nonnegative* t.

Moreover,

(a) *if* $y_0 < f(x^*)$, *then* x_t *increases to* x^* *and* c_t *decreases to* $f(x^*) - x^*$ *as* t *goes to infinity;*

(b) *if* $y_0 > f(x^*)$, *then* x_t *decreases to* x^* *and* c_t *increases to* $f(x^*) - x^*$ *as* t *goes to infinity;*

(c) *if* $y_0 = f(x^*)$, *then* $x_t = x^*$ *and* $c_t = f(x^*) - x^*$ *for all* t.

Proof. We first note that both x_t and c_t are bounded by the tth iterate $f(f(\dots f(y_0)\dots))$, which is bounded uniformly in t. Thus the difference between the sum (3.1) and the finite sum (3.4) has a bound of the form $K\delta^T$.

We denote the supremum of the sum (3.1) under the constraints (3.3) by W. Then for any positive ε there is an admissible sequence $\{\tilde{x}_t, \tilde{c}_t\}$ such that

$$\sum_{t=0}^{\infty} u(\tilde{c}_t) > W - \varepsilon.$$

Let W^T denote the maximum of the finite horizon sum (3.4) with the constraints (3.3) and the boundary condition $x_T = 0$, and let $\{x_t^T, c_t^T\}$ be the maximizing solution. Let

$$\hat{c}_t = \begin{cases} \tilde{c}_t & \text{for } \quad t < T, \\ \tilde{c}_T + \tilde{x}_T & \text{for } \quad t = T \end{cases}$$

and let $\hat{x}_T = 0$, so that $\{\hat{x}_t, \hat{c}_t\}$ is admissible for this finite horizon problem. Then

$$\sum_{t=0}^{T} \delta^t u(c_t^T) = W^T \geqslant \sum_{t=0}^{T} \delta^t u(\hat{c}_t) \geqslant \sum_{t=0}^{T} \delta^t u(\tilde{c}_t) > W - \varepsilon - K\delta^T.$$

Since this is true for arbitrary ε, we see that

$$\sum_{t=0}^{T} \delta^t u(c_t^T) \geqslant W^T - K\delta^T. \tag{3.8}$$

Since $x_T^{T+1} > 0 = x_T^T$, Lemma 3.1 shows that x_t^T is increasing and c_t^T is decreasing in T :for each fixed t. Since they are also bounded, these sequences converge to a limit sequence $\{x_t, c_t\}$ as $T \to \infty$. Because the sum in (3.8) converges uniformly in T, we may let T go to infinity to show that this limit sequence is a maximizing sequence. Its uniqueness follows as in the finite horizon case from the strict concavity of u.

Clearly if $(\{x_t\}, \{c_t\})$ is the maximizing sequence pair for the infinite-horizon problem, then for any T the restriction of this sequence to $t \leqslant T$ is the maximizer for the finite horizon problem with the value x_T specified. We observe that if the inequality (3.5) is violated, the only possible admissible sequence has $c_t = 0$ for $t < T$. Since T is arbitrary, and since a consumption stream with $c_t = 0$ for all t is clearly not optimal, we conclude that the inequality (3.5) is valid for all sufficiently large T. We therefore see from Theorem 3.1 that x_t and c_t are positive for all t, and that the sequence $\{x_t, c_t\}$ is again a solution of the equations (3.3) and (3.6). Then Lemma 3.2 shows that both x_t and c_t can have at most one maximum or minimum, so that they must eventually be monotone. Therefore they must have limiting values $\bar{x}$ and $\bar{c}$ as $t \to \infty$. Since $u' > 0$, we see from taking limits in (3.6) that $\delta f'(\bar{x}) \geqslant 1$, with the inequality possible only if $c_t = 0$ for all sufficiently large t. However, if this were the case, one could increase the sum (3.1) simply by consuming all of x_t at one of these large times. Therefore, $\bar{x} = x^*$. We then see by taking limits in the constraints (3.3) that

$$\bar{x} = x^*, \qquad \bar{c} = f(x^*) - x^*.$$

Thus we have established the limiting values in the Theorem. The monotonicity properties (a), (b), and (c) then follow immediately from Lemma 3.2.

We observe that the terms with $t \leqslant T$ of the maximizing sequence $\{x_t, c_t\}$ solve the T-horizon problem with the prescribed value x_T. If T is sufficiently large, then $c_T > 0$ and therefore x_T satisfies the admissibility inequality (3.5). We thus see from Theorem 3.1 that $x_t > 0$ and $c_t > 0$ for $t < T$. Since this is true for all sufficiently large T, the inequalities hold for all t. Therefore the conditions (3.6) reduce to (3.7).

We have thus shown that the maximizing sequence has all the properties stated in the theorem, with the possible exception of the uniqueness property. To prove this property, suppose there is a second sequence pair $(\{\hat{x}_t\}, \{\hat{c}_t\})$ with positive terms which also satisfies the conditions (3.3) and (3.7) for all nonnegative t. By the arguments given above, this sequence must converge to the same limit $(x^*, f(x^*) - x^*)$. Let $\mu_t = x_t(u, f) - \hat{x}_t$, $v_t = c_t(u, f) - \hat{c}_t$. We subtract the equations (3.3) and (3.7) for the two solutions and apply the mean value theorem to find that

$$-u'' v_t + \delta f' u'' v_{t+1} + \delta f'' u' \mu_t = 0,$$

$$\mu_{t+1} + v_{t+1} = f' \mu_t$$

where the functions u'' and f' are evaluated at values between the two solutions. We find from these equations that

$$(\mu_{t+1} - v_{t+1})^2 = a_t^2 \mu_t^2 + 2a_t b_t(-\mu_t v_t) + b_t^2 v_t^2 \tag{3.9}$$

where

$$a_t = f' + \frac{2f'' u'}{f' u''},$$

$$b_t = \frac{2u''}{\delta f' u''}.$$

Clearly, as t approaches infinity, b_t approaches 2 while a_t approaches a number above δ^{-1}. Hence a_t and b_t are both greater than one when t is sufficiently large. We see from Lemma 3.1 that $\mu_t v_t \leqslant 0$. Thus (3.9) shows that for all sufficiently large t

$$(\mu_{t+1} - v_{t+1})^2 \geqslant (\mu_t - v_t)^2. \tag{3.10}$$

Since the two original sequences have the same limits, both μ_t and v_t and hence also their difference must approach zero. By (3.10) this can only happen if $\mu_t - v_t = 0$ for all sufficiently large t. Since $\mu_t v_t \leqslant 0$, this means that $\hat{x}_t = x_t(u, f)$ and $\hat{c}_t = c_t(u, f)$ when t is large . Now given x_{t+1} and c_{t+1}, we can find x_t from (3.7) and then c_t from (3.3). By going backward in t, we conclude that any two solutions must coincide for all nonnegative t, and the Theorem is proved.

