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The analytic pricing of asymmetric defaultable swaps

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Abstract

Swaps where both parties are exposed to credit risk still lack convincing pricing mechanisms. This article presents a reduced-form model where the event of default is related to structural characteristics of each party. The cash flows submitted to credit risk are identified before the swap is priced. Analytical pricing formulas for interest rate and currency swaps are computed using a Gaussian model for risky bonds. Currency swaps exhibit additional correlation risk. The benefits from netting depend on the balance between exposures and market conditions in valuation. We show that sources of credit risk asymmetries are also likely to impact on credit spreads. © 2001 Elsevier Science B.V. All rights reserved.

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The valuation of undefaultable interest-rate swaps and swaptions under a specified term structure of interest rates is a classical exercise. However, the risk that either party defaults on its flow of payments should entail the addition of a

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spread to the coupon charged as a reward for the corresponding expected loss. Some authors have simplified the problem by considering the interest rate swap as a simple exchange of loans, as did Sundaresan (1991). Another simplification consists of taking into account the presence of one risky counterparty only, as in Baz (1995), Li (1998) and Cooper and Mello (1991).

These approaches overlook the presence of bilateral default risk. Whether its value is positive or negative for each party (i.e., corresponding to an asset or a liability), does not add much complexity in itself. However, the usual settlement rule in case of default creates asymmetry: for the defaulting party, the swap is terminated through payment in full of its market value if it is positive, but only through fractional payment if it is negative.

In order to deal with this issue, two solutions have been proposed. First, as in Duffie and Singleton (1997) and Nielsen and Ronn (1996), it is possible to deliberately restrict investigations to cases involving perfectly symmetric credit exposures. Under these conditions, the empirical use of market-wide data is possible. Second, allowing asymmetric default risk often leads to numerical approximations, as in Duffie and Huang (1996).

This paper uses the approach of default risk developed by Duffie and Singleton (1999) for the valuation of credit risky securities, as Duffie and Singleton (1999) do. Contrarily to these authors, who seek to simplify the swap valuation formula by directly modeling its market value, the model proposed here focuses on the interpretation of default risk for each party to endogenously provide the swap value. This more rigorous approach leads to analytic pricing equations. Thanks to this new valuation principle, the model yields pricing formulas for defaultable interest rate swaps and currency swaps with reasonable assumptions.

Obtaining closed-form solutions for swaps with bilateral credit risk allows us to isolate the various components of default spreads and to analyze the possible sources of asymmetry in credit qualities. In particular, the paper generalizes the results of Duffie and Huang (1996) about the use of a netting master agreement to reduce credit exposures by performing a systematic analysis of the impact of netting for any initial swap value with any additional notional.

The paper is organized as follows. Section 1 presents the general valuation framework. In Section 2, it is applied to interest rate and currency swaps, including the impact of netting. Section 3 concludes the article.

1. Valuation of swaps subject to bilateral credit risk

1.1. Principles

The starting point of the analysis is to identify how a defaultable swap differs from its default-free version. Obviously, any party can only be responsible for default on her own payments, as for any other obligation. A credit risk model must ultimately recognize this fact. But the swap contract involves bilateral risks which interact with each other, and the settlement rule in case of default may be asymmetric.

The swap contract is priced in two stages. First, the possible future occurrence of default of each party is explicitly integrated before the price is computed. Second, the pre-default contract value is obtained from the comparison of these defaultable components with the immunization of the one which is not subject to immediate default risk. This procedure substantially differs from the Duffie and Huang (1996) approach who directly model the pricing consequence of immediate default risk: they implicitly assume that default is tied to the contract value instead of underlying net payments.

The approach proposed here takes into account the contract settlement rule in case of default, which is a key factor of pricing differences. The most commonly used one is called the "full two-way payment rule" and obeys the International Swap Dealers Association (1992) Master Agreement: under this rule, the defaulting party receives a closed-out payment in full if the predefault value of the swap is positive for her, and pays a fractional amount otherwise. Duffie and Huang (1996) integrate this directly in the value by considering that default risk depends on its sign. However, each party also exhibits credit risk as she may not service the swap in the future. Only the party whose swap value is positive bears immediate default risk, but the other party is at risk too: if the swap remains alive, the sign of the price may reverse and she would then suffer from immediate default. This potential risk is introduced in the valuation and reflected in swap prices.

The pricing procedure also takes easily into account the "limited two-way payment rule", stipulating that the defaulting party receives a fractional termination payment if the swap value is positive for her at default time. Thus, the consequence of default is not tied to the sign of the swap price, and the probability of defaulting on one's obligations is also contingent on the other party's credit conditions.

1.2. Valuation model

This subsection proposes a two-stage valuation formula, with an emphasis of how it economically and technically differs from the results proposed by Duffie and Huang (1996).

The core difference is the way default risk is taken into account. This paper's approach focuses on assessing credit qualities prior to valuing the swap. First, one must identify the flows subject to credit risk and determine their present value. Secondly, the compensation for the undefaultable fraction of future payments is computed in order to derive the swap pricing equation.

1.2.1. Identification and valuation of cash flows subject to credit risk

The payments that are affected by credit risk considerations must logically be the ones that can actually occur. Therefore, the approach of identifying a swap to an exchange of loans is not sustainable in the case of interest-rate swaps, since principals are never exchanged. Besides, the settlement rule in case of early termination has a clear influence on who may default on which payments. Under the full two-way payment rule, a party can only default on her own obligations. Conversely, under the limited two-way payment rule, the consequence of the event of default is independent of the sign of the flow.

Once the relevant flows are identified, default risk has to be explicitly integrated in order to reach a present value for expected dividends. The proper method is to compute the expectation of all discounted payments under a risk-adjusted probability measure.

To consider default risk in valuation, two competing approaches can be distinguished. The "structural form approach" rests on the assumption that default is triggered when the value process of the firm gets lower than or equal to a specific value. It provides an economic interpretation of the event of default, but fails to explain the nature and magnitude of the associated loss. More recently, the "reduced-form approach" has been exploited by Duffie and Singleton (1999), Jarrow and Turnbull (1995) and Jarrow et al. (1997). It has a more descriptive point of view, and focuses on the spread added to the discount factor used in risk-neutral valuation. This spread turns out to be a function of the probability of default and of the loss experienced when default occurs.

Three different formulations of the loss function currently exist in the literature: the defaulted pay-off is either a fraction of par (Madan and Unal, 1998), of a risk-free bond with the same notional and coupons (Jarrow and Turnbull, 1995) or of the market value of the security just prior to default (Duffie and Singleton, 1999). The latter is especially suited to the case of swaps: in case of default by a party, the swap is unfolded but its price reflects credit conditions prevailing if it were renegotiated among similar parties, with a possibly reduced notional. This actually corresponds to the line followed in Duffie and Huang (1996).

The random variable representing the time elapsed before the occurrence of default is assumed to follow an exponential distribution with a stochastic time-dependent parameter whose cumulative density function is

$$F(t) = \Pr_Q[0 < \tau \leqslant t] = \int_0^t h(\mathbf{X}(\tau), \tau)\, \mathrm{e}^{-h(\mathbf{X}(\tau),\tau)\tau}\, \mathrm{d}\tau, \tag{1}$$

where τ is the timing of default, $\mathbf{X}(\tau)$ the vector of stochastic state variables determining default risk, Q a risk-adjusted probability measure and $h(\mathbf{X}(\tau), \tau)$ is the associated risk-neutral arrival rate of default or hazard rate.

