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METRIC SPACES OF FUZZY SETS

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Abstract: Two classes of metrics are introduced for spaces of fuzzy sets. Their equivalence is discussed and basic properties established. A characterisation of compact and locally compact subsets is given in terms of boundedness and p-mean equileft-continuity, and the spaces shown to be locally compact, complete and separable metric spaces.

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1. Introduction

Applications of fuzzy set theory very often involve the metric space $(\mathscr{E}^n, d_\infty)$, of normal fuzzy convex fuzzy sets over $\mathbf{R}^n$, where d_∞ denotes the supremum of the Hausdorff distances between corresponding level sets. This metric has been found very convenient in studying, for example, fuzzy random variables (Puri and Ralescu [11]), fuzzy differential equations (Kaleva [5]), dynamical systems (Kloeden [7]) and chaotic iterations of fuzzy sets (Diamond [1], Kloeden [8]). Indeed, many properties and applications of $(\mathscr{E}^n, d_\infty)$ can be regarded as generalisations of results involving the space $\mathscr{K}_{CO}(\mathbf{R}^n)$ of nonempty convex compacts endowed with the Hausdorff metric δ_∞. Both are complete metric spaces, and very recently compact sets have been completely characterised in $\mathscr{E}^n$ (Diamond and Kloeden [3]), thus extending the Blaschke selection theorem (see Lay [9]) to fuzzy sets.

However, the d_∞ metric fails to extend the Hausdorff metric topology in at least one important respect: the metric space $(\mathscr{E}^n, d_\infty)$ fails to be separable (Klement, Puri and Ralescu [6]). A different metric d_1 was introduced by Klement, Puri and Ralescu [6], such that the metric topology was separable, to prove a strong law of large numbers (SLLN) for fuzzy random variables. Both the d_1, d_∞ metrics share some of the less desirable properties of δ_∞. For example, variances of random variables taking values in $(\mathscr{K}_{CO}(\mathbf{R}^n), \delta_\infty)$ are not additive. Lyashenko [10] observed that an L_2 metric on $\mathscr{K}_{CO}(\mathbf{R}^n)$, defined by support functions, induces an appropriately additive variance, and this idea has been extended to the very special case of triangular fuzzy numbers by Diamond [2].

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The purpose of this note is to introduce and investigate two classes of metrics in $\mathscr{E}^n$. The first class d_p extends the Hausdorff metric and includes d_1, d_∞. The second class ρ_p is based upon L_p metrics for the support functions of compact convex sets (Vitale [12], and references therein) and includes the important L_2 case which induces additive variance. Our principal result is that, for each $1 \leq p < \infty$, the metric spaces $(\mathscr{E}^n, d_p)$ are equivalent to the corresponding ρ_p topology, and are complete, separable, locally compact metric spaces. Consequently, many important theorems (like the SLLN of Klement, Puri and Ralescu [6]) hold in all these equivalent spaces. A characterisation of the compact subsets in these spaces is also given.

Various definitions and preliminaries are set out in Section 2. Section 3 contains equivalence proofs, while the last section addresses compactness properties.

2. Preliminaries

As in [3], we restrict attention to the class of fuzzy sets $\mathscr{E}^n$, consisting of functions $u:\mathbf{R}^n \to I = [0, 1]$ for which

(1) u is normal, i.e. there exists an $x_0 \in \mathbf{R}^n$ such that $u(x_0) = 1$;

(2) u is fuzzy convex, i.e. for any $x, y \in \mathbf{R}^n$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\};$$

(3) u is uppersemicontinuous;

(4) the closure of $\{x \in \mathbf{R}^n: u(x) > 0\}$, denoted by $[u]^0$, is compact.

These properties imply that for each $0 < x \leq 1$, the α-level set $[u]^\alpha = \{x \in \mathbf{R}^n: u(x) \geq \alpha\}$ is a nonempty compact convex subset of $\mathbf{R}^n$, as is the support set $[u]^0$. The linear structure of $\mathscr{K}_{\mathrm{CO}}(\mathbf{R}^n)$ induces addition $u + v$ and scalar multiplication cu, $c \in \mathbf{R}^+$, on $\mathscr{E}^n$ in terms of the α-level sets, by

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \qquad [cu]^\alpha = c[u]^\alpha$$

for each $0 \leq \alpha \leq 1$.