Because Theorems 3.1 and 3.2 characterize the optimal solution $\{x_t(u, f), c_t(u, f)\}$ as the unique solution of a boundary value problem, we can realize it by a decentralized temporal process.

THEOREM 3.3. *Let*

$$\hat{\mathfrak{U}} = \{u \in C^3([0, X]) : u(0) = 0\},$$
$$\mathfrak{U} = \{u \in \hat{\mathfrak{U}} : u' > 0, u'' < 0\}$$
$$\hat{\mathfrak{F}} = \{f \in C^3([0, X]) : f(0) = 0\},$$
$$\mathfrak{F} = \{f \in \hat{\mathfrak{F}} : f' > 0, f'' < 0, \delta f'(0) > 1, f(X) < X\}.$$

The optimal solution $(\{x_t(u, f)\}, \{c_t(u, f))\}$ *of the finite or infinite horizon problem is realized by the following temporal process. The message space consists of positive sequence pairs* $(\{m_t\}, \{n_t\})$ *with* $m_t + n_t < X$, $m_0 + n_0 = y_0$, *and, if the horizon T is finite,* $m_T = x_T^*$. *The outcome function H is the defined by* $H_t(m) = (m_t, n_t)$. *Each function* G_t *has two-dimensional range and is defined by*

$$\begin{pmatrix} G_{t1} \\ G_{t2} \end{pmatrix} = \begin{pmatrix} 0 \\ -u'(n_t)/u'(n_{t+1}) \end{pmatrix} + \begin{pmatrix} m_{t+1} + n_{t+1} - f(m_t) \\ \delta f'(m_t) \end{pmatrix}.$$

We have made $\hat{\mathfrak{U}}$ and $\hat{\mathfrak{F}}$ subspaces of the space C^3 of three times continuously differentiable functions only to satisfy the requirement in the definition of mechanism that G be twice continuously differentiable in all its variables. It is clear that this particular temporal process still "works" if u and f are only required to be in $C^2([0, 1])$.

We note that each G_t only depends on a four-dimensional subspace of the message space.

Remark. We can see from the above construction that if one chooses an $x_0 < x_0(u, f)$ and solves the equations (3.3) and (3.7), there will eventually come a t such that (3.3) requires x_{t+1} to be negative. Similarly, if one takes $x_0 > x_0(u, f)$, there is a t such that (3.7) requires $u'(c_{t+1})$ to be larger than $u'(0)$, which is impossible.

If one wishes to extend Theorems 3.1, 3.2, and 3.3 to economies like the Cobb–Douglas economies in which u' and f' become infinite at 0, one needs to replace the smoothness conditions on the closed interval $[0, X]$ by the analogous conditions in the interval $(0, X]$, and the conditions $u(0) = 0$ and $f(0) = 0$ by $u(0+) = 0$ and $f(0+) = 0$, respectively. The proof of Theorem 3.1 goes through without alteration. However, the uniqueness statement of Theorem 3.2 must be altered, because there can be positive solutions of (3.3) and (3.7) which are not optimal and for which either x_t or c_t approaches zero as $t \to \infty$. Such solutions can be recognized by the fact that the product $(x_t - x^*)(c_t - f(x^*) + x^*)$ becomes negative for large t, while for the optimal solution it is nonnegative for all nonnegative t. Thus the statement of Theorem 3.2 is correct if the uniqueness statement is modified to say that the optimal solution is the only positive solution of (3.3) and (3.7) for which $(x_t - x^*)(c_t - f(x^*) + x^*)$ is nonnegative for all nonnegative t. The statement of Theorem 3.3 is correct for this case if we add an additional component $\hat{m}$ to the message and the additional component

$$G_{t3} = (x_t - x^*)(c_t + x^* - f(x^*)) - \hat{m}^2$$

in the verification function G_t.

4. The Nonexistence of a Decentralized Evolutionary Realization of the Temporal Investment Model

In this section we shall apply Theorem 2.1 to show that the infinite-horizon problem discussed in Section 3 cannot be realized by a decentralized evolutionary mechanism.

It is clear from the definition that if there were such a realization, then the mechanism $(\mathfrak{M}, G_0, H_0)$ would realize the first component $x_0(u, f), c_0(u, f)$ of the optimal sequence. Since $x_0(u, f) + c_0(u, f) = y_0$, the prescribed initial stock, it is sufficient to inquire whether the specific goal function $Q(u, f) = x_0(u, f)$ can be realized by a decentralized mechanism. Thus the allocation space $\mathfrak{A}$ for this problem is just the real line R^1. We shall consider y_0, and in the case of the finite-horizon problem also x_T^*, to be known and fixed.

We see from Definition 1.1 that if a mechanism $(\mathfrak{M}, G, H)$ realizes a goal function Q in an open subset $\mathfrak{E}$ of a linear vector space $\hat{\mathfrak{E}}$, then its restriction to the intersection of $\mathfrak{E}$ with a linear subspace of $\hat{\mathfrak{E}}$ also realizes the restriction of Q to this subset. That is, the smaller the space the easier it is to find a decentralized realization, and therefore the stronger is the statement that there is no such realization.

We shall only require $\hat{\mathfrak{U}}$ and $\hat{\mathfrak{F}}$ to contain the space C_0^∞ of all infinitely differentiable functions on the interval $[0, X]$ which vanish at 0. (See the discussion which follows the proof of Corollary 4.2).

