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Chapter 2

Finite State Securities Market Models and Arbitrage

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1. Introduction

A fundamental characteristic of security prices in a competitive securities market is that they do not permit arbitrage opportunities. The theory of arbitrage-based asset pricing is concerned with the implications of the lack of arbitrage opportunities. There are two basic questions that this theory addresses. The first is what restrictions can be placed on security price behavior if no arbitrage opportunities exist. The second question of interest is the converse of the first one; that is, what restrictions are needed to ensure that a given security price process is free of arbitrage opportunities. The resolution of these questions is of basic importance in financial economics. This is because, in the presence of investors who strictly prefer more wealth to less, a set of prices that permits arbitrage cannot possibly be sustained as an equilibrium. Not surprisingly then, the questions raised above have been the subject of intensive research in the last two decades. In this chapter we present the key results of this research.

In our analysis we assume that the number of states in the economy is finite. This allows for a simple and elegant characterization of arbitrage-free security prices. Not only is the finite state model an elegant theoretical construct, it also leads to very useful practical applications. For example, the celebrated binomial model of option pricing is an application of the finite state model of security prices.[1]

We provide an analysis of three topics related to the theory of arbitrage. Firstly, we analyze how one can characterize arbitrage-free prices in a frictionless securities market. To this end, we prove the fundamental theorem of asset pricing which links the lack of arbitrage opportunities to the existence of a risk-neutral probability measure. We also discuss the issue of dynamic completeness. We show how any contingent claim can be replicated by trading in existing securities in a dynamically complete market and, therefore, valued by arbitrage considerations alone.

[1] The binomial model was first presented by Sharpe [1978] and was fully developed in Cox, Ross & Rubinstein [1979] and Rendleman & Bartter [1979].

Secondly, we analyze how to characterize arbitrage-free asset prices in a market where there are costs of trading securities. Here we show how the fundamental theorem of asset pricing needs to be generalized to take account of market frictions. The questions of replicating state-contingent payoffs in the presence of transactions costs and market incompleteness are also analyzed. We show that if one's objective is to replicate a given payoff, it might be cheaper to replicate a payoff greater than the one originally targeted. We also show how the problem of minimum-cost replication of state-contingent payoffs can be formulated and solved when transaction costs are present.

The two questions mentioned above relate to the behavior of security prices and contingent claims valuation. We also discuss the question of optimal portfolio choice in a complete or incomplete market, with or without transactions costs. The focus here is on the optimal portfolio choice behavior of an individual investor who takes as given an arbitrage-free process for asset prices. This question is typically analyzed using the dynamic programming approach, as shown, for example, in Hakansson [1970] & Merton [1971]. Recently, another approach has been devised to solve the portfolio choice problem which reduces the dynamic problem to a static one using the properties of arbitrage free security prices.[2] The static approach leads to an elegant characterization of the optimal portfolio choice problem. Moreover, it is easier to apply than the dynamic programming approach in some cases. Our analysis of the portfolio problem is based on the static approach. Again it is seen that the finite state framework is quite useful in bringing forward important economic insights related to the portfolio choice question.

This chapter builds on several excellent articles that analyze the connection between arbitrage and security prices in a finite state model. We now mention some of these articles, noting, however, that the objective here is not to conduct an exhaustive survey. Our analysis is based on Ross [1976, 1978], Dybvig & Ross [1986], Huang & Litzenberger [1988] and Kreps [1979]. The articles by Ross, and Dybvig and Ross present an analysis of the fundamental theorem of asset pricing in a two date economy. An extension of this theorem to a multi-period set-up can be found in Chapter 8 of Huang and Litzenberger. However, in the interest of simplicity of analysis, Huang and Litzenberger assume that preferences of investors are given by time additive expected utility. The case of securities markets with frictions is also not considered in these articles. The analysis in the presence of transactions costs and other market frictions is presented in Garman & Ohlson [1981], Prisman [1986], Dammon & Green [1987] and Ross [1987]. Finally, we draw on the work of Kreps [1979] where a definitive analysis of dynamic completeness and its welfare implications is carried out in a finite state model.

This chapter is organized as follows. Section 2 contains a description of the finite state securities market model that is used in our analysis. Section 3 discusses arbitrage and security prices in frictionless markets. Section 4 considers the case

[2] For a derivation of this approach, see Cox & Leland [1982], Cox & Huang [1989], He & Pearson [1991a, b], Karatzas, Lehoczky & Shreve [1987] and Karatzas, Lehoczky, Shreve & Xu [1991].