To each $u \in \mathscr{E}^n$ there corresponds a support function $u^* \in C(I \times S^{n-1}, \mathbf{R})$, where S^{n-1} is the unit sphere in $\mathbf{R}^n$ (see [3] for details),

$$u^*(\alpha, x) = \sup_{a \in [u]^\alpha} \langle a, x \rangle, \quad \alpha \in I,\ x \in S^{n-1}.$$

Then u^* is well-defined for all $u \in \mathscr{E}^n$ and satisfies the following properties:

(1) u^* is uniformly bounded on $I \times S^{n-1}$,

$$|u^*(\alpha, x)| \leq \sup_{a \in [u]^0} |a| \quad \text{for all } \alpha \in I \text{ and all } x \in S^{n-1};$$

(2) $u^*(\cdot, x)$ is nonincreasing and left-continuous in $\alpha \in I$ for each $x \in S^{n-1}$;

(3) $u^*(\alpha, \cdot)$ is Lipschitz continuous in x uniformly in $\alpha \in I$,

$$|u^*(\alpha, x) - u^*(\alpha, y)| \leq \Big(\sup_{a \in [u]^0} |a|\Big)\, |x - y|,$$

for all $\alpha \in I$ and all $x, y \in S^{n-1}$.

In particular, if a is a nonempty compact, convex set in $\mathbf{R}^n$ and χ_A its characteristic function, then χ_A^* is the usual support function of A with domain S^{n-1}.

Let δ_∞ denote the Hausforff metric in $\mathscr{K}_{\mathrm{CO}}(\mathbf{R}^n)$,

$$\delta_\infty(A, B) = \min\left\{\inf_{a\in A}\sup_{b\in B}\|a-b\|,\ \inf_{b\in B}\sup_{a\in A}\|a-b\|\right\}.$$

We denote by δ_p, $1 \leqslant p < \infty$, the L_p metric on $\mathscr{K}_{\mathrm{CO}}(\mathbf{R}^n)$,

$$\delta_p(A, B) = \left(\int_{S^{n-1}} |\chi_A^*(x) - \chi_B^*(x)|^p \,\mu(\mathrm{d}x)\right)^{1/p},$$

where $\mu(\cdot)$ is unit Lebesgue measure on S^{n-1}. Then for each $\alpha \in I$ and $u, v \in \mathscr{E}^n$,

$$\delta_\infty([u]^\alpha, [v]^\alpha) = \sup_{x\in S^{n-1}} |u^*(\alpha, x) - v^*(\alpha, x)|.$$

Definition 1. For each $1 \leqslant p < \infty$ define

$$d_p(u, v) = \left(\int_0^1 \delta_\infty([u]^\alpha, [v]^\alpha)^p \,\mathrm{d}\alpha\right)^{1/p}.$$

and $d_\infty(u, v) = \sup_{0\leqslant\alpha\leqslant 1} \delta_\infty([u]^\alpha, [v]^\alpha)$.

Clearly d_p is defined for all $u, v \in \mathscr{E}^n$ by properties (1)–(3) immediately above, and $d_\infty(u, v) = \lim_{p\to\infty} d_p(u, v)$, with $d_p \leqslant d_q$ if $p \leqslant q$.

The other class of metrics is defined directly from L_p metrics on support functions.

Definition 2. For $1 \leqslant p < \infty$ put

$$\rho_p(u, v) = \left(\int_0^1 \delta_p([u]^\alpha, [v]^\alpha)^p \,\mathrm{d}\alpha\right)^{1/p}.$$

Again, properties (1)–(3) above imply that ρ_p is well-defined on $\mathscr{E}^n$. Observe that $\rho_p \leqslant \rho_q$ for all $1 \leqslant p \leqslant q < \infty$, and $\rho_p \leqslant d_p \leqslant d_\infty$ for $1 \leqslant p < \infty$. We shall see later as a consequence of of Theorem 2 stated below, that $\lim_{p\to\infty} \rho_p = \rho_\infty = d_\infty$.