When a party defaults, the current market value of the claim considered is instantaneously reduced by a fraction $l(\mathbf{X}(t), t)$, which represents magnitude risk.

In a risk-neutral world, the discount rate adjusted for default risk $R(\mathbf{X}(t), t)$ of the promised pay-off should leave the same present value as does the expected pay-off discounted at the riskless rate. This characterization corresponds to the framework developed by Duffie and Singleton (1999), who show that

$$R(\mathbf{X}(t), t) = r(t) + h(\mathbf{X}(t), t)l(\mathbf{X}(t), t) = r(t) + s(\mathbf{X}(t), t). \tag{2}$$

The discount factor for a risky cash-flow is just the spot rate plus a default spread, which is itself the product of the arrival rate and the loss rate.

This approach can be used to differentiate the settlement rules in case of default. If the arrival and loss rates for the parties A and B of a swap are $h_A(\mathbf{X}_A(t), t)$, $l_A(\mathbf{X}_A(t), t)$, $h_B(\mathbf{X}_B(t), t)$ and $l_B(\mathbf{X}_B(t), t)$, respectively, where $\mathbf{X}_A(t)$ and $\mathbf{X}_B(t)$ can be different or identical, and considering as negligible the probability of simultaneous default, the spread associated to payments by A is:

$$s_A(\mathbf{X}(t), t) = \begin{cases} s_A(\mathbf{X}_A(t), t) = h_A(\mathbf{X}_A(t), t)l_A(\mathbf{X}_A(t), t) \\ \quad \text{for full two-way,} \\ s_A(\mathbf{X}_A(t), \mathbf{X}_B(t), t) = h_A(\mathbf{X}_A(t), t)l_A(\mathbf{X}_A(t), t) \\ \qquad\qquad\qquad\qquad + h_B(\mathbf{X}_B(t), t)l_B(\mathbf{X}_B(t), t) \\ \quad \text{for limited two-way.} \end{cases} \tag{3}$$

Under the full two-way payment rule, the spread for a party is only influenced by the state variables determining her own credit risk, while in the limited two-way payment rule, the spread is affected by both parties' default risk. For the latter settlement rule, the sign of the flow actually does not matter, because then: $s_A(\mathbf{X}_A(t), \mathbf{X}_B(t), t) = s_B(\mathbf{X}_A(t), \mathbf{X}_B(t), t)$.

Introducing credit risk at the level of defaultable flows yields the following standard expression for the present value equation:

$$V_j(t) = E_Q\left[\int_t^T \mathrm{e}^{-\int_t^s R_j(\mathbf{X}(u), u)\,\mathrm{d}u}\, \mathrm{d}D_j(s) \,\middle|\, \mathscr{F}_t \right], \tag{4}$$

where D_j is the nonnegative process of all promised payments which are defaultable to j, $j = A, B$, and $R_j(\mathbf{X}(t), t)$ is the default-adjusted discount rate for these flows at time t.

1.2.2. Swap pricing equation

The original feature involved in the second stage is the use of a compensation principle. Since the contract is priced after credit risk has been taken into account, part of it cancels out from comparing exposures and compensating them.

It is important to note that the swap value $V(t)$ is a market value that results from discounting cash flows subject to default risk, but it is still defaultable by both parties. As cash flows have already been identified, it does not make sense anymore to expose them fully to credit risk because they can be partly hedged by netting the flows. Only the unnetted part of the discounted cash flows should be ultimately vulnerable to default risk.

Denote $V_A(t)$ and $V_B(t)$ the present value of all payments to parties A and B of a swap. Credit risk has already been accounted for in each expression. The lower of these two value is actually compensated by a fraction of the higher, because valuation takes place before the possible occurrence of default is considered. This netted part is

$$I(t) = \min[V_A(t); V_B(t)] \tag{5}$$

and the vulnerable part, which remains subject to credit risk at t is equal to

$$\max[V_A(t); V_B(t)] - I(t) = |V_A(t) - V_B(t)|. \tag{6}$$

The swap value for party A is the vulnerable part if $V_A \geqslant V_B$ and the opposite of the vulnerable part otherwise. It gives her exposure to B's immediate default. The swap pricing formula simplifies to

$$V(t) = V_A(t) - V_B(t). \tag{7}$$

$V_j(t)$ is continuous when $D_j(t)$ is continuous, and so is $V(t)$. It is crucial to stress that $V(t)$ also encloses a latent component of default risk: through the netted part, default risks of both parties have been introduced in the swap price. Even when $V(t) < 0$, A bears some credit risk under the limited two-way payment rule: the higher B's credit risk, the lower A's compensated part, and the more negative the vulnerable part. In terms of replicating securities, this swap is equivalent to a portfolio of a long position on a defaultable cap and a short position on a defaultable floor (or conversely) with the same party, where the lower of the values of these positions is compensated through netting with a fraction of the higher one.

As $R_A(\mathbf{X}(t), t) = R_B(\mathbf{X}(t), t) = R(\mathbf{X}(t), t)$ in the limited two-way payment rule, and since $\mathrm{d}D_A(t) + \mathrm{d}D_B(t) = \mathrm{d}D(t)\ \forall t$, $V(t)$ can immediately be simplified to

$$V(t) = E_Q\left[\int_t^T \mathrm{e}^{-\int_t^s R(\mathbf{X}(u),u)\,\mathrm{d}u}\,\mathrm{d}D(s)\,\middle|\,\mathscr{F}_t\right] \quad \text{under limited two-way payment rule,} \tag{8}$$

which can easily be solved by the classical martingale approach for swap pricing under a multi-factor interest rate model. Due to its limited interest and use in practice, this type of settlement need not be examined further in the paper.

This reduced-form modeling of the swap is easy to compare with Proposition 1 in Duffie and Huang (1996). It presents the price as the unique solution of a recursive integral equation:

$$V^{\mathrm{DH}}(t) = E_Q\left[\int_t^T \mathrm{e}^{-\int_t^s R^{\mathrm{DH}}(\mathbf{X}(u),u)\,\mathrm{d}u}\, \mathrm{d}D(s) \middle| \mathscr{F}_t \right], \tag{9}$$

where $R^{\mathrm{DH}}(\mathbf{X}(t),t) = r(t) + s_A(\mathbf{X}(t),t)1_{\{V^{\mathrm{DH}}(t)<0\}} + s_B(\mathbf{X}(t),t)1_{\{V^{\mathrm{DH}}(t)\geqslant 0\}}$. Unfortunately, the modeling choice of introducing the default asymmetry within the risky rate, and taking the net value process for the risk-neutral expectation precludes analytical results. The main outputs are driven for the general behavior of swaps (presence of netting, change in the information structure, marginal impact of credit quality).