We shall prove the following two theorems. We recall that x^* is defined in terms of δ and f by (3.2).

THEOREM 4.1. *If the decentralized mechanism* $(\mathfrak{M}, G, H)$ *realizes the time* 0 *entry* $x_0(u, f)$ *of the optimal solution* $(\{x_t\}, \{c_t\})$ *of the T-horizon problem and if* $\hat{\mathfrak{U}}$ *and* $\hat{\mathfrak{F}}$ *contain* C_0^∞, *then the dimension k of the range of G must satisfy*

$$k \geq \tfrac{1}{2}T.$$

THEOREM 4.2. *If* $\hat{\mathfrak{U}}$ *and* $\hat{\mathfrak{F}}$ *contain* C_0^∞, *the time zero entry* $x_0(u, f)$ *of the optimal solution* $(\{x_t\}, \{c_t\})$ *for the infinite-horizon problem cannot be realized by a decentralized mechanism.*

Theorem 4.2 is a consequence of the fact that Definition 1.2 requires k to be finite and of Theorem 4.1, which implies that no finite k will do.

From Theorem 4.2 and the definition of evolutionary process we immediately obtain the following corollary.

COROLLARY. *If* $\hat{\mathfrak{U}}$ *and* $\hat{\mathfrak{F}}$ *contain* C_0^∞, *the optimal sequence pair* $(\{x_t(u, f)\}, \{c_t(u, f)\})$ *of the infinite-horizon problem cannot be realized by a decentralized evolutionary process.*

Proof of Theorem 4.1. We write the T-horizon problem in the form

$$\begin{aligned} -u'(c_t) + \delta f'(x_t)\, u'(c_{t+1}) &= 0 \qquad \text{for} \quad t = 0, ..., T-1, \\ x_{t+1} + c_{t+1} - f(x_t) &= 0 \qquad \text{for} \quad t = 0, ..., T-1, \\ x_0 + c_0 &= y_0, \\ x_T &= x_T^*, \end{aligned} \tag{4.1}$$

where y_0 and x_T^* are prescribed.

The Jacobian matrix L of the left-hand sides of the $2T+2$ equations (4.1) with respect to the $2T+2$ unknowns (x_t, c_t) is defined by the relation

$$L\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -u''(c_t)\, q_t + \delta f''(x_t)\, u'(c_{t+1})\, p_t + \delta f'(x_t)\, u''(c_{t+1})\, q_{t+1}]_{t=0,...,T-1} \\ p_{t+1} + q_{t+1} - f'(x_t)\, p_t]_{t=0,...,T-1} \\ p_0 + q_0 \\ p_T \end{pmatrix} \tag{4.2}$$

for an arbitrary $2T+2$-vector $\{p_0, p_1, ..., p_T, q_0, q_1, ..., q_T\}$. (Each of the first two rows of the right-hand side is a T-dimensional column vector.)

The following analogue of Lemma 3.2 will imply that this matrix is nonsingular.

LEMMA 4.1. *If the vector $\{p, q\}$ satisfies the equation*

$$L\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} r \\ s \\ 0 \\ 0 \end{pmatrix} \tag{4.3}$$

and if for some $t \leqslant T-1$ the inequalities

$$\begin{aligned} p_t &\geqslant 0, \\ q_t &\leqslant 0, \\ r_t &\geqslant 0, \\ s_t &\geqslant 0 \end{aligned} \tag{4.4}$$

are satisfied, then the system (4.3) *implies that also*

$$\begin{aligned} p_{t+1} &\geqslant 0, \\ q_{t+1} &\leqslant 0. \end{aligned} \tag{4.5}$$

Moreover, if one of the inequalities (4.4) *is strict, then the first inequality in* (4.5) *is strict.*

Proof. We recall that the first derivatives of f and u are positive and the second derivatives are negative. If we solve the first equation in (4.3) for q_{t+1} and use (4.4), we find the second inequality in (4.5). The second equation in (4.3) then leads to the first part of (4.5). If one of the inequalities in this chain is strict, we wind up with a stict inequality.

COROLLARY. *The matrix L defined by* (4.2) *is nonsingular.*

Proof. Lemma 4.1 applied to $(\{p_t\}, \{q_t\})$ and to $(\{-p_t\}, \{-q_t\})$, and the last two equations in (4.3) immediately show that when $r=s=0$, the only solution of (4.3) is $p=q=0$.

In view of Theorem 2.1, we can prove Theorem 4.1 by showing that if $T>2k$, then for any verification function G with k-dimensional range and one point w_0 at which (b.ii) of Definition 1.1 is satisfied one can find $\zeta \in \mathfrak{U}$ and $\eta \in \mathfrak{F}$ such that (2.2) and (2.3) hold but (2.5) is violated. Suppose, then, that we have such a G and such a point $w_0 = (u_0, f_0, m_0)$. We choose some $\zeta \in \hat{\mathfrak{U}}$ and some $\eta \in \hat{\mathfrak{F}}$, and we confine our attention to the two-dimensional subset $u = u_0 + \sigma\zeta$, $f = f_0 + \rho\eta$ of $\hat{\mathfrak{U}} \times \hat{\mathfrak{F}}$.

We place this (u, f) in the system (4.1). Because the Jacobian L is nonsingular, we can solve the resulting system for (x_t, c_t) as twice continuously differentiable functions of σ and ρ in some neighborhood of the origin. In order to compute the derivatives of these functions, we differentiate the system (4.1) with respect to σ and set $\sigma = \rho = 0$. We obtain the linear systems[8]

$$L\begin{pmatrix} x_f(u_0, f_0; \eta) \\ c_f(u_0, f_0; \eta) \end{pmatrix} = \begin{pmatrix} -\delta u'(c_{t+1})\,\eta'(x_t)]_{t=0,\dots,T-1} \\ \eta(x_t)]_{t=0,\dots,T-1} \\ 0 \\ 0 \end{pmatrix}. \tag{4.6}$$

and

$$L\begin{pmatrix} x_u(u_0, f_0; \zeta) \\ c_u(u_0, f_0; \zeta) \end{pmatrix} = \begin{pmatrix} \zeta'(c_t) - \delta f'(x_t)\,\zeta'(c_{t+1})]_{t=0,\dots,T-1} \\ 0]_{t=0,\dots,T-1} \\ 0 \\ 0 \end{pmatrix}. \tag{4.7}$$

If $f_0(x^*) = y_0$, we choose an $\eta \in \hat{\mathfrak{F}}$ with $\eta(x^*) \neq 0$ and $\eta'(x^*) = 0$, and replace f_0 by $f_0 + \rho\eta$ with η so small that the new point $(u_0, f_0 + \rho\eta, m_0)$ is in $\mathfrak{U} \times \mathfrak{F} \times \mathfrak{M}$ and G_w is still nondegenerate there. Thus we shall assume without loss of generality that $f_0(x^*) \neq y_0$.