Theorem 1. *$(\mathscr{E}^n, d_p)$, $(\mathscr{E}^n, \rho_p)$, $1 \leqslant p < \infty$, are metric spaces.*

Proof. The following argument is a modification of that of Proposition 3.1 in [6], which is for the d_1 metric. Symmetry of both d_p, ρ_p is clear, while the triangle inequality follows easily from Minkowski's inequality. It remains to show that $d_p(u, v) = 0$ implies $u = v$, and likewise for ρ_p. The result for d_p is a trivial extension of [6]. If $\rho_p(u, v) = 0$, then $\delta_p([u]^\alpha, [v]^\alpha) = 0$ a.e. in I. But δ_p is a metric on $\mathscr{K}_{\mathrm{CO}}(\mathbf{R}^n)$, so $[u]^\alpha = [v]^\alpha$ a.e. The argument of [6] again applies to give equality for all α, and hence the result follows.

The following estimate will be central to much of our considerations:

Theorem 2 (Vitale [12]). *Let $K, L \in \mathscr{K}_{CO}(\mathbf{R}^n)$ with $H = \operatorname{diam}(K \cup L)$. Then*

$$c_p(K, L)(\delta_\infty(K, L))^{(p+n-1)/p} \leqslant \delta_p(K, L) \leqslant \delta_\infty(K, L)$$

where $1 \leqslant p < \infty$,

$$c_p(K, L) = (B(p+1, n-1)/(H^{n-1}B(\tfrac{1}{2}, \tfrac{1}{2}(n-1))))^{1/p},$$

and $B(\cdot, \cdot)$ is the beta function.

It follows that $\lim_{p\to\infty} \rho_p(u, v) = d_\infty(u, v)$. For, if u, v are identical singletons, all metric distances are zero and equality holds trivially, while if this is not the case

$$\left(\int_0^1 c_p([u]^\alpha, [v]^\alpha)^p \delta_\infty([u]^\alpha, [v]^\alpha)^p \, d\alpha\right)^{1/p} \leqslant \rho_p(u, v) \leqslant d_\infty(u, v).$$

But $[u]^\alpha \cup [v]^\alpha \subseteq [u]^0 \cup [v]^0$, so $c_p([u]^\alpha, [v]^\alpha) \geqslant c_p([u]^0, [v]^0) = r_p(u, v)$, say. It thus suffices to show that $\lim_{p\to\infty} r_p(u, v) = 1$ and this follows since

$$\lim_{p\to\infty} B(p+1, n-1)^{1/p} = \lim_{p\to\infty} (\Gamma(p)/\Gamma(p+n))^{1/p} = 1.$$

Finally, the following notions will help characterise compactness in $\mathscr{E}^n$, and may be found in further detail in [3]. Say that $U \subset \mathscr{E}^n$ is *uniformly support bounded* if there is a constant $K > 0$ such that the support sets lie within a ball of radius K in $\mathbf{R}^n$ for all $u \in U$. A family of support functions $U^* = \{u^* : u \in U\}$ is called *equi-left-continuous* in $\alpha \in I$ uniformly in $x \in S^{n-1}$ if, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $u^*(\beta, x) \leqslant u^*(\alpha, x) < u^*(\beta, x) + \epsilon$ for all $\beta - \delta < \alpha \leqslant \beta$, $x \in S^{n-1}$ and $u^* \in U^*$. In addition, a set U of $\mathscr{E}^n$ is said to have the *Blaschke property* iff it is uniformly support bounded and U^* is equi-left-continuous. Diamond and Kloeden [3] showed that a closed set in $(\mathscr{E}^n, d_\infty)$ is compact iff it has the Blaschke property.