The proposed treatment of default risk differs sensibly from the interpretation given in Duffie and Huang (1996). In the full two-way payment rule, the swap value can be written

$$V(t) = E_Q\left[\int_t^T \phi_{A,B}(s)\, \mathrm{e}^{-\int_t^s R(\mathbf{X}(u),u)\,\mathrm{d}u}\, \mathrm{d}D(s) \middle| \mathscr{F}_t \right], \tag{10}$$

where

$$D(t) = D_A(t)1_{\{\phi_{A,B}(t)\geqslant 0\}} + D_B(t)1_{\{\phi_{A,B}(t)<0\}},$$

$$R(t) = r(t) + s_B(\mathbf{X}(t),t)1_{\{\phi_{A,B}(t)\geqslant 0\}} + s_A(\mathbf{X}(t),t)1_{\{\phi_{A,B}(t)<0\}}$$

and $\phi_{A,B}(t)$ represents the (unknown) fraction of $D(t)$ which is vulnerable at t. This expression exhibits two major differences with respect to (9): first, the intervention of a stochastic factor ϕ, which depends on A and B, translates the interaction of both credit risks in valuation and imposes to separate its components to get an analytical solution; second, the elimination of $V(t)$ from the switching regime conditions, making it unnecessary to solve a recursive integral as in Duffie and Huang (1996). Overall, the formulation proposed in this paper is more appealing than their formula because it provides a tractable decomposition.

2. Application to plain-vanilla swaps

From now on, swaps are supposed to be submitted to the full two-way payment rule, as the alternative limited two-way payment rule has been shown to correspond to a classical pricing model.

Primarily for the purposes of tractability, it is assumed that the spot interest rate process follows, under the risk-adjusted probability measure, a Gaussian process as described by the Vasicek (1977) model:

$$\mathrm{d}r(t) = a_r\left[\bar{r} - r(t)\right]\mathrm{d}t + \sigma_r\,\mathrm{d}Z_r(t), \tag{11}$$

where $Z_r(t)$ is a Wiener process adapted to the market filtration F.

The reduced-form approach does not specify a priori the relationship between the market value of the firm's assets and the critical level that affects credit risk at any point in time, such as in the structural form. The application is developed under the assumption that both the arrival risk and the magnitude risk for each party j, $j = A, B$, of the swap are considered to be functions of a one-dimensional state variable $X_j(t)$. Obviously, if one considers that a party may want to strategically default depending on the other party's credit risk, the framework has to be extended to a multi-dimensional set. This does not cause any major technical or conceptual difficulty, but is not the purpose of the paper.

This state variable follows a lognormal mean-reverting process, whose risk-adjusted drift, diffusion and mean-reversion parameters are specific to each defaultable party:

$$\mathrm{d}\ln X_j(t) = a_j\left[\bar{\gamma}_j - \ln X_j(t)\right]\mathrm{d}t + \sigma_j\,\mathrm{d}Z_j(t), \tag{12}$$

where Z_j is adapted to F, $\mathrm{corr}(Z_A(t), Z_B(t)) = \rho_{AB}$ and $\mathrm{corr}(Z_r(t), Z_j(t)) = \rho_{jr}$. In correspondence to the structural form approach, $X_j(t)$ should be thought of as a ratio of the market value of the firm's assets $A_j(t)$ over an economically interpretable counterpart $K_j(t)$. Unlike in the structural approach, this counterpart should not be viewed as a measure of debt, but rather as another way to assess the value of the firm's investments than through market data: book value, discounted earnings or cash flows or any other measure related to the firm that can provide information about its fundamental financial health and that analysts can compare with a market value to assess how well the firm performs with respect to this benchmark. Furthermore, this ratio exhibits mean-reversion which can empirically be explained by phenomena such as overreaction, market-to-book or price-earnings effects; or theoretically be justified by the reduction in the risk premium required on the firm's investments as financial distress becomes unlikely. The adjustment for risk performed at the level of the state variable allows to use the martingale approach with the corresponding risk-adjusted arrival rate.

The issue considered within this class of models is the pricing of a credit-risky discount bonds. Denote $P_c(r(t), X_j(t), t, T)$ the price of a pure discount defaultable bond with maturity T. Provided that default has not yet occurred at time t, it is given by

$$P_c(r(t), X_j(t), t, T) = E_Q\left[\exp\left(-\int_t^T R_j(r(u), X_j(u))\,\mathrm{d}u\right)\middle|\mathscr{F}_t\right], \tag{13}$$

where $E_Q[\cdot]$ stands for the expectation under the risk-adjusted probability measure, $\mathscr{F}_t$ is a sub-sigma-algebra, representing information available at time t, of the filtration F.

In order to introduce correlation analytically into the pricing formulas, $R_j(r(t), X_j(t))$ should be normally distributed. Thanks to the additive property of the normal distribution, this only requires $s_j(X_j(t))$ to be linear in $\ln X_j(t)$. The most sustainable calibration of this linear function goes as follows: if $X_j(t) \to 0$, $h_j(X_j(t)) \to \infty$ and $l_j(X_j(t)) \to 1$ (bankruptcy state); if $X_j(t) = 1$, $h_j(1) = \lambda_j$ and $l_j(1) = 1 - \alpha_j$ (benchmark state); if $X_j(t) = \pi_j$ for some constant π_j reached with negligible probability, $h_j(\pi_j) \to 0$ and $l_j(\pi_j) = 0$ (riskless state). The constants λ_j, α_j and π_j may be interpreted, respectively, as the arrival rate of reference, the recovery rate of reference and the threshold of risklessness for party j.

These three conditions together may lead to a great variety of functional forms for $h_j(X_j(t))$ and $l_j(X_j(t))$. To get a Gaussian spread, their product has to result in

$$s_j(X_j(t)) = C_{0j} + C_{1j} \ln X_j(t) = \lambda_j(1-\alpha_j) + \frac{\lambda_j(1-\alpha_j)}{\ln 1/\pi_j} \ln X_j(t), \tag{14}$$

but of course, relaxing the convenient assumption of a Gaussian spread opens the way to any kind of arrival and loss rate, to the expense of a reduction in tractability.

Using the spread defined in (14), the price of the risky discount bond is given by

Proposition 1. *The price at time t of a corporate discount bond maturing at T, when credit risk is recurrent, where the risk-adjusted default spread is given by* (14) *and the processes for $r(t)$ and $\ln X_j(t)$ are given by* (11) and (12), *respectively, is given by*:

$$P_c(r(t), X_j(t), t, T) = \exp\left[\frac{1}{2}K_j^2(t,T) - N_j(r(t), X_j(t), t, T)\right], \tag{15}$$

$$\begin{aligned} K_j^2(t,T) = {} & \kappa_1 \,\mathrm{e}^{-a_r(T-t)} + \kappa_2 \,\mathrm{e}^{-a_j(T-t)} + \kappa_3 \,\mathrm{e}^{-2a_r(T-t)} + \kappa_4 \,\mathrm{e}^{-2a_j(T-t)} \\ & + \kappa_5 \,\mathrm{e}^{-(a_r+a_j)(T-t)} + \kappa_6(T-t) + \kappa_7, \end{aligned} \tag{16}$$

$$\begin{aligned} N_j(r(t), X_j(t), t, T) = {} & \Big(\bar{r} + C_{1j}\bar{\gamma}_j + \lambda_j(1-\alpha_j)\Big)(T-t) \\ & + \frac{1}{a_r}\left(1 - \mathrm{e}^{-a_r(T-t)}\right)\Big(r(t) - \bar{r}\Big) \\ & + \frac{1}{a_j} C_{1j}\left(1 - \mathrm{e}^{-a_j(T-t)}\right)\Big(\ln X_j(t) - \bar{\gamma}_j\Big), \end{aligned} \tag{17}$$