Then Theorem 3.1 shows that either the sequence $\{x_t\}$ or the sequence $\{c_t\}$ is strictly monotone, while the other sequence has at most one local maximum or minimum. Suppose first that $\{x_t\}$ is monotone so that no two of its members coincide. There may, however, be pairs of c_t which are equal. If so, choose the sequence $\{c_{t,f}(u_0, f_0; \eta)\}$ in such a way that it is equal to one at one member of each pair (t, t') where $c_t = c_{t'}$ and zero for all other t. Set $x_{t,f} = 0$ for all t, and use (4.6) to find corresponding values of $\eta(x_t)$ and $\eta'(x_t)$. Since $\hat{\mathfrak{F}}$ contains all infinitely differentiable functions which vanish at 0, one can choose an η in this space with these prescribed values. In the same way as we made $f_0(x^*) \neq y_0$ in the preceding paragraph, we can now alter w_0 slightly to keep the conditions of Definition 1.1 and make the c_t all distinct.

If the sequence $\{c_t\}$ is monotone, we can make the x_t distinct by a similar process. In this case, we prescribe the values of $x_{t,u}(u_0, f_0; \zeta)$ in a

[8] Here x_f denotes the Gâteaux partial derivative with respect to f of the optimal vector x. The t component of x_f, which is the Gâteaux derivative of x_t, will be denoted by $x_{t,f}$.

suitable fashion, determine $c_{t,u}$ from the second set of equations in (4.7) and the next-to the last equation, and then find values of $\zeta'(x_t)$ so that the first set of equations in (4.7) is satisfied. We now choose a function ζ with these values and proceed as before. Thus we shall assume without loss of generality that *the x_t are all distinct and the c_t are all distinct.*

In view of this fact, we can prescribe any values p_t of the $x_{t,u}(u_0, f_0; \zeta)$ with $p_T = 0$, find corresponding $c_{t,u}(u_0, f_0; \zeta)$ from the second set of equations and the next-to-the-last equation in (4.7), and then use the first set of equations of (4.7) and the added condition $\zeta'(c_T) = 0$ to determine the values of $\zeta'(c_t)$ for $0 \leqslant t \leqslant T$ uniquely.

In particular, for any integer $s \in [0, 2k]$ we determine the derivatives $\zeta_s'(c_t)$ such that

$$x_{t,u}(u_0, f_0; \zeta_s) = \delta_{ts}, \tag{4.8}$$

the Kronecker delta.[9] We see, in particular, that $\zeta_s'(c_t) = 0$ for $t > s+1$. Because $\mathfrak{U}$ contains all infinitely differentiable functions which vanish at 0, we can certainly choose a function $\zeta_s \in \hat{\mathfrak{U}}$ whose derivatives take on the prescribed values at the $T+1$ distinct points c_t. For this function ζ_s the Gâteaux derivatives $x_{t,u}$ are given by (4.8).

In order to construct a function ζ which gives a contradiction with Theorem 2.1, we shall use a linear combination

$$\zeta = \sum_{j=0}^{2k} a_j \zeta_j$$

of these functions. Since we wish to apply Theorem 2.1, we impose the k conditions

$$G_u(u_0, f_0, m_0; \zeta) = 0. \tag{4.9}$$

These give at most k linearly independent linear homogeneous equations for the coefficients a_j. Hence there is a set S of at most k (and possibly 0) of the integers in $[0, 2k]$ with the property that given any values of p_t with $t \leqslant 2k$ and $t \notin S$, one can set $a_t = p_t$ for $t \notin S$ and solve for the corresponding a_t with $t \in S$. We see from (4.8) and the form of ζ that $x_{t,u}(u_0, f_0; \zeta) = a_t$. Thus we have shown that given any vector p_t, there is a linear combination ζ of the ζ_j which satisfies (4.9) and such that $x_{t,u}(u_0, f_0; \zeta) = p_t$ for $t \notin S$.

We now choose some functions $\eta_s \in \hat{\mathfrak{F}}$ with the properties that

$$\left.\begin{aligned} \eta_s(x_t) = \eta_s'(x_t) = 0, \\ \eta_s''(x_t) = \delta_{ts} \end{aligned}\right\} \quad \text{for} \quad t = 0, ..., T, \; s = 0, ..., 2k, \tag{4.10}$$

[9] That is, δ_{ts} has the value 1 when $t = s$ and is 0 otherwise.

and let η be a linear combination

$$\eta = \sum_{s=0}^{2k} \alpha_s \eta_s.$$

We see that $\eta(x_t) = \eta'(x_t) = 0$ for all $t \leqslant T$, so that (4.6) implies that

$$x_{t,f}(f_0, u_0; \eta) = 0 \qquad \text{for all } t. \tag{4.11}$$

We now impose the conditions

$$\begin{aligned} \alpha_s &= 0 \qquad \text{for} \quad s \in S, \\ G_f(f_0, u_0, m_0; \eta) &= 0. \end{aligned} \tag{4.12}$$

Because there are at most k points in S, these constitute at most $2k$ linearly independent linear homogeneous equations in the coefficients α_s. In the same way as we constructed S, we now obtain a (possibly empty) set $S' \supset S$ of at most $2k$ points in the interval $[0, 2k]$ such that for any prescribed values of the α_s for $s \leqslant 2k$ and $s \notin S'$ there is a linear combination η of the $2k$ functions η_s which satisfies (4.12). Since S' contains at most $2k$ points, there is at least one $s_0 \notin S'$ with $s_0 \leqslant 2k < T$. We choose

$$\alpha_s = \delta_{ss_0} \qquad \text{for} \quad s \notin S'$$

and construct the corresponding η. As above, we construct a linear combination ζ of the ζ_j which satisfies (4.9) and the conditions

$$x_{t,u}(u_0, f_0; \zeta) = \delta_{ts_0} \qquad \text{for} \quad t \notin S.$$

Since $\eta''(x_t) = 0$ for $t \in S$, we see that

$$\eta''(x_t)\, x_{t,u}(u_0, f_0; \zeta) = \delta_{ts_0} \tag{4.13}$$

for all $t \leqslant T$.