3. Equivalence of metrics

Our principal result is:

Theorem 3. *For each given p, $1 \leqslant p < \infty$, d_p and ρ_p induce equivalent topologies on $\mathscr{E}^n$ and yield complete, separable and locally compact metric spaces, in which closed sets with the Blaschke property are compact.*

The proof will be accomplished by showing the equivalence of the ρ_p topology to that induced by d_p, and then demonstrating that $(\mathscr{E}^n, d_p)$ has the requisite properties, through the following three lemmas. The local compactness is established as a corollary in the next section.

Lemma 1. *d_p, ρ_p induce the same topology on $\mathscr{E}^n$.*

Proof. From Theorem 2, for $u, v \in \mathscr{E}^n$, $\rho_p(u, v) \leqslant d_p(u, v)$, and

$$c_p([u]^0, [v]^0) d_{p+n-1}(u, v)^{(p+n-1)/p} \leqslant \rho_p(u, v)$$

provided $[u]^0$ and $[v]^0$ are not the same singleton set. Suppose v is fixed and $\{u_k\}$ a sequence in $\mathscr{E}^n$. If $[v]^0$ is not a singleton in $\mathbf{R}^n$, then $d_p(u_k, v) \to 0$ iff $\rho_p(u_k, v) \to 0$ as $k \to \infty$, since $d_{p+n-1}(u_k, v) \geq d_p(u_k, v)$. If v is a singleton, this argument breaks down if $u_k = v$, when the constant c_p is infinite. But then $d_p(u_k, v) = \rho_p(u_k, v) = 0$.

Lemma 2. *For* $1 \leq p < \infty$, $(\mathscr{E}^n, d_p)$ *is separable.*

Proof. This is adapted from [6], but uses a somewhat different construction to ensure that the approximating fuzzy sets are fuzzy convex. Take any $u \in \mathscr{E}^n$ and $\epsilon > 0$. Since $[u]^0$ is compact, there exists a minimal cover $\{S_i\}$ of cubes $S_i = \prod_{j=1}^n [a_{ij}, b_{ij})$, $i = 1, \ldots, r$, with $a_{ij}, b_{ij} \in \mathbf{Q}$, $0 < b_{ij} - a_{ij} < \varepsilon/(4n^{1/2})$ and $[u]^0 \subseteq \bigcup_{i=1}^r S_i$. From fuzzy convexity, for each $0 \leq \alpha \leq 1$, $[u]^\alpha$ has a minimal sub-cover $\{S_{i(\alpha)}\} \subseteq \{S_i\}$. Write $V_k = [u]^0 \cup (\bigcup_k^r S_i)$. Note that $\delta_\infty([u]^0, \bigcup_1^r S_i) \leq \frac{1}{4}\epsilon$, $\delta_\infty([u]^\alpha, \bigcup_{i(\alpha)} S_{i(\alpha)}) < \frac{1}{4}\epsilon$ and $\delta_\infty(V_k, \bigcup_k^r S_i) \leq \frac{1}{4}\epsilon$. Write $\alpha_i = \sup_{x \in \bar{S}_i} u(x)$ and relabel $S_1, \ldots, S_r$ so that $0 = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_r = 1$. Define the fuzzy set ϕ_0 by

$$\phi_0(x) = \begin{cases} x_i & \text{if } x \in S_i,\ 1 \leq i \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

For any $0 < \alpha \leq 1$, there exists $1 \leq k \leq r$ so that $\alpha_{k-1} \leq \alpha \leq \alpha_k$. If k is the largest integer such that u is constant on $[u]^0 \cup (\bigcup_1^k S_i)$, then $\delta_\infty([u]^\alpha, [\phi_0]^\alpha) = \delta_\infty(V_k, \bigcup_k^r S_i) < \frac{1}{4}\epsilon$. If $\alpha_{k-1} = \alpha < \alpha_k$, then

$$\delta_\infty([u]^\alpha, [\phi_0]^\alpha) \leq \delta_\infty\left([u]^{\alpha_{k-1}}, \bigcup_k^r S_i\right) = \delta_\infty\left(V_{k-1}, \bigcup_k^r S_i\right) < \tfrac{1}{4}\epsilon,$$

and similarly if $\alpha_{k-1} < \alpha = \alpha_k$. For $\alpha_{k-1} < \alpha < \alpha_k$,

$$\delta_\infty([u]^\alpha, [\phi_0]^\alpha) \leq \delta_\infty\left([u]^{\alpha_{k-1}}, \bigcup_k^r S_i\right) \leq 2 \max_{1 \leq i \leq r} \operatorname{diam}(S_i) < \tfrac{1}{2}\epsilon.$$

