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Chapter 8

Algorithmic Implications of the Graph Minor Theorem

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1. Introduction

In the course of roughly the last ten years, Neil Robertson and Paul Seymour have led the way in developing a vast body of work in graph theory. One of their most celebrated results is a proof of an old and intractable conjecture in graph theory, previously known as Wagner's Conjecture, and now known as the Graph Minor Theorem. The purpose of this chapter is to describe some of the algorithmic ramifications of this powerful theorem and its consequences.

Significantly, many of the tools used in the proof of the Graph Minor Theorem can be applied to a very broad class of algorithmic problems. For example, Robertson and Seymour have obtained a relatively simple polynomial-time algorithm for the disjoint paths problem (described in detail later), a task that had eluded researchers for many years. Other applications include combinatorial problems from several domains, including network routing, utilization and design. Indeed, it is a critical measure of the value of the Graph Minor Theorem that so many applications are already known for it. Only the tip of the iceberg seems to have surfaced thus far. Many more important applications are being reported even as we write this.

The entire graph minors project is immense, containing almost 20 papers whose total length may exceed 600 pages. Thus we focus here primarily on some of the main algorithmic ideas, although a brief sketch of related issues is necessary. We assume the reader is familiar with basic concepts in graph theory [Bondy & Murty, 1976]. Except where noted otherwise, all graphs we consider are finite, simple and undirected.

2. A brief outline of the graph minors project

Three of the key notions employed are *minors*, *obstructions* and *well-quasi-orders*, and we examine them in that order.

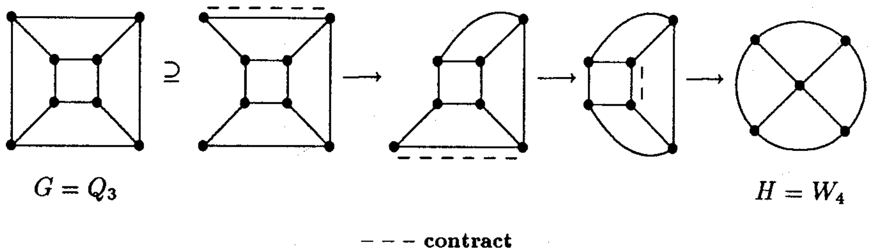

Fig. 1.

Minors. Given graphs H and G, we say that H is a minor of G (or that G contains H as a minor) if a graph isomorphic to H can be obtained by removing from G some vertices and edges and then contracting some edges in the resulting subgraph. Thus every graph is a minor of itself, and the single vertex graph is a minor of every nonempty graph. For a slightly less trivial example, see Figure 1, which illustrates that the wheel with four spokes (W_4) is a minor of the binary three-cube (Q_3).

A concept related to minor containment is *topological* containment. We say that a graph G is a *subdivision* of a graph H if G may be obtained by subdividing edges of H (an edge $\{u, v\}$ is subdivided by replacing $\{u, v\}$ with a path with ends u and v and whose internal vertices are new). We say that G topologically contains H if G contains a subgraph that is a subdivision of H. Thus topological containment is a special case of minor containment (we can only contract edges at least one of whose endpoints have degree two). Observe that W_4 is not topologically contained in Q_3.

Topological containment has been heavily studied by graph theorists. Perhaps the most famous theorem in this regard is Kuratowski's [1930]: a graph is planar if and only if it does not topologically contain K_5 or $K_{3,3}$. We note here that these two graphs are *minimally* nonplanar, that is, every graph topologically (and properly) contained in either of them is planar.

For the sake of exposition, let us view this theorem in terms of minors. Clearly, every minor of a planar graph is also planar. That is, the class of planar graphs is *closed* in the minor order. Consequently, no planar graph contains a K_5 or $K_{3,3}$ minor. Moreover, every proper minor of either of these two graphs is planar, and neither one contains the other as a minor. But can there be other minimal excluded minors? The answer is negative, for if G were such a purported graph, then G would be nonplanar, and thus it would contain, topologically (and therefore as a minor), either K_5 or $K_{3,3}$. In summary, a graph is planar if and only if it does not contain a K_5 or $K_{3,3}$ minor.

We note in passing two other points of interest concerning planarity. One is that planarity can be tested in polynomial time (in fact in linear time [Hopcroft & Tarjan, 1974]). The other is that a problem of natural interest is to try to extend Kuratowski's theorem to higher surfaces. (A surface is obtained from the sphere by 'gluing' onto it a finite number of 'handles' and/or 'crosscaps' [Massey, 1967].) A graph can be embedded on a given surface if it can be drawn on that