and

$$\kappa_1 = \frac{2}{a_r^2}\left(C_{2j}\rho_{jr}\sigma_r + \frac{\sigma_r^2\left(1-\rho_{jr}^2\right)}{a_r}\right), \quad \kappa_2 = \frac{2}{a_j^2}C_{1j}C_{2j}\sigma_j, \quad \kappa_3 = \frac{-\sigma_r^2}{2a_r^3},$$

$$\kappa_4 = \frac{-C_{1j}^2\sigma_j^2}{2a_j^3}, \quad \kappa_5 = \frac{-2C_1\rho_{jr}\sigma_j\sigma_r}{a_r a_j(a_r+a_j)}, \quad \kappa_6 = C_2^2 + \frac{\sigma_r^2\left(1-\rho_{jr}^2\right)}{a_r^2},$$

$$\begin{aligned}\kappa_7 = {} & \frac{\sigma_r^2}{2a_r^3} + C_{1j}^2\frac{\sigma_j^2}{2a_j^3} - \frac{2}{a_r^2}C_{2j}\rho_{jr}\sigma_r - \frac{2\sigma_r^2\left(1-\rho_{jr}^2\right)}{a_r^3} \\ & - \frac{2}{a_j^2}C_{1j}C_{2j}\sigma_j + \frac{2C_{1j}\rho_{jr}\sigma_j\sigma_r}{a_r a_j(a_r+a_j)},\end{aligned}$$

$$C_{1j} = \frac{\lambda_j(1-\alpha_j)}{\ln\frac{1}{\pi_j}} \quad \textit{and} \quad C_{2j} = \left(\frac{\rho\sigma_r}{a_r} + \frac{C_{1j}\sigma_j}{a_j}\right).$$

Proof. See Appendix 1 in Hübner (1997).

This proposition will be exploited in the development of the pricing formula for interest rate swaps and currency swaps. The assumptions leading to it are also maintained throughout for each defaultable party.

2.1. Interest rate swaps

Consider a plain-vanilla interest rate swap between two parties subject to recurrent default risk. Fixed payment of $r_f\tau$ by A are exchanged for floating payments by B at the rate r for a period τ at times $t_1,\ldots,t_n$. The promised dividend to the swap contract to party A at time t_i is $D(t_i) = \tau(Y_i(\tau) - r_f)$. Consider that the floating rate is determined at the time of actual payment. Hence, denoting $V(t)$ the predefault value of an interest rate swap for A, paying fixed and receiving floating, Eq. (4) specializes to

$$\begin{aligned}V(t) = {} & E_Q\left[\sum_{i=1}^{n} \mathrm{e}^{-\int_t^{t_i} R_B(X_B(u),u)\,\mathrm{d}u} \max[\tau(Y_i(\tau)-r_f),0]\middle|\mathscr{F}_t\right] \\ & - E_Q\left[\sum_{i=1}^{n} \mathrm{e}^{-\int_t^{t_i} R_A(X_A(u),u)\,\mathrm{d}u} \max[\tau(r_f-Y_i(\tau)),0]\middle|\mathscr{F}_t\right],\end{aligned} \tag{18}$$

where $Y_i(\tau)$ denotes the instantaneous yield-to-maturity of a riskless bond maturing at $t_i+\tau$ taken at time t_i. This expression indicates that the direct correlation between $X_A(t)$ and $X_B(t)$ is not involved in the pricing of the swap, as the two state variables appear in different expectation operators. The corresponding formula is given in Proposition 2.

Proposition 2. *If the parties are exposed to the recurrent credit risk model with* X_j *respecting* (12) *for* $j = A, B$, *and if the credit exposures partly offset each other upon default, the undefaulted interest rate swap price to party* A *is given by*:

$$\begin{aligned} V(t) = {} & \sum_{i=1}^{n} P_c(r(t), X_B(t), t, t_i)\Big[(\alpha(\tau) + \beta(\tau) m_B(t, t_i) - r_f)\mathcal{N}(-d_{Bi}) \\ & + \beta(\tau) v(t, t_i)\mathcal{N}'(d_{Bi})\Big]\tau - \sum_{i=1}^{n} P_c(r(t), X_A(t), t, t_i)\Big[(r_f - \alpha(\tau) \\ & - \beta(\tau) m_A(t, t_i))\mathcal{N}(d_{Ai}) - \beta(\tau) v(t, t_i)\mathcal{N}'(d_{Ai})\Big]\tau, \end{aligned} \tag{19}$$

$$d_{ji} = \frac{r_f - \alpha(\tau) - \beta(\tau) m_j(t, t_i)}{\beta(\tau) v(t, t_i)} \quad \textit{for } j = A, B, \tag{20}$$

where $\beta(\tau)$, $\alpha(\tau)$, $m_j(t, t_i)$ *and* $v^2(t, t_i)$ *are given by Eqs.* (A.3)–(A.6), *and* $\mathcal{N}$ *and* $\mathcal{N}'$ *denote, respectively, the c.d.f. and the p.d.f. of the standard normal distribution.*

Proof. See Appendix A.

This option-like formula allows for any correlation pattern between the spreads and the riskless spot rate, any difference in credit exposures, and only requires to observe and parametrize the three state variables of the model.

2.2. Currency swaps

The same notation for the spot interest rate and the default spread of each party is kept in order to price defaultable currency swaps. Parties A and B exchange at time 0 a notional amount in two currencies d and f. The real process for the exchange rate $E(t)$ (the number of units of f that can be bought by 1 unit of d), follows a geometric Brownian motion:

$$\mathrm{d}E(t) = \mu_E E(t)\,\mathrm{d}t + \sigma_E E(t)\,\mathrm{d}Z_E'(t), \tag{21}$$

where the correlation coefficients between $Z_E'(t)$ and $Z_r(t)$, $Z_A(t)$ and $Z_B(t)$ are ρ_{Er}, ρ_{AE} and ρ_{BE}, respectively.

Normalize to 1 the notional amount paid by party A at time 0, and the amount paid by B is the corresponding $1/E(0)$, which is expressed in foreign currency. At times t_i, $i = 1, \ldots, n$, coupon payments of c_d are made by party A in exchange of $c_f/E(0)$ received in foreign currency from party B. The principals are paid back at time t_n. Furthermore, the instantaneous international interest rate parity holds: $r_f(t) = r(t) - \mu_E + \lambda(t)\sigma_E$ in the risk-neutral measure, where $\lambda(t)$ is the price of exchange risk. This allows to express the swap value in terms of the domestic currency and interest rate.

In this case, the dividend promised to party A at time t_i is, expressed in domestic currency, $D(t_i) = ((E(t_i)/E(0))C_{fi} - C_{di})$, where $C_{ki} = c_k$ when $i = 1, \dots, n-1$ and $C_{kn} = c_k + 1$, $k = d, f$, and is exactly the opposite for party B. Since discounting may take place for both dividend processes in domestic currency terms, the swap valuation equation given in (4) becomes

$$\begin{aligned} V(t) = {} & E_Q\left[\sum_{i=1}^{n} \mathrm{e}^{-\int_t^{t_i} R_B(u)\,\mathrm{d}u} \max\left[\left(\frac{E(t_i)}{E(0)}C_{fi} - C_{di}\right), 0\right] \middle| \mathscr{F}_t\right] \\ & - E_Q\left[\sum_{i=1}^{n} \mathrm{e}^{-\int_t^{t_i} R_A(u)\,\mathrm{d}u} \max\left[\left(C_{di} - \frac{E(t_i)}{E(0)}C_{fi}\right), 0\right] \middle| \mathscr{F}_t\right]. \end{aligned} \tag{22}$$

The lognormal distribution of $E(t)$ makes the solution to each term of this sum look like a Black–Scholes formula, as shown in Proposition 3.