Define the fuzzy convex set $\phi \in \mathscr{E}^n$ by $[\phi]^\alpha = \overline{\text{co}}[\phi_0]^\alpha$, $0 \leq \alpha \leq 1$, where $\overline{\text{co}}$ denotes the closed convex hull. Then

$$\delta_\infty([\phi]^\alpha, [\phi_0]^\alpha) = \max_{1 \leq k \leq r} \delta_\infty\left(\bigcup_{i(\alpha_k)} S_{i(\alpha_k)}, \overline{\text{co}} \bigcup_{i(\alpha_k)} S_{i(\alpha_k)}\right) < \tfrac{1}{4}\epsilon.$$

Thus $d_p(\phi, \phi_0) = (\int_0^1 \delta_\infty([\phi]^\alpha, [\phi_0]^\alpha)^p \, d\alpha)^{1/p} < \frac{1}{4}\epsilon$ and so $d_p(u, \phi) \leq d_p(u, \phi_0) + d_p(\phi, \phi_0) < \frac{3}{4}\epsilon$.

Now let $M > 4(r-1)\operatorname{diam}([u]^0)$. For $i = 1, \ldots, r$, relabel $\alpha_1, \ldots, \alpha_r$, and exclude duplicates if necessary, so that $0 \leq \alpha_0 < \alpha_1 \cdots < \alpha_s = 1$, with $s \leq r$. If $\alpha_i \notin \mathbf{Q}$, choose $\beta_i \in \mathbf{Q}$ such that

$$\max\{\alpha_{i-1}, \alpha_i - \epsilon^p/M^p\} < \beta_i < \alpha_i$$

and if $\alpha_i \in \mathbf{Q}$, set $\beta_i = \alpha_i$. Define $\psi \in \mathscr{E}^n$ by

$$\psi(x) = \begin{cases} \beta_i & \text{if } \phi(x) = \alpha_i, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the class of all such ψ is countable, and that

$$d_p(\phi, \psi) \leqslant \sum_{i=1}^{s-1} \left(\int_{\beta_i}^{\alpha_i} \delta_\infty([\phi]^\alpha, [\psi]^\alpha)^p \, d\alpha \right)^{1/p}$$

$$\leqslant \mathrm{diam}([u]^0) \sum_{1}^{s-1} (\alpha_i - \beta_i)^{1/p} < \tfrac{1}{4}\epsilon.$$

Finally, $d_p(u, \psi) \leqslant d_p(u, \phi) + d_p(\phi, \psi) < \epsilon$ and the result follows.

Lemma 3. *Every closed set in* $(\mathscr{E}^n, d_p)$ *which has the Blaschke property is compact.*

Proof. Each d_p topology, $1 \leqslant p < \infty$, embeds continuously into $(\mathscr{E}^n, d_\infty)$, since $d_p \leqslant d_\infty$. Consequently, the compact sets in the latter space, which are precisely the closed Blaschke sets [3], are compact in $(\mathscr{E}^n, d_p)$.

4. Compactness in d_p topology

The Blaschke property is sufficient for compactness of a closed set (Lemma 3), in the d_p metric topology, but it is too strong to be also necessary. This is because $d_p \leqslant d_\infty$ and we seek a condition more appropriate for L_p type spaces, related to equi-left-continuity in the stronger topology. Let $u \in \mathscr{E}^n$. If for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon, u) > 0$ such that for all $0 \leqslant h < \delta$,

$$\int_h^1 \delta_\infty([u]^\alpha, [u]^{\alpha-h})^p \, d\alpha < \epsilon^p, \quad 1 \leqslant p < \infty,$$

