D.P. Heyman, M.J. Sobel, Eds., *Handbooks in OR & MS*, *Vol. 2*

Chapter 1

Point Processes

*Richard F. Serfozo**

Department of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA 30332, U.S.A.

1. Introduction

1.1. Literature on point processes

A point process is a model for describing the random numbers of occurrences of a certain event in time intervals or of the numbers of points in regions of a space. Some archetypal examples are as follows.

Times at which a certain event occurs. Times of births, police emergencies, failures of a machine, insurance claims, earthquakes, etc.

Random flows or streams of items. Times at which items enter or leave a certain place such as telephone calls arriving to a switching center, data packets entering a computer, parts leaving a manufacturing work station, and cash flows in a company.

Random locations of points in an Euclidean space. Galaxies in space, errors in a computer code, animals in a forest, nerve cells in a brain and aircraft over a city.

Times of special events in a stochastic process. The instants when a Gaussian process crosses a certain level or when a pure-jump Markov process makes a certain type of transition. See Figure 1.

Time-space models of events. In addition to recording the occurrence times of a certain event there may be a need to record other attendant information about the event. Examples are the time and location of a demand for a service or product, the time and size of an insurance claim, or the time and type of a data packet entering a computer. This information can be modeled by a point process on the time-space set $\mathbb{R}_+ \times S$, where a point at the location (t, s) means that the event occurs at time $t \in \mathbb{R}_+ \equiv [0, \infty)$ and its attendant information is $s \in S$. See Figure 2.

Random locations of elements in an abstract set. One can talk of a point process in which the points are functions in a space of functions, lines in a set of lines on the plane, graphs in a set of graphs, etc.

* This work was sponsored in part by the Air Force Office of Scientific Research under contracts 84-0367 and F49620 85 C 0144.

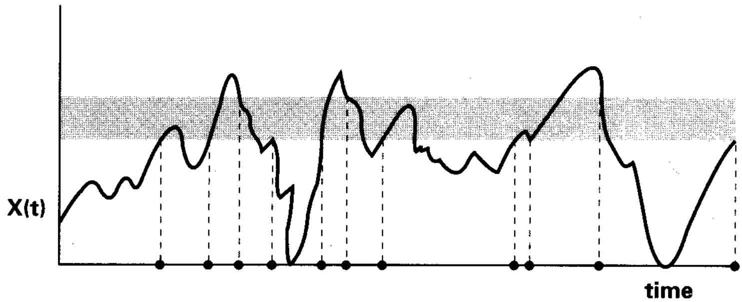

Fig. 1. Point process of times at which a process enters a region. The times at which X enters the special region form a point process on the time axis.

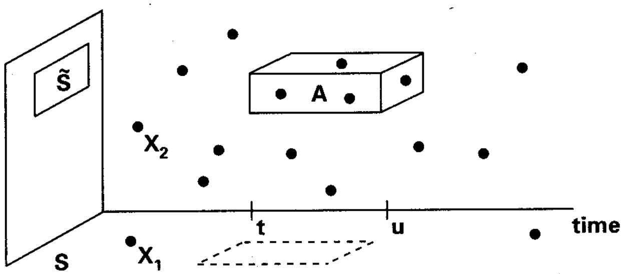

Fig. 2. Point process of electronic-mail originations. The $X_1, X_2, \ldots$ are point locations of a point process. An $X_n = (t, s_1, s_2)$ represents an e-mail origination at time t at the site (s_1, s_2) in S. The number $N(A)$ of these X_n's in the rectangle A is the number of e-mail originations in the region $\tilde{S}$ in the time interval $[t, u]$. Here $N(A) = 4$.

Although the theory of point processes has been developed only in the last three decades, its origins go back several centuries. Here are its major roots.

Poisson phenomena. Poisson [63] showed that the Poisson distribution is the limit of a binomial distribution of rare events. This led to numerous applications of the Poisson distribution in the nineteenth century and the eventual development of the Poisson process. Three notable Poisson applications before the modern era of probability were Abbe's (1878) work on spatial statistics, Erlang's [20] model of telephone calls to a trunk line and Bateman's [3] model of α-particle emitted from a radioactive substance. The preeminence of the Poisson process in the area of point processes is similar to that of the Brownian motion process in the area of real-valued stochastic processes. Their importance is due largely to the central limit phenomenon for stochastic processes. Brownian motion processes arise as limits of processes of sums of random variables. Similarly, Poisson processes arise as limits of sums of uniformly sparce point processes, or as limits of processes of rare events.

Life-tables, system reliability and renewal phenomena. The numerous studies of mortality based on life-tables from Graunt [26] to Lotka [53] are the