We now take the mixed partial derivative of (4.1) with respect to σ and ρ, and set $\sigma = \rho = 0$. In view of (4.11), we find that

$$L\begin{pmatrix} x_{uf}(u_0, f_0; \zeta, \eta) \\ c_{uf}(u_0, f_0; \zeta, \eta) \end{pmatrix} = \begin{pmatrix} -\delta u'(c_{t+1})\, \eta''(x_t)\, x_{t,u}]_{t=0,\dots,T-1} \\ 0]_{t=0,\dots,T-1} \\ 0 \\ 0 \end{pmatrix}. \tag{4.14}$$

The property (4.13) shows that the vector in the top line on the right is zero except at the point s_0, where it is negative. The last statement of Lemma 4.1 then shows that $x_{0,uf}(u_0, f_0; \zeta, \eta) = 0$ implies that

$x_{T,uf}(u_0, f_0; \eta) < 0$, contrary to the last equation in (4.14). We conclude that $x_{0,uf}(u_0, f_0; \zeta, \eta) \neq 0$, so that the equation (2.5) with $Q = x_0$ is violated, while by (4.9) and (4.12) the conditions (2.2) and (2.3) are satisfied. Thus Theorem 2.1 shows that there cannot be a decentralized mechanism with $2k < T$, and Theorem 4.1 is established.

Proof of Theorem 4.2. In order to extend the above proof to the infinite-horizon problem, we delete the last row and set $T = \infty$ in the definition (4.2) of L, so that the first two lines on the right are infinite-dimensional vectors. We shall show in the Appendix that the infinite-dimensional matrix L is invertible, and that consequently the optimal sequence pair $(\{x_t(u, f)\}, \{c_t(u, f)\})$ of the infinite-horizon problem is twice continuously differentiable. In particular, then, the goal function $x_0(u, f)$ is twice continuously differentiable. We proceed to copy the above proof of Theorem 4.1.

As before, we can introduce a small change in f_0 if necessary to make $f_0(x^*) \neq y_0$. Then by Theorem 3.2 the sequences $\{x_t\}$ and $\{c_t\}$ are strictly monotone, so that their elements are distinct.

There are now infinitely many conditions on η_s in (4.10). However, since the x_t are strictly motone, we can find an infinitely differentiable function which satisfies all these conditions by making $\eta_s = 0$ on the interval with end points x_{s+1} and x^* and satisfying the finite set of remaining conditions outside this interval, together with the condition $\eta_s(0) = 0$. Similarly, we can find an infinitely differentiable ζ_s for which (4.8) is valid by making $\zeta_s = 0$ on the interval with end points c_{s+2} and $F(x^*) - x^*$ and satisfying finitely many conditions outside this interval.

Once this is done, the proof goes through exactly as before, and Theorem 4.2 is established.

The corollary of Theorem 4.2 follows immediately from the theorem and the earlier remark that if there is a decentralized evolutionary process which realizes the optimal sequence pair, then the decentralized mechanism $(\mathfrak{M}, G_0, H_0)$ realizes x_0.

We note that we have used the fact that $\mathfrak{U}$ and $\mathfrak{F}$ contain all infinitely differentiable functions which vanish at 0 only to make sure that we can find ζ_s and η_s for which ζ_s', η_s, η_s', and η_s'' take on prescribed values. In the *finite-horizon* problem there are only finitely many such conditions. Therefore, it is sufficient to assume that $\mathfrak{U}$ and $\mathfrak{F}$ contain all the polynomials which vanish at 0, or, more generally, any Chebyshev system.

In the *infinite-horizon* problem the requirement that $\zeta_s'(c_t) = 0$ for $t > s + 1$ means that all derivatives of ζ_s vanish at the limit point $f(x^*) - x^*$ of the c_t. If the space $\mathfrak{U}$ contains only analytic functions, the only function which meets these requirements and vanishes at 0 is $\zeta_s \equiv 0$, for which the equation (4.8) is violated at $t = s$, and the proof of Theorem 4.2 fails. A similar argument using the requirements (4.10) for the function η_s shows that the proof also fails when $\hat{\mathfrak{F}}$ contains only analytic functions.

We do not know whether the conclusion of Theorem 4.2 is still valid when one of the spaces contains only analytic functions.

However, we note that if one of the spaces, say $\hat{\mathfrak{U}}$, has a finite dimension l, then $x_0(u, f)$ can be realized by the following decentralized parameter transfer mechanism with $k = l + 1$. We suppose that u has the form

$$u = \sum_{j=1}^{l} \alpha_j u_j$$

where the linearly independent functions $u_j(c)$ are commonly known while the coefficients α_j are known only to the consumer. The message space is R^{l+1}, the verification function is

$$G_k(u, f, m) = \begin{cases} \alpha_k - m_k & \text{for} \quad k \leqslant l \\ x_0\left(\sum_{j=1}^{l} m_j u_j, f\right) - m_{l+1} & \text{for} \quad k = l+1, \end{cases}$$

and the outcome function is m_{l+1}.

We also note that while the functional $x_0(u, f)$ for the infinite horizon problem cannot be realized by a decentralized mechanism, it is certainly realized by the nondecentralized mechanism $(R^1, x_0(u, f) - m, m)$. In fact, the nondecentralized evolutionary mechanism with $\mathfrak{M}$ the set of sequence pairs $(\{m_t\}, \{n_t\})$, $G_t(u, f, m) = (x_t(u, f) - m_t, c_t(u, f) - n_t)$, and H the identity map clearly realizes $Q = \{x_t(u, f), c_t(u, f)\}$.