say that u is *p-mean left-continuous*. If for nonempty $U \subset \mathscr{E}^n$ this holds uniformly in $u \in U$, we say U is *p-mean equi-left-continuous*. If, in addition, U is uniformly support bounded, then U is said to have the *p-Blaschke property*. Observe that for the corresponding family of support functions, this property translates as

$$\int_h^1 (u^*(\alpha - h, x) - u^*(\alpha, x))^p \, d\alpha < \epsilon^p$$

for all $0 \leqslant h < \delta$, $x \in S^{n-1}$ and $u^* \in U^*$, and that in the limit $p = \infty$ this concept is just the Blaschke property of the previous section. However, p-Blaschke$\not\Rightarrow$ Blaschke (although the converse is true), as the following example shows.

Counterexample. Define $U \subset \mathscr{E}^n$ by $U = \{\tilde{u}, u_1, u_2, \ldots\}$ where

$$u_n(x) = \begin{cases} x^n & \text{if } 0 \leqslant x \leqslant 1, \\ 0 & \text{if } x \notin [0, 1], \end{cases}$$

and $\tilde{u}(x) = 1$ if $x = 1$, and 0 otherwise. Clearly U is uniformly support bounded. We show u to be 1-mean equi-left-continuous, and thus 1-Blaschke, but not equi-left-continuous. For $0<h<1$,

$$\begin{aligned}\int_h^1 d_\infty([u_n]^{\alpha-h}, [u_n]^\alpha)\,\mathrm{d}\alpha &= \int_h^1 (\alpha^n-(\alpha-h)^n)\,\mathrm{d}\alpha\\ &=(n+1)^{-1}[\alpha^{n+1}-(\alpha-h)^{n+1}]_h^1\\ &=(n+1)^{-1}(1-(1-h)^{n+1}-h^{n+1})\\ &\leqslant \tfrac{1}{2}(1-(1-h)^2-h^2)\\ &=h-h^2\leqslant h\end{aligned}$$

for all $0\leqslant h\leqslant 1$ and all $n\geqslant 1$. Thus $\int_h^1 d_\infty([u_n)^{\alpha-h}, [u_n]^\alpha)\,\mathrm{d}\alpha\leqslant\epsilon$ if $0<h<\delta(\epsilon)=\epsilon$ and this is the required condition. On the other hand,

$$\sup_{h\leqslant\alpha\leqslant 1} d_\infty([u]^{\alpha-h}, [u]^\alpha)=\sup_{h\leqslant\alpha\leqslant 1}|\alpha^n-(\alpha-h)^n|=1-(1-h)^n$$

and for $h\geqslant 0$ this has supremum 1 as $n\to\infty$, while for $h=0$ the supremum is 0. Consequently, u is not equi-left-continuous, and thus not Blaschke.

Lemma 4. *Any $u\in(\mathscr{E}^n, d_p)$, $1\leqslant p<\infty$, is p-mean left-continuous.*

Proof. Let $\alpha\in[0, 1]$ and suppose $\{\alpha_n\}$ is a nondecreasing sequence converging to α. Then $[u]^\alpha=\bigcap_{n=1}^\infty [u]^{\alpha_n}$ and $\delta_\infty([u]^{\alpha_n}, [u]^\alpha)\to 0$, and the result follows from left-continuity on the compact interval $[0, 1]$.

Theorem 4. *A closed set U of $(\mathscr{E}^n, d_p)$, $1\leqslant p\leqslant\infty$, is compact iff U has the p-Blaschke property.*

Proof. For $p=\infty$, the result is that of [3], so suppose $1\leqslant p<\infty$.

Necessity. Let u be a compact set in $(\mathscr{E}^n, d_p)$. If u were not uniformly support bounded, then there would exist a sequence of compact convex sets in $\mathbf{R}^n$, $V_j=[u_j]^0$, $u_j\in U$, such that $\delta_\infty(V_j, \{0\})>j$. Clearly $\{V_j\}$ has no subsequence with limit in $\mathscr{K}_{\mathrm{CO}}(\mathbf{R}^n)$. But since U is compact, there is a subsequence $\{u_{j(k)}\}$ converging to $u\in U$, and $\lim_{k\to\infty} V_{j(k)}=[u]^0\in\mathscr{K}_{\mathrm{CO}}(\mathbf{R}^n)$ which is impossible. Hence U must be uniformly support bounded.

