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Chapter 3

# Capital Growth Theory

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## 1. Introduction

Even casual observation strongly suggests that capital growth is not just a catch-phrase but something which many actively strive to achieve. It is therefore rather surprising that capital growth *theory* is a relatively obscure subject. For example, the great bulk of today's MBA's have had little or no exposure to the subject, having had their attention focussed almost exclusively on the single-period mean–variance model of portfolio choice. The purpose of this essay is to review the theory of capital growth, in particular the so-called growth-optimal investment strategy, its properties, its uses, and its links to betting and other investment models. We also discuss several applications that have tended to refine the basic theory.

The central feature of the growth-optimal investment strategy, also known as the geometric mean model and the Kelly criterion, is the logarithmic shape of the objective function. But the power and durability of the model is due to a remarkable set of properties. Some of these are unique to the growth-optimal strategy and the others are shared by all the members of the (remarkable) small family to which the growth optimal strategy belongs.

Investment over time is multiplicative, not additive, due to the compounding nature of the process itself. This makes a number of results in dynamic investment theory appear nonintuitive. For example, in the single-period portfolio problem, the optimal investment policy is very sensitive to the utility function being used; the set of policies that are inadmissible or dominated across all utility functions is relatively small. The same observation holds in the dynamic case when the number of periods is not large. But as the number of periods does become large, the set of investment policies that are optimal for *current* investment tends to shrink drastically, at least in the basic reinvestment case without transaction costs. As we will see, many strikingly different investors will, in essence, invest the same way when the horizon is distant and will only begin to part company as their horizons near.

It is tempting to conjecture that all long-run investment policies to which risk-averse investors with monotone increasing utility functions will flock, under a favorable return structure, insure growth of capital with a very high probability. Such a conjecture is false; many investors will, even in this case, converge on investment policies which almost surely risk ruin in the long run, in effect ignoring feasible policies which almost surely lead to capital growth. Similarly, the relationship between the behavior of capital over time and the behavior of the expected utility of that same capital over time often appears strikingly nonintuitive.

Section 2 reviews the origins of the capital growth model while Section 3 contains a derivation and identifies its key properties. The conditions for capital growth are examined in Section 4. The model's relationship to other long-run investment models is studied in Section 5 and Section 6 contains its role in intertemporal investment/consumption models. Section 7 adds various constraints for accomplishing tradeoffs between growth and security, while Section 8 reviews various applications. A concluding summary is given in Section 9.

## 2. Origins of the model

The approach to investment commonly known as the growth-optimal investment strategy has a number of apparently independent origins. In particular, Williams [1936], Kelly [1956], Latane [1959], and Breiman [1960, 1961] seem to have been unaware of each other's papers. But one can also argue that Bernoulli (1738) unwittingly stumbled on it in 1738 in his resolution of the St. Petersburg Paradox — see the 1954 translation — and Samuelson's survey [1977].

Samuelson [1971] appears to be the earliest to have related the geometric mean criterion to utility theory — and to find it wanting. The growth optimal strategy's inviolability in the larger consumption–investment context when preferences for consumption are logarithmic was first noted by Hakansson [1970]. Finally, models considering tradeoffs between capital growth and security appear to have been pioneered by MacLean & Ziemba [1986].

## 3. The model and its basic properties

The following notation and basic assumptions will be employed:

$w_t$ = amount of investment capital at decision point $t$ (the end of the $t$th period);

$M_t$ = the number of investment opportunities available in period $t$, where $M_t \leq M$;

$S_t$ = the subset of investment opportunities which it is possible to sell short in period $t$;

$r_{1t}$ = rate of interest in period $t$;