5. The Dimension of the Message Space

We observe that the dimension of the message space is not involved in the above results. Because much of the previous literature in decentralization is focussed on this dimension, we shall show that a solvability hypothesis can force the dimension of the message space to be as large as that of the range of the verification function in the mechanism. In particular, such a condition and Theorem 4.1 give a lower bound for the dimension of the message space in the finite-horizon problem.

We begin with a condition which works when the space of economies $\mathfrak{E}$ is finite-dimensional.

Definition 5.1. Let $G: \mathfrak{E} \times \mathfrak{M} \rightarrow R^k$. The equation $G(e, m) = 0$ is said to be *globally solvable* in the open subset $\mathcal{O}$ of $\mathfrak{E}$ if for each e in $\mathcal{O}$ there is an m in $\mathfrak{M}$ such that $G(e, m) = 0$.

Proposition 5.1. *Let the space $\hat{\mathfrak{E}}$ have a finite dimension p. Suppose that G is globally solvable in an open subset $\mathcal{O}$ of $\mathfrak{E}$. Also suppose that G is* *nondegenerate at each point* (e, m) *where* $e \in \mathcal{O}$ *and* $G=0$. *Then the dimension of the message space* $\mathfrak{M}$ *is at least the dimension* k *of the range of* G.

Proof. Assume that the dimension l of $\mathfrak{M}$ is less than k. Choose a compact subset $\mathscr{K}$ of $\mathcal{O}$ with nonempty interior. The implicit function theorem states that each point of $\mathscr{K} \times \mathfrak{M}$ where $G=0$ has a neighborhood in which the $p+l$ coordinates of these points where $G=0$ can be written as smooth functions of $p+l-k$ parameters. The projection of this part of the null space on $\mathfrak{E}$ is obtained by keeping only the e-part of this representation. Thus it is a piece of a manifold of dimension $p+l-k$, which is less than p. Therefore the Lebesgue measure of the projection of each such coordinate patch is zero.

Since for any integer n the set $\mathscr{K} \times \{m \in \mathfrak{M} : n \leqslant \|m\| \leqslant n+1\}$ is compact, its intersection with the null set of G can be covered with finitely many such coordinate patches. Therefore the intersection of the null set of G with the set $\mathscr{K} \times \mathfrak{M}$ is a denumerable union of the coordinate patches. Its projection on $\mathfrak{E}$ is thus a denumerable union of sets of measure zero, so that this projection has measure zero. Because K has nonempty interior, it has positive Lebesgue measure. Thus there are points e of K for which there is no m such that $G(e, m)=0$.

We have shown that when $l<k$, G is not globally solvable in $\mathcal{O}$, which proves the proposition.

When the space $\hat{\mathfrak{E}}$ is infinite-dimensional, we need a stronger condition to prove this result. (We do not know whether the conclusion of Proposition 5.1 is true when $\hat{\mathfrak{E}}$ is infinite-dimensional and G is globally but not locally solvable.)

DEFINITION 5.2. Let $G: \mathfrak{E} \times \mathfrak{M} \to R^k$. The equation $G(e, m)=0$ is said to be *locally solvable* for e at (e_0, m_0) if for any neighborhood $\mathcal{N}_1$ of m_0 there is a neighborhood $\mathcal{N}_2$ of e_0 such that for any e in $\mathcal{N}_2$ there is an m in $\mathcal{N}_1$ which makes $G(e, m)=0$.

PROPOSITION 5.2. *Let* $(\mathfrak{M}, G, H)$ *be a mechanism as defined in Section* 1. *Suppose that there is at least one point* (e_0, m_0) *at which* G *is nondegenerate and locally solvable for* e.

Then the dimension of the message space $\mathfrak{M}$ *is at least as large as the dimension* k *of the range of* G.

Proof. Let the dimension l of $\mathfrak{M}$ be $l<k$. Let $r \leqslant l$ be the dimension of the range of $G_m(u_0, f_0, m_0; \mu)$. We replace the components of G by k linearly independent linear combinations in such a way that the projection onto the first r components of G_m is already r-dimensional and that $[G_m(u_0, f_0, m_0; \mu)]_i = 0$ for all $\mu \in \hat{\mathfrak{M}}$ when $i>r$.

Because of these special properties and because G is nondegenerate, we can choose $v_1, \ldots, v_k$ which satisfy (2.1) and which have the additional property that if

$$v_j = (e_j, m_j),$$

then $e_j = 0$ for $j \leqslant r$. Moreover, it is easily verified that $e_{r+1}, \ldots, e_k$ are linearly independent. If $r < l$, we choose $\mu_1, \ldots, \mu_{l-r}$ so that $m_1, \ldots, m_r, \mu_1, \ldots, \mu_{l-r}$ form a basis for $\mathfrak{M}$.

Then to any set of coefficients $a_{r+1}, \ldots, a_k$ and any element $m \in \mathfrak{M}$ there correspond coefficients $a_1, \ldots, a_r, \rho_1, \ldots, \rho_{l-r}$ such that

$$\left(e_0 + \sum_{j=r+1}^{k} a_j e_j, m\right) = (e_0, m_0) + \sum_{j=1}^{k} a_j v_j + \left(0, \sum_{\nu=1}^{l-r} \rho_\nu \mu_\nu\right).$$

Thus the question of finding an m to make $G(e_0 + \sum a_j e_j, m)) = 0$ is reduced to solving the equation

$$G\left((e_0, m_0) + \sum_{j=1}^{k} a_j v_j + \left(0, \sum_{\nu=1}^{l-r} \rho_\nu \mu_\nu\right)\right) = 0.$$

As we saw in the proof of Theorem 2.1, this equation can be solved for the k coefficients a_j as twice continuously differentiable functions α_j of the $l-r$ variables ρ_ν, when the latter are sufficiently small. Thus to an element of the form $e_0 + \sum a_j e_j$ with the a_j sufficiently small there corresponds an m which makes $G = 0$ if and only if the $k-r$-vector $\{a_{r+1}, \ldots, a_k\}$ lies on an $(l-r)$-dimensional smooth surface.[10] Since $l < k$, it is easy to find a vector $\{a_j\}$ arbitrarily close to the origin for which this is not the case. Thus $l < k$ implies that $G = 0$ is not locally solvable at (e_0, m_0), which proves the proposition.