Let $\epsilon>0$ and let $u_1, u_2, \ldots, u_k\in\mathscr{E}^n$ be a $\frac{1}{3}\epsilon$-cover of U, that is for any $u\in U$ one of the sequence elements u_i satisfies $d_p(u, u_i)<\frac{1}{3}\epsilon$. Such a sequence exists by compactness of U. By Lemma 4, $u_1, \ldots, u_k$ are p-mean left-continuous and so there exists $\delta(\epsilon)=\min_{1\leqslant i\leqslant k}\delta(\epsilon, u_i)>0$ such that $\int_h^1 \delta_\infty([u_i]^{\alpha-h}, [u_i]^\alpha)^p\,\mathrm{d}\alpha<(\frac{1}{3}\epsilon)^p$ for $i=1, \ldots, k$ and $0\leqslant h<\delta(\epsilon)$. Thus for $u\in U$, the triangle inequality gives

$$\begin{aligned}\left(\int_j^1 \delta_\infty([u]^{\alpha-h}, [u]^\alpha)^p\,\mathrm{d}\alpha\right)^{1/p} &\leqslant d_p(u, u_i)+\left(\int_h^1 \delta_\infty([u_i]^{\alpha-h}, [u_i]^\alpha)^p\,\mathrm{d}\alpha\right)^{1/p}+d_p(u_i, u)\\ &\leqslant\epsilon,\end{aligned}$$

so U is p-mean equi-left-continuous.

Sufficiency. This adapts an argument of [3] and is only sketched. Let $\{u_k\}$ be a sequence in U and $\{u_k^*\}$ the corresponding sequence in U^*. Let $D_1 = \{\alpha_i\}$, $D_2 = \{X_j\}$ be countable dense subsets of I and S^{n-1} respectively. The usual diagonalisation construction gives a subsequence $\{u_{k(k)}^*\}$ and a function $g : D_1 \times D_2 \to \mathbf{R}$ such that $u_{k(k)}^*(\alpha_i, \alpha_j) \to g(\alpha_i, x_j)$ uniformly in $(\alpha_i, x_j) \in D_1 \times D_2$ as $k \to \infty$. For notational simplicity write $w_k^* = u_{k(k)}^*$, $w_k = u_{k(k)}$.

Since U is uniformly support bounded, there exists $K > 0$ such that

$$|w_k^*(\alpha_i, x) - w_k^*(\alpha_i, y)| \leqslant \left(\sup_{a \in [w_k]^0} |a| \right) |x - y| = K(x - y|$$

for all $\alpha_i \in D_1$ and any $w_k^* \in U^*$. That is, the $w_k^*(\alpha_i, \cdot)$ are equicontinuous on S^{n-1}, uniformly in $\alpha_i \in D_1$. Hence the sequence $\{w_k^*(\alpha_i, x)\}$ converges for each $\alpha_i \in D_1$ and $x \in S^{n-1}$ in the d_∞ and hence d_p norms, and we denote the limits by $g(\alpha_i, x)$. As in [3] (see also [4]) such convergence is uniform in S^{n-1}, and moreover is uniform in D_1 as well, for the sup norm, and thus for d_p norm.

From the properties of the $w_k^* \in U^*$ it follows that

(1) $|g(\alpha_i, x)| \leqslant K$ for all $\alpha_i \in D_1$ and $x \in S^{n-1}$;

(2) $|g(\alpha_i, x) - g(\alpha_i, y)| \leqslant K\, |x - y|$ for all $\alpha_i \in D_1$;

(3) $g(\alpha_i, x) \leqslant g(\beta_i, x)$ for all $\beta_i \leqslant \alpha_i$ in D_1 and $x \in S^{n-1}$.