Proposition 3. *If the parties are exposed to the recurrent credit risk model with* X_j *respecting* (12) *for* $j = A, B$, *and if the credit exposures partly offset each other upon default, the undefaulted currency swap price to party A is given by*:

$$\begin{aligned} V(t) = {} & \sum_{i=1}^{n}\left[\frac{P_c(r(t), X_B(t), t, t_i)}{D_B(t, t_i)}\frac{C_{fi}E(t)}{E(0)}\mathcal{N}(d_{Bi})\right. \\ & \left. - C_{di}P_c(r(t), X_B(t), t, t_i)\mathcal{N}\left(d_{Bi} - \sigma_E\sqrt{t_i - t}\right)\right] \\ & - \sum_{i=1}^{n}\left[-\frac{P_c(r(t), X_A(t), t, t_i)}{D_A(t, t_i)}\frac{C_{fi}E(t)}{E(0)}\mathcal{N}(-d_{Ai})\right. \\ & \left. + C_{di}P_c(r(t), X_A(t), t, t_i)\mathcal{N}(-d_{Ai} + \sigma_E\sqrt{t_i - t})\right], \end{aligned} \tag{23}$$

$$\begin{aligned} d_{ji} &= \frac{\ln\left[\frac{C_{fi}E(t)}{C_{di}E(0)D_j(t,t_i)}\right]}{\sigma_E\sqrt{t_i - t}} + \frac{\sigma_E\sqrt{t_i - t}}{2}, \\ D_j(t, t_i) &= \exp\left\{\frac{C_{1j}\rho_{Ej}\sigma_E\sigma_j}{a_j}\left(t_i - t - \frac{1 - \mathrm{e}^{-a_j(t_i - t)}}{a_j}\right)\right\} \quad \textit{for } j = A, B, \end{aligned} \tag{24}$$

$$C_{ki} = \begin{cases} c_k & \textit{for } i = 1, \dots, n-1 \textit{ and } k = d, f, \\ c_k + 1 & \textit{for } i = n \textit{ and } k = d, f. \end{cases} \tag{25}$$

Proof. See Appendix B.

It turns out from Proposition 3 that the correlation between the exchange rate and the processes $\ln X_j$, $j = A, B$, has an impact, through D_j, on swap valuation. If there is no correlation between E and X_j, $D_j = 1$ and $\mathcal{N}(d_{ji})$ is exactly similar to $\mathcal{N}(d_1)$ in the Black–Scholes formula for European currency options. In fact, each coupon payment is priced like a riskless currency option, except that it is reduced by a factor $P_c(r(t), X_j(t), t, t_i)/P(r(t), t, T) < 1$. This is the direct impact of the credit risk exposure.

But another type of risk appears in this formula through the factor D_j dividing the price of the risky bond. Indeed, Eq. (24) suggests that the sign of the argument of the exponential is the opposite of the sign of ρ_{Ej}. This means that $D_j <$ (resp. $>$) 1 if $\rho_{Ej} >$ (resp. $<$) 0.

Surprisingly, as D_j does not depend on any bond price, it could become so low that $P(r(t), t, t_i) D_j(t, t_i) < P_c(r(t), X_j(t), t, t_i)$: correlation risk might produce a higher option value than in the riskless case, simply because high values in pay-offs (e.g., when $E(t)$ increases) tend to go along with a decrease of the default spread, increasing the discount factors and making the option more interesting.

Clearly, the differences between currency swaps and interest rate swaps are of three kinds. First, a currency swap involves the exchange of principals, making it presumably more sensitive to credit risk characteristics. Second, the modeling choice for each type of swap involves two different distributions, namely, the normal and the lognormal, producing different types of risk-adjusted probabilities and expectations. Finally, a third state variable appears in the pricing of the currency swap, and its correlation with default characteristics considerably affects the results.

2.3. The impact of netting

A swap master agreement can stipulate that the creation of a new swap with reversed obligations may partly offset the promised dividend payments before default is considered. This netting agreement represents a hedging device for default risk. In Proposition 2 in Duffie and Huang (1996), it is shown to bring additional value to the counterparty for which the old swap value is positive, if compared with unnetted swaps. In their analysis, they consider the case of a zero-value swap, and conclude that the coupon rate of the new swap is a linear function of k for $0 \leqslant k \leqslant 1$.

The analysis may start here with any initial swap value. Without loss of generality, consider a swap with unit notional marked to market to its value $V(t) = V_{A,B}(t) \geqslant 0$ for counterparty A . The notation $V_{A,B}(t)$ means that party A pays fixed (for interest rate swaps) or domestic coupons (for currency swaps) of c ($c = r_f$ or c_d). The value of the same swap with reversed payments between A and B is denoted $V_{B,A}(t)$. A new swap, whose notional is k for the interest rate swap and $k/E(0)$ for the currency swap, and with identical floating or foreign coupon, is assorted with a netting master agreement: the promised dividend payments are compensated with those of the original contract. The netted value of the combined swaps is denoted $V(c^k(t), t)$, and equates $V(t)$ at contract settlement. The variable under interest is the fixed (for interest rate swaps) or domestic (for currency swaps) coupon rate $c^k(t) = \arg[V(c^k, t) = V(t)]$ in the new swap. Its unnetted value with coupon $c^k(t)$, when normalized to a unit notional, is denoted $V^k_{B,A}(t)$ if payment obligations are reversed $(k > 0)$ and $V^k_{A,B}(t)$ if they are reinforced $(k < 0)$. If this swap value is negative, netting is profitable to party A as the netted swap is worth more than the sum of the contract values, while the opposite case leaves B better off.

The valuation formula $V(c^k, t)$ of the netted swap is of course similar to the unnetted swap $V(t)$, except that, for an interest rate swap, $\alpha(\tau)$ and $\beta(\tau)$ are replaced by $(1 - k)\alpha(\tau)$ and $(1 - k)\beta(\tau)$ and r_f is replaced by $r_f - kc^k(t)$. For a currency swap, c_f is replaced by $(1 - k)c_f$ and c_d is replaced by $c_d - kc^k(t)$. The next result shows the impact of netting for any default risk asymmetry and any notional added to the existing swap.