We remark that Proposition 5.2 remains valid when the dimension of the verification function G is infinite, provided the smoothness of G is interpreted as the smoothness of each of its components, and the nondegeneracy is taken to mean that for every positive integer k there are k components of G such that the mapping which consists of only these components is nondegenerate. Since the equation $G = 0$ implies that these k components of G are zero, Proposition 5.2 then shows that $l \geqslant k$ for every positive integer k, which means that the message space $\mathfrak{M}$ cannot be finite-dimensional.

APPENDIX: The Differentiability of the Optimal Sequence

In this Appendix we shall prove the following result, which is needed for the proof of Theorem 4.2.

[10] If $r = l$, this "surface" is the single point $a_{r+1} = \cdots = a_k = 0$.

THEOREM A.1. *The optimal sequence pair* $(\{x_t(u, f)\}, \{c_t(u, f)\})$ *of the infinite-horizon problem solved in Section* 3 *is twice continuously differentiable in the sense defined in Section* 2.

Proof. We recall that the optimal sequence is characterized as the unique positive solution of the conditions (3.7) and (3.3). That is,

$$P(u, f, \{x_t\}, \{c_t\}) \equiv \begin{pmatrix} \{-u'(c_t) + \delta f'(x_t)\, u'(c_{t+1})\} \\ \{x_{t+1} + c_{t+1} - f(x_t)\} \\ x_0 + c_0 - y_0 \end{pmatrix} = 0. \tag{A.1}$$

We define the Banach space[11]

$$b_0 = \{\{p_t\}, \{q_t\} : \lim_{t \to \infty} \delta^t(|p_t| + |q_t|) = 0\}$$

with the norm

$$\|\{p_t\}, \{q_t\}\| = \max_{t \geq 0} \delta^t(|p_t| + |q_t|).$$

It is easily seen that for fixed twice continuously differentiable functions u and f the transformation P takes a pair of sequences $(\{x_t\}, \{c_t\})$ with $(\{x_t - x^*(f)\}, \{c_t - f(x^*) + x^*\}) \in b_0$ into a pair of sequences in b_0 followed by a number. That is, we can think of P as a nonlinear operator on b_0.

We now write the linearization

$$L\begin{pmatrix} \{p_t\} \\ \{q_t\} \end{pmatrix} = \begin{pmatrix} \{-u_0''(c_t^0)\, q_t + \delta f_0'(x_t^0)\, u_0''(c_{t+1}^0)\, q_{t+1} + \delta f_0''(x_t^0)\, u_0'(c_{t+1}^0)\, p_t\} \\ \{p_{t+1} + q_{t+1} - f_0'(x_t^0)\, p_t\} \\ p_0 + q_0 \end{pmatrix} \tag{A.2}$$

of $P(u_0, f_0, \{x_t\}, \{c_t\})$ about the point $(u_0, f_0, \{x_t^0\}, \{c_t^0\})$, where we have introduced the abbreviations

$$x_t^0 = x_t(u_0, f_0), \qquad c_t^0 = c_t(u_0, f_0).$$

The linear transformation L again takes b_0 into b_0. We wish to show that it is a one-to-one transformation with bounded inverse defined on all of b_0.

[11] A Banach space is a normed linear vector space with the additional completeness property that every sequence which satisfies the Cauchy criterion has a limit in the space.

Lemma A.1. *The equation*

$$L\begin{pmatrix}\{p_t\}\\ \{q_t\}\end{pmatrix}=\begin{pmatrix}\{r_t\}\\ \{s_t\}\\ 0\end{pmatrix} \tag{A.3}$$

has a unique solution $(\{p_t\}, \{q_t\})\in b_0$ *for every* $(\{r_t\}, \{s_t\})\in b_0$, *and the norm of the solution is bounded by a constant times the norm of* $(\{r_t\}, \{s_t\})$.

Proof. We begin by defining the sequence

$$\gamma_t=\prod_{s=0}^{t-1} f_0'(x_t^0) \tag{A.4}$$

and the new variables

$$\mu_t=p_t/\gamma_t. \tag{A.5}$$

It is easily seen that $x_t^0-x^*$ approaches zero at an exponential rate as t goes to infinity, where $\delta f_0'(x^*)=1$. It follows that the sequence $\delta^t\gamma_t$ converges to a positive number as $t\to\infty$. Therefore there are positive constants A and B such that

$$A\delta^{-t}\leqslant\gamma_t\leqslant B\delta^{-t}. \tag{A.6}$$

In terms of the new variables μ_t the equation (A.3) becomes

$$-u_0''(c_t^0)\,q_t+\delta f_0'(x_t^0)\,u_0''(c_{t+1}^0)\,q_{t+1}+\delta f_0''(x_t^0)\,u_0'(c_{t+1}^0)\,\gamma_t\mu_t=r_t,$$
$$\gamma_{t+1}[\mu_{t+1}-\mu_t]+q_{t+1}=s_t,$$
$$p_0+q_0=0.$$

Because $\delta^t p_t$ goes to zero as $t\to\infty$, we see from (A.5) and (A.6) that μ_t goes to zero. Thus if μ_t has any positive values, it must take on a positive maximum. If such a maximum occurs at $t\geqslant 1$, then $\mu_t-\mu_{t-1}\geqslant 0$ so that by the second set of equations $q_t\leqslant s_t$, and for a similar reason $q_{t+1}\geqslant s_{t+1}$. We substitute these inequalities in the first set of equations and recall that the first derivatives of u_0 and f_0 are positive and their second derivatives are negative to conclude that the positive maximum must satisfy the inequality

$$p_t\leqslant\frac{r_t+u_0''(c_t^0)\,s_t-\delta f_0'(x_t^0)\,u_0''(c_{t+1}^0)\,s_{t+1}}{\delta f_0''(x_t^0)\,u_0'(c_{t+1}^0)\,\gamma_t}.$$