Then for each $(\alpha, x) \in I \times S^{n-1}$, define

$$g(\alpha, x) = \lim_{\alpha_i \to \alpha -} g(\alpha_i, x), \quad \alpha_i \in D_1.$$

Each such exists because $g(\cdot, x)$ is nonincreasing in $\alpha_i \in D_1$ and bounded. This defines g on all of $I \times S^{n-1}$, and in such a way that the three properties, immediately above, hold for g on all of $I \times S^{n-1}$. These, together with the left-continuity of $g(\cdot, x)$, show that $g(\cdot, \cdot)$ is the support functioin of a well-defined fuzzy set w whose support lies in $\bigcup_{u \in U} [u]^0$. It remains to show that $d_p(w_k, w) \to 0$ as $k \to \infty$.

By p-mean equi-left-continuity, for a monotonic nondecreasing sequence $\alpha_i = \alpha - h_i \in D_1$,

$$\int_{h_1}^{1} \delta_\infty([w_k]^{\alpha - h_i}, [w_k]^\alpha)^p \, \mathrm{d}\alpha < (\tfrac{1}{2}\epsilon)^p$$

provided $0 \leqslant h_i < \delta$ for $\delta = \delta(\epsilon) > 0$, uniformly in $w_k \in U$. But for $k > N(\frac{1}{2}\epsilon)$, $g(\alpha_i, x) - \frac{1}{2}\epsilon < w_k^*(\alpha_i, x) < g(\alpha_i, x)$ uniformly in S^{n-1} and since g is non-decreasing,

$$g(\alpha, x) - \tfrac{1}{2}\epsilon \leqslant g(\alpha_i, x) - \tfrac{1}{2}\epsilon < w_k^*(\alpha_i, x) < g(\alpha, x).$$

Thus $\delta_\infty([w_k]^{\alpha_i}, [w]^\alpha) = \sup_{x \in S^{n-1}} |w_k^*(\alpha_i, x) - g(\alpha, x)| < \frac{1}{2}\epsilon$. Hence

$$d_p([w_k]^\alpha, [w]^\alpha) \leqslant \left(\int_{h_i}^{1} \delta_\infty([w_k]^\alpha, [w_k]^{\alpha - h_i})^p \, \mathrm{d}\alpha \right)^{1/p} + \left(\int_{h_i}^{1} \delta_\infty([w_k]^{\alpha - h_i}, [w]^\alpha)^p \, \mathrm{d}\alpha \right)^{1/p} < \epsilon$$

for all $k > N(\frac{1}{2}\epsilon)$.

Corollary. *The space* $(\mathscr{E}^n, d_p)$, $1 \leqslant p < \infty$, *is locally compact. Moreover a subset U is locally compact iff every uniformly support bounded and closed subset of U is p-Blaschke.*

Proof. For sufficiency, let $U \subset \mathscr{E}^n$ be such that any uniformly support bounded and closed set is p-Blaschke, and take $u \in U$. Since u has compact support, there exists $K > 0$ such that $d_p(u, \{0\}) \leqslant K$. Then $N_\eta(u) = \{v: d_p(u, v) < \eta\}$ form a neighbourhood basis of u, and for every $w \in N_\eta(u)$, $d_p(w, \{0\}) \leqslant d_p(w, u) + d_p(u, \{0\}) \leqslant K + \eta$. So $N_\eta(u)$ is uniformly support bounded, and hence p-Blaschke. So $\mathrm{cl}(N_\eta(u))$ is compact, and U is locally compact. For necessity, note that $(\mathscr{E}^n, d_p)$, $1 \leqslant p < \infty$, is actually a locally compact space, since the same argument shows every point of the metric space has a compact neighbourhood. Since, for $1 \leqslant p < \infty$, the space is also separable, $\mathscr{E}^n = \bigcup_{k \geqslant 1} U_k$ where $U_1 \subseteq \cdots \subseteq U_k \subseteq U_{k+1} \subseteq \cdots$ and the U_k are p-Blaschke. So any closed subset of U that is uniformly support bounded lies in one of the u_k, for some sufficiently large k, and is thus p-mean equi-left-continuous, and so p-Blaschke.

Remark. The space $(\mathscr{E}^n, d_\infty)$ is not locally compact, in contrast to the above.

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