Proposition 4. *Let $V(t) \geqslant 0$ be the value of the existing swap for party A. Denote $c^*_{A,B}(t) = \arg[V^k_{A,B}(t) = 0]$ and $c^*_{B,A}(t) = \arg[V^k_{B,A}(t) = 0]$. Then $c^0(t) = c$ and $c^1(t) = V(t)/\sum_{i=1}^{n} P_c(r(t), X_B(t), t, t_i) + c \geqslant c$. For $k \leqslant 0$, $V^k_{A,B}(t) \leqslant 0$ and $\lim_{k\downarrow-\infty} kV^k_{A,B}(t) = -V_{A,B}(t)$; for $k > 0$, $V^k_{B,A}(t) \leqslant 0$ when $c^k(t) \leqslant c^*_{B,A}(t)$ and $V^k_{B,A}(t) > 0$ otherwise, and $\lim_{k\uparrow-\infty} kV^k_{B,A}(t) = V_{B,A}(t)$. When $0 < k < 1$, if $c^*_{B,A}(t) \geqslant c^1(t)$, $V^k_{B,A}(t) < 0$ and if $c^*_{B,A}(t) \leqslant c$, $V^k_{B,A}(t) > 0$.*

Proof. The coupon rate $c^k(t)$ is a smooth increasing continuous function of k for $k \geqslant 0$ with a minimum of c at 0. If $k = 1$, promised floating/foreign dividends cancel out, and $\mathcal{N}(d_{Bi}) = 1$. Eqs. (19) and (23) both yield $V(t) = \sum_{i=1}^{n} P_c(r(t), X_B(t), t, t_i)(c^1(t) - c)$. For $k \leqslant 0$, the unnetted swaps have value $V(t) - kV^k_{A,B}(t)$. The make it equal to the old swap, the coupon paid by party A should then be $c^*_{A,B}(t) \geqslant c$. If swaps are netted, the derivative with respect to $|k|$ of the corresponding swap value $(1 - k)V(c - kc^*_{A,B}(t), t)$ is $-V(t) < 0$, and is equal to $V(t)$ for $k = 0$. In order to verify $V(c^k(t), t) = V(t)$, one must have $c^k(t) \geqslant c^*_{A,B}(t)$ and therefore $V^k_{A,B}(t) \leqslant 0$. The limit is obtained through l'Hospital's rule. For $k > 0$, $V^k_{B,A}(t)$ is decreasing in $c^k(t)$. If some value k^* is such that $c^*_{B,A}(t) = c^{k*}(t)$, the corresponding unnetted swap value is $V(t) - k^* V^*_{B,A}(t) = V(t)$. For higher values of k, the coupon $c^k(t) \geqslant c^{k*}(t)$ is such that $V^k_{B,A}(t) \leqslant 0$. The same reasoning holds for values of $k \leqslant k^*$. The limit is similar to the case when $k < 0$. The special case when $0 < k < 1$ is straightforward. □

Proposition 4 can be confronted with the more restrictive Proposition 2 in Duffie and Huang (1996). These authors state that netting is profitable to a party when her credit spread is lower than the one of the other party. The subsequent hedging results are confirmed in this subsection: if $V(t) = 0$ and A is always more exposed to credit risk than B, then $c^*_{A,B}(t) = c$ and $c^*_{B,A}(t) > c^1(t) = c$, so that netting is always profitable to party A when $k > 0$ and the parties are indifferent if $k < 0$. Beyond this result, Proposition 4 suggests that the hedging properties of netting are not to be generalized. As the defaultable swap can be decomposed in a pure market component and a credit risk component, the single observation of the swap value is not sufficient in order to extract the importance of credit exposure. Its impact is related to the level of $c^*_{B,A}(t)$, whose high value implies that A is severely exposed to credit risk and is likely to be better of by netting the swap. But if $c^*_{B,A}(t) < c$, netting has no hedging role for party A in spite of the positive swap price which is solely due to favorable market conditions.

3. Conclusion

On the basis of a set of assumptions that allow us to use a recurrent credit risk framework, but where defaultable cash flows may partly offset each other, useful analytical expressions for the valuation of interest rate and currency swaps are provided. Yet, they are not obtained at the expense of a lack of sustainability of the analysis. Indeed, the approach is intuitively closer than the Duffie and Huang (1996) paper to traditional credit risk models, and is flexible enough so as to address the question of strategic default by adapting the spreads.

The swap valuation equations display various sources of credit asymmetries. These may have the power to explain the contradictory predictions of a stream of theoretical literature and of some reported empirical evidence concerning the role of credit risk for interest rate swaps. While simulations by Li (1998) and Duffie and Huang (1996) hypothesize a single source of credit risk asymmetry and obtain that swap yields are almost insensitive to credit conditions, tests on actual data performed by Sun et al. (1993), Chen and Selender (1995), Cossin and Pirotte (1997) and Minton (1997) suggest that credit risk is not negligible in real transactions. This paper takes the view that part of the spread can be due to the sources of asymmetries in credit quality. They may affect the value of the swap even if the level of the credit spread charged on each party's bonds is identical. In particular, for currency swaps, the presence of correlation risk may be extremely influent, though it is not considered as a component of credit risk in traditional approaches.

The basic results on netting proposed by Duffie and Huang (1996) are confirmed in this setup, as a special case of our detailed analysis. Stylized facts concerning the profits to be made from netting are generalized to any initial swap value, new notional and credit conditions.

This paper outlines the multi-dimensionality of the notion of "credit risk", which requires efforts to decompose credit risk factors. In particular, the mere assumption of a sole state variable determining corporate credit risk, as in structural form models, creates at least three dimensions of credit risk (volatility of the state variable, arrival or magnitude risk, and correlation risk). Expanding to more state variables would lead to a considerably more thorough analysis of the components of default risk.

This model should of course ideally be confronted with real data. Yet, it has a normative content too: the account of default risk in swaps is desirable, but it is not sufficient to ensure correct pricing, because of the complexity of the swap spread formation. It is also a step towards recognition of the importance of the asymmetry in bilateral credit risk, which is likely to affect coupon rates, and so the undefaulted swap value.

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Appendix A. Proof of Proposition 2

Eq. (18) is indeed the difference between two yield options, whose formula is given by Longstaff (1990) for a CIR specification. It is adapted by the following lemma:

Lemma 1. *Let $F(Y_i(\tau))$ denote the pay-off function for a contingent claim on $Y(\tau)$ maturing at t_i submitted to credit risk of party j. The value of the claim can be written as*

$$\Gamma(r, X_j, t, t_i, \tau) = P_c(r, X_j, t, t_i) E_{Q'}[F(Y(\tau)) \mid \mathscr{F}_t]. \tag{A.1}$$

The expectation is taken with respect to $Y(\tau)$ which is distributed as

$$\alpha(\tau) + \beta(\tau)\tilde{n}(m_j(t, t_i), v^2(t, t_i)), \tag{A.2}$$

where $\tilde{n}(m_j(t, t_i), v^2(t, t_i))$ is a normal variate with expectation m_j and variance v^2, and

$$\beta(\tau)=\frac{B_1(\tau)}{\tau}=\frac{1-\mathrm{e}^{-a_r\tau}}{a_r\tau}, \tag{A.3}$$

$$\alpha(\tau)=-\frac{1}{\tau}\left[\frac{(B_1(\tau)-\tau)\left(a_r^2\bar{r}-\sigma_r^2/2\right)}{a_r^2}-\frac{\sigma_r^2B_1(\tau)^2}{4a_r}\right], \tag{A.4}$$

$$\begin{aligned}m_j(t,t_i)={}&r(t)\,\mathrm{e}^{-a_r(t_i-t)}+\left(\bar{r}-\frac{\sigma_r^2}{a_r^2}-\frac{C_{1j}\rho_j\sigma_r\sigma_j}{a_ra_j}\right)\left(1-\mathrm{e}^{-a_r(t_i-t)}\right)\\&+\frac{\sigma_r^2}{2a_r^2}\left(1-\mathrm{e}^{-2a_r(t_i-t)}\right)+\frac{C_{1j}\rho_j\sigma_r\sigma_j}{a_j(a_r+a_j)}\left(1-\mathrm{e}^{-(a_r+a_j)(t_i-t)}\right),\end{aligned} \tag{A.5}$$

$$v^2(t,t_i)=\frac{\sigma_r^2}{2a_r}\left(1-\mathrm{e}^{-2a_r(t_i-t)}\right). \tag{A.6}$$

