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Chapter 6

# Probabilistic Networks and Network Algorithms

*Timothy Law Snyder*
*Department of Computer Science, Georgetown University, Washington, DC 20057, U.S.A.*

*J. Michael Steele*
*Department of Statistics, The Wharton School, University of Pennsylvania, Philadelphia, PA 19104, U.S.A.*

## 1. Introduction

The uses of probability in the theory of networks are extensive, and new applications emerge at an increasing rate. Still, when compared with the purely deterministic aspects of network theory, the part that calls upon probability theory is in its infancy. Certainly there are areas where the uses of probability have developed into a reasonably complete theory, but in many instances the results that have been obtained have to be regarded as fragmented and incomplete. This situation presents considerable opportunity for researchers, and the purpose of this chapter is to highlight aspects of the current state of the theory with an eye toward the developments and the tools that seem most likely to be of value in further investigations.

Probability enters into the theory of networks and network algorithms in several different ways. The most direct way is through probabilistic modeling of some aspect of the network. For example, in some freight management models the cost of transportation along the arcs of the network are modeled by random variables. In models such as these probability helps us grasp a little better a world that comes with its own physical randomness.

A second important way probability enters is through more stylized stochastic models where the aim is to provide deeper insight into our technical understanding of the methods of operations research. Here there is considerably less emphasis on building detailed models that hope to capture aspects of randomness that live in a specific application context; rather, the aim is to provide mathematically tractable models of reasonable generality that can be used to explore a variety of different computational or estimation methods. Among the types of issues that have been studied in such models are the efficacies of deterministic algorithms and of deterministic heuristic methods. Many of the 'average case' analyses of algorithms would fit into this second role for probability.

The third path by which probability enters into network theory is through randomized algorithms. This is the newest of the roles for probability, but it is a role that is of increasing importance. To make certain of the distinction that makes an algorithm 'randomized,' consider a version of depth-first search where one chooses the next vertex to be explored by selecting it at random from a set of candidates. Here one does not call on any modeling of the network, which may in fact be specified in a way that is completely deterministic. The use of probability here is purely technical in the sense that it is employed to serve our computations, not to model some external physical randomness, or even to capture the notion of an 'average case.'

In the material that follows, one does well to keep these differing uses of probability in clear sight. Still, the distinctions may not always be pristine, mostly because two or more roles for probability can be present in the same problem. As an example, consider the computation or estimation of the reliability polynomial $R(p)$ of a network. Here one begins with a simple, physically motivated stochastic model. Given a specific graph intended to represent a communication network, one models the possibility of degraded communication in the network by allowing edges to 'fail' with probability $p$. The key problem is the determination of the probability $R(p)$ that for each pair of vertices $a$ and $b$ in the graph there exists a path from $a$ to $b$ that consists only of edges that have not failed. As the problem sits, it offers a simple but useful stochastic model, and one can go about the calculation or estimation of $R(p)$ by whatever tools are at one's disposal. The multiplicity of roles for probability enters exactly when one starts to notice that there are randomized *algorithms* for the estimation of $R(p)$. This is just one example where there are several roles for probability in the context of a single problem.

There are even dicier instances where the role of probability in the design and analysis of algorithms starts to offer some ambiguity. For example, close cousins of the randomized algorithms are the algorithms that (a) assume that the input follows some stochastic model and (b) exploit that assumption in the computational choices that it makes. A natural example of this design is Karp's algorithm for the Euclidean traveling salesman problem which we take up in Section 4. Such algorithms are fairly called *probabilistic algorithms*, but in the absence of internally generated random choices the best practice is to preserve the distinction made above and to avoid calling them randomized algorithms; though, admittedly, there is no reason to press for a rigid nomenclature.

The central aim of this chapter is to engage at least some aspect of each of the major roles for probability in the theory of network algorithms. When choices must be made, an emphasis is placed on those ideas one can expect to continue to be used and developed in the future. In Section 2 we engage the probability theory of network characteristics, where one mainly sees probability in either of the first two roles described above, as elements of either a physical or an idealized stochastic model. The section first develops the background for several inequalities that have evolved in the area of percolation theory. The FKG inequality is the best known and most widely used of these; but, as applications in percolation theory