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Chapter 4

The Traveling Salesman Problem

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1. Introduction

A traveling salesman wants to visit each of a set of towns exactly once, starting from and returning to his home town. One of his problems is to find the shortest such trip.

The traveling salesman problem, TSP for short, has model character in many branches of Mathematics, Computer Science, and Operations Research. Heuristics, linear programming, and branch and bound, which are still the main components of todays most successful approaches to hard combinatorial optimization problems, were first formulated for the TSP and used to solve practical problem instances in 1954 by Dantzig, Fulkerson and Johnson.

When the theory of NP-completeness developed, the TSP was one of the first problems to be proven NP-hard by Karp in 1972. New algorithmic techniques have first been developed for or at least have been applied to the TSP to show their effectiveness. Examples are branch and bound, Lagrangean relaxation, Lin–Kernighan type methods, simulated annealing, and the field of polyhedral combinatorics for hard combinatorial optimization problems (polyhedral cutting plane methods and branch and cut).

This chapter presents a self-contained introduction into algorithmic and computational aspects of the traveling salesman problem along with their theoretical prerequisites as seen from the point of view of an operations researcher who wants to solve practical instances. Lawler, Lenstra, Rinnooy Kan & Shmoys [1985] motivated considerable research in this area, most of which became apparent at the specialized conference on the TSP which took place at Rice University in 1990. This chapter is intended to be a guideline for the reader confronted with the question of how to attack a TSP instance depending on its size, its structural properties (e.g., metric), the available computation time, and the desired quality of the solution (which may range from, say, a 50% guarantee to optimality). In contrast to previous surveys, here we are concerned with practical problem solving, i.e., theoretical results are presented in a form which make clear their importance in the design of algorithms for approximate but provably good, and optimal solutions of the TSP. For space reasons, we concentrate on the symmetric TSP and discuss related problems only in terms of their practical importance and the structural and algorithmic insights they provide for the symmetric TSP.

For the long history of the TSP we refer to Hoffman & Wolfe [1985]. The relevant algorithmic approaches, however, have all taken place in the last 40 years. The developments until 1985 are contained in Lawler, Lenstra, Rinnooy Kan & Shmoys [1985]. This chapter gives the most recent significant developments. Historical remarks are confined to achievements which appear relevant from our point of view.

Let $K_n = (V_n, E_n)$ be the complete undirected graph with $n = |V_n|$ nodes and $m = |E_n| = \binom{n}{2}$ edges. An edge e with endpoints i and j is also denoted by ij, or by (i, j). We denote by $\mathbb{R}^{E_n}$ the space of real vectors whose components are indexed by the elements of E_n. The component of any vector $z \in \mathbb{R}^{E_n}$ indexed by the edge $e = ij$ is denoted by z_e, z_{ij}, or $z(i, j)$.

Given an objective function $c \in \mathbb{R}^{E_n}$, that associates a 'length' c_e with every edge e of K_n, the *symmetric traveling salesman problem* consists of finding a *Hamiltonian cycle* (a cycle visiting every node exactly once) such that its c-length (the sum of the lengths of its edges) is as small (large) as possible. Without loss of generality, we only consider the minimization version of the problem. From now on we use the abbreviation *TSP* only for the symmetric traveling salesman problem.

Of special interest are the Euclidean instances of the traveling salesman problem. In these instances the nodes defining the problem correspond to points in the 2-dimensional plane and the distance between two nodes is the Euclidean distance between their corresponding points. More generally, instances that satisfy the triangle inequality, i.e., $c_{ij} + c_{jk} \geq c_{ik}$ for all three distinct i, j, and k, are of particular interest.

The reason for using a complete graph in the definition of the TSP is that for such a graph the existence of a feasible solution is always guaranteed, while for general graphs deciding the existence of a Hamiltonian cycle is an NP-complete problem. Actually, the number of Hamiltonian cycles in K_n, i.e., the size of the set of feasible solutions of the TSP, is $(n-1)!/2$. The TSP defined on general graphs is shortly described in Section 2 along with other combinatorial optimization problems whose relation to the TSP is close enough to make the algorithmic techniques covered in this chapter promising for the solution with various degrees of suitability. In Section 3 we discuss a selection of practical applications of the TSP or one of its close relatives. The algorithmic treatment of the TSP starts in Section 4 in which we cover approximation algorithms that cannot guarantee to find the optimum, but which are the only available techniques for finding good solutions to large problem instances. To assess the quality of a solution, one has to be able to compute a lower bound on the value of the shortest Hamiltonian cycle. Section 5 presents several relaxations on which lower bound computations can be