Proof (*Proof of Lemma 1*)**.** Writing the price of the riskless bond as $P(r,t_i,t_i+\tau)=\mathrm{e}^{-\tau Y_i(\tau)}$, one immediately gets from Vasicek (1977) $Y_i(\tau)=\alpha(\tau)+\beta(\tau)r(t_i)$. In order to price any derivative security $\Gamma(r(t),X_j(t),t,t_i)$ whose pay-off is $F(Y_i(\tau))$, notice first it must satisfy the partial differential equation:

$$\begin{aligned}&-\Gamma_{t_i}+a_r\left(\bar{r}-r(t)\right)\Gamma_r+a_j\left(\bar{\gamma}_j-\ln X_j(t)\right)\Gamma_j\\&\quad+\frac{1}{2}\left(\sigma_r^2\Gamma_{rr}+2\rho_{jr}\sigma_j\sigma_r\Gamma_{rj}+\sigma_j^2\Gamma_{jj}\right)-r(t)\Gamma=0,\end{aligned} \tag{A.7}$$

where subscripts t_i, r, and j represent partial derivatives with respect to t_i, $r(t)$ and $\ln X_j(t)$.

Rewriting $\Gamma(r(t),X_j(t),t,t_i)=P_c(r(t),X_j(t),t,t_i)G(r(t),t_i)$ and noticing that $G_j=G_{jj}=0$, this partial differential equation is decomposed in

$$\begin{aligned}&G\left[-P_{t_i}+a_r\left(\bar{r}-r(t)\right)P_r+\frac{1}{2}\left(\sigma_r^2P_{rr}+2\rho_{jr}\sigma_j\sigma_rP_{rj}+\sigma_j^2P_{jj}\right)-r(t)P\right]\\&\quad+P\left[-G_{t_i}+\left[a_r\left(\bar{r}-r(t)\right)+\frac{P_r}{P}\sigma_r^2+\frac{P_j}{P}\rho_{jr}\sigma_j\sigma_r\right]G_r+\frac{1}{2}\sigma_r^2G_{rr}\right]=0.\end{aligned} \tag{A.8}$$

The term multiplied by G satisfies the partial differential equation for P , and is thus equal to 0. Since Eq. (15) gives that

$$P_r/P=-B_1(t_i-t)=-\frac{1-\mathrm{e}^{-a_r(t_i-t)}}{a_r}$$

and

$$P_j/P = -B_2(t_i - t) = -\frac{C_{1j}\left(1 - \mathrm{e}^{-a_j(t_i-t)}\right)}{a_j} \quad \text{for } X = X_j,$$

we have the following condition for G:

$$-G_{t_i} + \left[a_r\left(\bar{r} - r(t)\right) - B_1(t_i - t)\sigma_r^2 - B_2(t_i - t)\rho_{jr}\sigma_j\sigma_r\right]G_r + \frac{1}{2}\sigma_r^2 G_{rr} = 0 \tag{A.9}$$

subject to unchanged maturity conditions. From Theorem 5.2 of Friedman (1975),

$$\Gamma(r(t), X_j(t), t, t_i) = P_c(r(t), X_j(t), t, t_i)E_{Q'}[F(Y(\tau) \mid \mathscr{F}_t], \tag{A.10}$$

where the expectation is taken with respect to the process satisfying the stochastic differential equation,

$$\begin{aligned} \mathrm{d}r(t) = {} & a_r\left[\bar{r} - \frac{\sigma_r^2\left(1 - \mathrm{e}^{-a_r(t_i-t)}\right)}{a_r^2} - \frac{C_{1j}\rho_{jr}\sigma_j\sigma_r\left(1 - \mathrm{e}^{-a_j(t_i-t)}\right)}{a_r a_j} - r(t)\right]\mathrm{d}t \\ & + \sigma_r\,\mathrm{d}Z_r(t), \end{aligned} \tag{A.11}$$

which corresponds to a unique value for $r(t_i)$:

$$\begin{aligned} r(t_i) = {} & r(t)\,\mathrm{e}^{-a_r(t_i-t)} + \left(\bar{r} - \frac{\sigma_r^2}{a_r^2} - \frac{C_{1j}\rho_{jr}\sigma_r\sigma_j}{a_r a_j}\right)\left(1 - \mathrm{e}^{-a_r(t_i-t)}\right) \\ & + \frac{\sigma_r^2}{2a_r^2}\left(1 - \mathrm{e}^{-2a_r(t_i-t)}\right) + \frac{C_{1j}\rho_{jr}\sigma_r\sigma_j}{a_j(a_r + a_j)}\left(1 - \mathrm{e}^{-(a_r+a_j)(t_i-t)}\right) \\ & + \sigma_r\int_t^{t_i} \mathrm{e}^{-a_r(t_i-u)}\,\mathrm{d}Z_r(u), \end{aligned} \tag{A.12}$$

whose expectation and variance correspond to expressions (A.5) and (A.6). □

Eq. (19) follows then from the application of Lemma 1 to Eq. (18) and pointing out that $E[\tilde{n}1_{\{\tilde{n} \geqslant k\}}] = m\mathscr{N}(-d) + v\mathscr{N}'(d)$, where $d = k - m/v$ (Amemiya, 1973).

Appendix B. Proof of Proposition 3

In order to solve Eq. (22), the use of two lemmas is necessary.

Lemma 2. *If the process for the exchange rate under the actual probability measure is given by Eq.* (21) *and if the domestic spot interest rate follows a Vasicek process, the risk-adjusted process for the exchange rate is described by the following Itô process*:

$$\ln E(t_i) = \ln E(t) + \bar{\mu}_E(t, t_i) - \frac{1}{2}\sigma_E(t_i - t) + \sigma_E[Z_E(t_i) - Z_E(t)], \tag{B.1}$$

where

$$\bar{\mu}_E(t, t_i) = \frac{\rho_{Er}\sigma_E\sigma_r}{a_r}\left(t_i - t - \frac{1 - \mathrm{e}^{-a_r(t_i - t)}}{a_r}\right), \tag{B.2}$$

and Z_E is a standard Wiener process under the risk-adjusted probability measure Q, with unchanged correlations with the other Wiener processes.