If the positive maximum occurs at $t=0$, we again have $q_{t+1}\geqslant 0$, and the last equation gives the equality $q_0 = -p_0$. By substituting in the $t=0$ equation of the first set of (A.3) we find that

$$p_0 \leqslant \frac{r_0 - \delta f_0'(x_0^0)\, u_0''(c_1^0)\, s_0}{u_0''(c_0^0) + \delta f_0''(x_0)\, u_0'(c_1^0)}.$$

The same reasoning leads to the same bounds with the inequalities reversed for a negative minimum. Because of the factor γ_t on the right and the inequalities (A.6), we see that there is a constant D such that

$$\max_t |\mu_t| \leqslant D\, \|(\{r_t\}, \{s_t\})\|.$$

It then follows from the transformation (A.5) and the inequalities (A.6) that

$$\delta^t\, |p_t| \leqslant BD\, \|(\{r_t\}, \{s_t\})\|.$$

We use the second set of equations in (A.3) and the last equation to solve for each q_t as a linear combination with bounded coefficients of p_t, p_{t-1}, and s_{t-1}. This together with the preceding inequality shows that there is a constant C such that

$$\|\{p_t\}, \{q_t\}\| \leqslant C\, \|\{r_t\}, \{s_t\}\|.$$

This inequality immediately shows that there is at most one solution of (A.3), that L has a bounded inverse on its range, and that this range is a closed linear subspace of b_0. Thus if the range of L is not the whole space, there must be a bounded linear functional which vanishes on the range. It is easily seen that any linear bounded linear functional on b_0 can be written in the form

$$l[(\{r_t\}, \{s_t\})] = \sum_{t=0}^{\infty} (m_t r_t + n_t s_t),$$

where $(\{m_t\}, \{n_t\})$ is some sequence pair with the property

$$\sum_{t=0}^{\infty} \delta^{-t}(|m_t| + |n_t|) < \infty.$$

If this linear functional vanishes on the range of L, then for any choice of $(\{p_t\}, \{q_t\}) \in b_0$ the result of applying it to $L(\{p_t\}, \{q_t\})$ must be zero. If, in particular, we successively let $p_t = \delta_{ts}$, $q_t = 0$ for $s = 1, 2, \ldots$, then let $p_t = 0$, $q_t = \delta_{ts}$, and finally set $p_t = \delta_{t0}$, $q_t = -\delta_{t0}$, we obtain the system of equations

$$\delta f_0''(x_s^0)\, u_0'(c_{s+1}^0)\, m_s + n_{s-1} - f_0'(x_s^0)\, n_s = 0,$$
$$-u_0''(c_s^0)\, m_s + \delta f_0'(x_{s-1}^0)\, u_0''(c_s^0)\, m_{s-1} + n_{s-1} = 0,$$
$$u_0''(c_0^0)\, m_0 + f_0'(x_0^0) n_0 = 0.$$

The reasoning used above shows that the sequence n_s/γ_{s+1} vanishes at infinity and cannot attain a positive maximum or a negative minimum. Therefore all the n_s are zero, and it follows from the first set of equations and the last equation that all the m_s are also zero.

Thus there is no nontrivial bounded linear functional which vanishes on the range of L. We conclude that this range is the whole space b_0, and Lemma A.1 is established.

We return to the proof of Theorem A.1. Lemma A.1 shows that the operator L has a bounded inverse operator L^{-1}. We now let $u = u_0 + \sum \sigma_j \zeta_j$ and $f = f_0 + \sum \rho_k \eta_k$. As in the standard proof of the implicit function theorem, we use L^{-1} to rewrite the equation (A.1) in the form

$$\begin{pmatrix} x_t - x_t^0 \\ c_t - c_t^0 \end{pmatrix} = -L^{-1}[P(u_0, f_0; \{x_t\}, \{c_t\})$$

$$- L(\{x_t - x_t^0\}, \{c_t - c_t^0\}) - P(u_0, f_0; \{x_t^0\}, \{c_t^0\})]$$

$$-L^{-1}\begin{pmatrix} \sum \sigma_j \zeta_j(c_t^0) + \delta f_0'(x_t^0) \sum \sigma_j \zeta_j(c_{t+1}^0) \\ + \delta \sum \rho_k \eta_k(x_t^0)[u_0'(c_{t+1}^0) + \sum \sigma_j \zeta_j(c_{t+1}^0)] \\ -\sum \rho_k \eta_k(x_t^0) \\ 0 \end{pmatrix}, \qquad (A.7)$$

The norm of the first term on the right is clearly of order $\|(\{x_t - x_t^0\}, \{c_t - c_t^0\})\|^2$, while that of the other term is of order $|\sigma| + |\rho|$, the sum of the Euclidean norms. It follows from (A.7) that $\|(\{x_t - x_t^0\}, \{c_t - c_t^0\})\|$ is of order $|\sigma| + |\rho|$ when this quantity is small. Therefore the left-hand side is equal to the second term on the right, which is a polynomial in σ and ρ, plus a term of order $(|\sigma| + |\rho|)^2$.

If we substitute this fact back into the equation (A.7), we can write the right-hand side as a quadratic polynomial in σ and ρ plus a term of order $(|\sigma| + |\rho|)^3$. Thus we see that the function $(\{x_t(u_0 + \sum \sigma_j \zeta_j, f_0 + \sum \rho_k \eta_k)\}, \{c_t(u_0 + \sum \sigma_j \zeta_j, f_0 + \sum \rho_k \eta_k)\})$ is twice continuously differentiable at $\rho = \sigma = 0$. Since we can replace (u_0, f_0) by any nearby point of the form $(u_0 + \sum \sigma_j \zeta_j, f_0 + \sum \rho_k \eta_k)$ in the above argument, we find that the function $(\{x_t(u_0 + \sum \sigma_j \zeta_j, f_0 + \sum \rho_k \eta_k)\}, \{c_t(u_0 + \sum \sigma_j \zeta_j, f_0 + \sum \rho_k \eta_k)\})$ is a twice continuously differentiable function of σ and ρ. According to our definition, then, the function $(\{x_t(u, f)\}, \{c_t(u, f)\})$ is twice continuously differentiable.

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