Proof (*Proof of Lemma 2*)**.** Since $E(t)$ is a traded asset, its risk-neutral process must have the martingale property under the Vasicek process for the interest rate,

$$E_Q\left[E(t_i)\exp\left\{-\int_t^{t_i} r(u)\,\mathrm{d}u\right\}\middle|\mathscr{F}_t\right] = E(t)P(r(t), t, t_i). \tag{B.3}$$

The argument of the expectation is lognormally distributed, so only the expectation and variance of its natural logarithm have to be known. With some simple algebra, the Itô process solving this equality is given by formulas (B.1) and (B.2). □

Lemma 3. *Let $F(E(t_i))$ denote the pay-off function for a contingent claim on E maturing at t_i submitted to credit risk of party j. Let $P_c(r, X_j, t, t_i)$ denote the value, under a Vasicek process for the instantaneous short rate, of a pure discount bond maturing at t_i and submitted to recurrent credit risk of party j. The value of the claim can be written as*

$$\Gamma(r, X_j, E, t, t_i) = P_c(r, X_j, t, t_i)E_{Q_{ij}}[F(E(t_i))\,|\,\mathscr{F}_t]. \tag{B.4}$$

The expectation is taken with respect to $E(t_i)$ which is distributed as

$$\exp\left[\tilde{n}(\mu_{Ej}(t, t_i), \sigma_E^2(t_i - t))\right], \tag{B.5}$$

where $\tilde{n}(\mu_{Ej}(t, t_i), \sigma_E^2(t_i - t))$ is a normal variate with expectation μ_{Ej} and variance σ_E^2, and

$$\begin{aligned}\mu_{Ej}(t, t_i) = {} & \ln E(t) - \left(\frac{\sigma_E^2}{2} + \frac{C_{1j}\rho_{Ej}\sigma_E\sigma_j}{a_j}\right)(t_i - t) \\ & + \frac{C_{1j}\rho_{Ej}\sigma_E\sigma_j}{a_j^2}\left(1 - \mathrm{e}^{-a_j(t_i - t)}\right).\end{aligned} \tag{B.6}$$

Proof (*Proof of Lemma 3*)**.** First orthogonalize the Wiener processes,

$$\begin{pmatrix} Z_E \\ Z_j \\ Z_r \end{pmatrix} = \begin{pmatrix} \sqrt{(1-\rho_{Er}^2)(1-\rho_{Ej}^{*2})} & \sqrt{(1-\rho_{Er}^2)}\rho_{Ej}^* & \rho_{Er} \\ 0 & \sqrt{1-\rho_{jr}^2} & \rho_{jr} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U_E \\ U_j \\ U_r \end{pmatrix}, \tag{B.7}$$

where

$$\rho_{Ej}^{*2} = \frac{\rho_{Ej} - \rho_{Ej}\rho_{jr}}{\sqrt{(1-\rho_{Er}^2)(1-\rho_{jr}^2)}}.$$

It allows to work with the orthogonalized process,

$$\ln E(t) = \ln E(0) + \bar{\mu}_E(0,t) - \frac{1}{2}\sigma_E t + \sigma_E \Bigg[\sqrt{(1-\rho_{Er}^2)(1-\rho_{Ej}^{*2})}\,U_E(t) + \sqrt{1-\rho_{Er}^2}\,\rho_{Ej}^* U_j(t) + \rho_{Er} U_r(t) \Bigg]. \tag{B.8}$$

The risk-adjusted expectation that has to be computed is of the form

$$\Gamma = E_Q\left[\exp\left\{ -\int_t^{t_i} R_j(u) \right\} F(E(t_i)) \,\middle|\, \mathscr{F}_t \right]. \tag{B.9}$$

Defining a new vector of prices of risks

$$\mathbf{\Lambda}(t) = \begin{pmatrix} 0 \\ \lambda_j(t) \\ \lambda_r(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\sqrt{1-\rho_{jr}^2}\,C_{1j}\sigma_j}{a_j}\left(1-\mathrm{e}^{-a_j(t_i-t)}\right) \\ \frac{\rho_{jr}C_{1j}\sigma_j}{a_j}\left(1-\mathrm{e}^{-a_j(t_i-t)}\right) + \frac{\sigma_r}{a_r}\left(1-\mathrm{e}^{-a_r(t_i-t)}\right) \end{pmatrix} \tag{B.10}$$

allows to apply Girsanov's theorem to yield a new probability measure Q_{ij}, with its associated density

$$\eta[\mathbf{\Lambda}](t,t_i) = \exp\left\{ -\frac{1}{2}\left(\int_t^{t_i} \mathbf{\Lambda}(u)^{\mathrm{T}} \Lambda(u)\,\mathrm{d}u - \int_t^{t_i} \mathbf{\Lambda}(u)^{\mathrm{T}} \right) \mathbf{dU}(u) \right\}$$

and three-dimensional Wiener process $\mathbf{dU}' = \mathbf{dU} + \mathbf{\Lambda}^{\mathrm{T}}\,\mathbf{dt}$. Eq. (B.9) can be rewritten as

$$\Gamma = E_{Q_{ij}}\left[\exp\left\{ -\int_t^{t_i} R_j(u)\,\mathrm{d}u \right\} \frac{1}{\eta[\mathbf{\Lambda}](t,t_i)} F(E(t_i)) \,\middle|\, \mathscr{F}_t \right] \tag{B.11}$$

$$= P_c(r(t), X_j(t), t, t_i) E_{Q_{ij}}[F(E(t_i)) \,|\, \mathscr{F}_t]. \tag{B.12}$$

This leads to

$$E_{Q_{ij}}[\ln E(t_i) \,|\, \mathscr{F}_t] = \ln E(t) + \bar{\mu}_E(t,t_i) - \frac{1}{2}\sigma_E(t_i - t) - \sigma_E \times \int_t^{t_i} \left(\sqrt{1-\rho_{Er}^2}\rho_{Ej}^*\lambda_j(u) + \rho_{Er}\lambda_r(u)\right) \mathrm{d}u. \tag{B.13}$$

Substituting λ_j and λ_r by the values given in (B.10), integrating, and simplifying produces

$$\mu_{Ej}(t,t_i) = \ln E(t) - \left(\frac{\sigma_E^2}{2} + -\frac{C_{1j}\rho_{Ej}\sigma_E\sigma_j}{a_j}\right)(t_i - t) + \frac{C_{1j}\rho_{Ej}\sigma_E\sigma_j}{a_j^2}\left(1 - \mathrm{e}^{-a_j(t_i - t)}\right), \tag{B.14}$$

which corresponds to formula (B.6). □

Successively applying Lemmas 2 and 3 to (22) yields the following equation:

$$\sum_{i=1}^{n} P_c(r(t), X_B(t), t, t_i)\frac{C_{fi}}{E(0)}E_{Q_{iB}}\left[\left(E(t_i) - \frac{C_{di}E(0)}{C_{fi}}\right)1_{\{E(t_i) \geqslant (C_{di}E(0))/C_{fi}\}}\middle|\mathscr{F}_t\right] - \sum_{i=1}^{n} P_c(r(t), X_A(t), t, t_i)\frac{C_{fi}}{E(0)} \times E_{Q_{iA}}\left[\left(-E(t_i) + \frac{C_{di}E(0)}{C_{fi}}\right)1_{\{-E(t_i) < (C_{di}E(0))/C_{fi}\}}\middle|\mathscr{F}_t\right]. \tag{B.15}$$

Noticing that

$$E\left[\exp\left\{\tilde{n}\right\}1_{\{\exp\{\tilde{n}\} \geqslant k\}}\right] = \exp\left\{m + \frac{v^2}{2}\right\}\mathscr{N}\left(v - \frac{\ln k - m}{v}\right)$$

for $\tilde{n}$ normally distributed with expectation m and variance v^2 (see Nielsen, 1993) leads to formula (23). □

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